Entanglement in Elastic and Inelastic Two-Hadron Scattering

Conference in honor of Andrzej Bialas on his 90^{th} *anniversary*

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At high energy (S-Matrix): R.P. and Shigenori Seki (Setsunan U.)
At low energy (QM): Bertrand Giraud and R.P (both Saclay) .

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Part I: Entanglement in a S-Matrix Framework

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PLB(2016), PRD(2019), PRD(2026)

Keywords: S-Matrix, two-body collisions, high energy,
Entanglement entropy in momentum from density matrices.

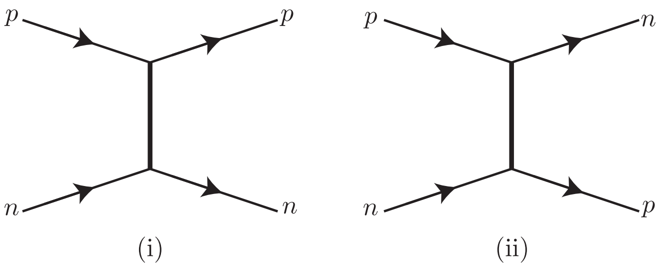


Fig.: (i) the elastic channel: $pn \to pn$. (ii) the inelastic channel: $pn \to np$.

Goal: How to measure the entanglement between two hadrons, A and B, in a final state generated by a collision, elastic or inelastic.

Method:

Using

- S-Matrix theory (high energy)
- Reduced density matrices
- Von Neumann Entanglement Entropy

$$S_A = Tr(\rho_A \log \rho_A) \quad \rho_A = Tr_{A,\,fixed}(\rho \log \rho)$$

- and Quantum Mechanics for comparison (see Part II)

Note: *S-Matrix formalism inspired by L. Van Hove, Nuovo Cim.* **28** *(1963) 2344, and A. Białas and L. Van Hove, Nuovo Cim.* **38** *(1965) 1385.*

Formalism

$$A_1 B_1 \to \begin{cases} A_1 B_1 & \text{(elastic channel)} \\ A_2 B_2 & \text{(two-particle inelastic channel)} \\ X & \text{(more than two-particle channels)} \end{cases}$$

The inner product of states in the center-of-mass frame is

$${}_i\langle \boldsymbol{p} | \boldsymbol{q} \rangle_j = \delta_{ij}\ 2E_{A_i \boldsymbol{p}}\ 2E_{B_i \boldsymbol{p}}\ \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \delta^{(3)}(0) \quad (i, j = 1, 2)$$

The complete set of orthogonal basis is

$$\mathbf{1} = \sum_{h=1}^{2} \int \frac{d^3 \boldsymbol{p}}{2E_{A_h \boldsymbol{p}}\, 2E_{B_h \boldsymbol{p}} \delta^{(3)}(0)} |\boldsymbol{p}\rangle_h\, {}_h\langle \boldsymbol{p}| + \int dX\, |X\rangle \langle X|$$

Density Matrices

$$\rho_1 = \frac{1}{\mathcal{N}_1} |A_1\rangle \langle A_1| , \quad \rho_2 = \frac{1}{\mathcal{N}_2} |A_2\rangle \langle A_2|$$

$\mathcal{N}_i$ $(i = 1, 2)$ are normalization factors such that $\mathrm{tr}_{A_i} \mathrm{tr}_{B_i} \rho_i = 1$

Reduced density matrices (w.r.t.) particle $A_i, i = 1, 2$

$$\rho_{A_1} = \frac{1}{\mathcal{N}_1} \int \frac{d^3\boldsymbol{p}}{2E_{A_1\boldsymbol{p}}} \frac{\delta(E_{A_1\boldsymbol{p}} + E_{B_1(\boldsymbol{k}+\boldsymbol{l}-\boldsymbol{p})} - E_{A_1\boldsymbol{k}} - E_{B_1\boldsymbol{l}})\delta(0)}{2E_{A_1\boldsymbol{p}}\, 2E_{B_1(\boldsymbol{k}+\boldsymbol{l}-\boldsymbol{p})}} \times |{}_1\langle \boldsymbol{p}, \boldsymbol{k} + \boldsymbol{l} - \boldsymbol{p}|\mathbf{S}|\boldsymbol{k}, \boldsymbol{l}\rangle_1|^2 \, |\boldsymbol{p}\rangle_{A_1\ A_1}\langle \boldsymbol{p}|$$

$$\rho_{A_2} = \frac{1}{\mathcal{N}_2} \int \frac{d^3\boldsymbol{p}}{2E_{A_2\boldsymbol{p}}} \frac{\delta(E_{A_2\boldsymbol{p}} + E_{B_2(\boldsymbol{k}+\boldsymbol{l}-\boldsymbol{p})} - E_{A_1\boldsymbol{k}} - E_{B_1\boldsymbol{l}})\delta(0)}{2E_{A_2\boldsymbol{p}}\, 2E_{B_2(\boldsymbol{k}+\boldsymbol{l}-\boldsymbol{p})}} \times |{}_2\langle \boldsymbol{p}, \boldsymbol{k} + \boldsymbol{l} - \boldsymbol{p}|2\mathbf{T}|\boldsymbol{k}, \boldsymbol{l}\rangle_1|^2 |\boldsymbol{p}\rangle_{A_2\ A_2}\langle \boldsymbol{p}|$$

By definition of the S-Matrix: $\mathbf{S} = \mathbf{1} + 2i\mathbf{T}$ and $\langle A_1|\mathbf{1}|A_2\rangle = 0$

Entangled Entropy

$$S_i = -\,\mathrm{tr}_{A_i}\,\rho_{A_i} \ln \rho_{A_i} = -\lim_{n\to 1} \frac{\partial}{\partial n}\,\mathrm{tr}_{A_i}(\rho_{A_i})^n,$$

where $\mathrm{tr}_{A_i}(\rho_{A_i})^n$ is the Renyi entropy. After more or less tedious calculations one finds

$$\mathrm{tr}_{A_i}(\rho_{A_i})^n = \left(\frac{2}{V}\right)^{n-1} \int_{-1}^{1} d\cos\theta\,(\mathcal{P}_i(\cos\theta))^n$$

with

$$\mathcal{P}_1(\cos\theta) = \delta(1-\cos\theta) \times \left\{1 - \frac{\sigma_1}{\frac{\pi}{k^2}V - (\sigma^{\mathrm{tot}} - \sigma_1)}\right\} + \frac{2k^2}{\frac{\pi}{k^2}V - (\sigma^{\mathrm{tot}} - \sigma_1)}\,\frac{d\sigma_1}{dt}$$

$$\mathcal{P}_2(\cos\theta) = \frac{2k^2}{\sigma_2}\frac{d\sigma_2}{dt}$$

V : Hilbert space volume

needs regularization

$$V = \langle A_1|\mathbf{1}|A_1\rangle = \sum_{\ell=0}^{\infty}(2\ell+1) = 2\delta(0)$$

In the equation for $\mathcal{P}_1$ the $\delta(1-\cos\theta)$ term is not physical: we thus have to define a regularized volume $\tilde{V}$ such that

$$0 = 1 - \frac{\sigma_{11}}{\frac{\pi}{k^2}\tilde{V} - (\sigma_1^{\rm tot} - \sigma_1)}$$

$$\tilde{V} = \frac{k^2}{\pi}\sigma_1^{\rm tot}$$

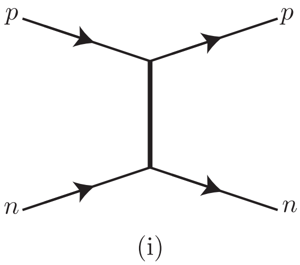
Finally, we get the entanglement entropies

$$\tilde{S}_i = \ln\frac{k^2\sigma_1^{\rm tot}}{2\pi} - \int_{-1}^{1} d\cos\theta\,\tilde{\mathcal{P}}_i\ln\tilde{\mathcal{P}}_i\,,\quad \tilde{\mathcal{P}}_i = \frac{1}{\sigma_i}\frac{d\sigma_i}{d\cos\theta}$$

for elastic ($i=1$) and inelastic ($i=2$) 2-hadron scattering

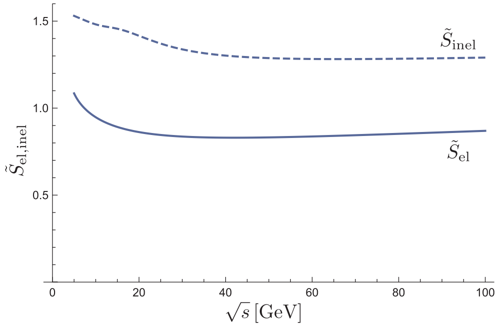
Evaluation of entanglement entropy

$$pn \to \begin{cases} pn & \text{(elastic channel)} \\ np & \text{(inelastic channel)} \end{cases}$$



Results

Using the useful np charge exchange analysis (Bouquet and Diu, 1978), one finds



Why $\tilde{S}_{\rm inel} > \tilde{S}_{\rm el}$?

$$\text{Suppose :} \qquad \frac{d\sigma_{\rm el}}{dt} \sim \sigma_{\rm el}(s)\, e^{\bar{R}_{\rm el}^2 t}\,, \quad \frac{d\sigma_{\rm inel}}{dt} \sim \sigma_{\rm inel}(s)\, e^{\bar{R}_{\rm inel}^2 t}$$

$$\bar{R} = \quad \text{average radius}$$

$$\tilde{S}_{\rm el} = \ln\left(\frac{\sigma_{\rm tot}}{4\pi \bar{R}_{\rm el}^2}\right) + 1$$

$$\tilde{S}_{\rm inel} = \ln\left(\frac{\sigma_{\rm tot}}{4\pi \bar{R}_{\rm inel}^2}\right) + 1$$

$$\Delta\tilde{S} = \tilde{S}_{\rm inel} - \tilde{S}_{\rm el} = \ln\left(\frac{\bar{R}_{\rm el}^2}{\bar{R}_{\rm inel}^2}\right)$$

$$\bar{R}_{\rm inel} \quad < \quad \bar{R}_{\rm el} \quad \Rightarrow \quad \tilde{S}_{\rm inel} \quad > \quad \tilde{S}_{\rm inel}$$

Density of entanglement entropy in transverse momentum

$$\tilde{S}_{el,inel} = \ln \frac{k^2 \sigma_1^{\text{tot}}}{2\pi} - \int_{-1}^{1} d\cos\theta \, \tilde{\mathcal{P}}_{el,inel} \ln \tilde{\mathcal{P}}_{el,inel}$$

$$\tilde{\mathcal{P}}_{el,inel} = \frac{1}{\sigma_{el,inel}} \frac{d\sigma_{el,inel}}{d\cos\theta}$$

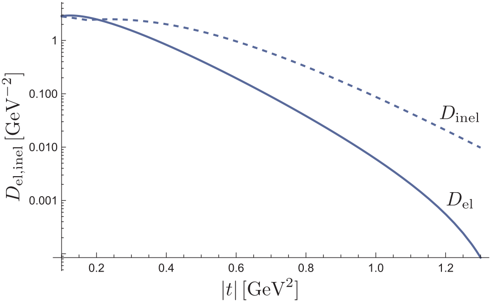
Change of variable: $\cos\theta \to t$, transverse momentum squared

$$\tilde{\mathcal{P}}(\cos\theta)\, d\cos\theta = P(t)\, dt$$

$$\tilde{S}_{\text{el}} = \int_0^{4k^2} d|t|\, D_{\text{el}}(t)\,, \quad D_{\text{el}}(t) \equiv -P_{\text{el}}(t) \ln\left(\frac{4\pi}{\sigma_{\text{tot}}} P_{\text{el}}(t)\right)$$

$$\tilde{S}_{\text{inel}} = \int_0^{4k^2} d|t|\, D_{\text{inel}}(t)\,, \quad D_{\text{inel}}(t) \equiv -P_{\text{inel}}(t) \ln\left(\frac{4\pi}{\sigma_{\text{tot}}} P_{\text{inel}}(t)\right)$$

Results for $pn \to np$



The elastic density $D_{\rm el}$ (solid line) and the inelastic one $D_{\rm inel}$ (dashed line) as functions of $|t|$ at $\sqrt{s} = 20$ GeV

Note: Not much energy dependence

Conclusions:

• Parallel treatment of entanglement formulation to two-hadron elastic and inelastic scattering
• Formulation in terms of cross-section observables: e.g. $np \to pn$ $np \to np$
• More entanglement for $np \to pn$ than $np \to np$
• Description of the entanglement entropy density as a function of momentum transfer

Questions:

• Is it a general rule that inelastic collisions (i.e. with nonvacuum exchange) lead to more entanglement ?
• Can we observe the "flow of entanglement" using the entanglement entropy densities?
• Is it a non trivial coincidence that $D_{inel}(t=0) \sim D_{inel}(t=0)$?

More generally

• Can entanglement after a two-hadron collision can teach us something brand new about strong interactions ... could it be true also for other two-body scattering interactions ?

Remain tuned !

Part II Toy Model for Entanglement in a Quantum Mechanical Collision

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Keywords: Time dependent Shroedinger, Wave packet collision, Entanglement entropy from density matrices in one spatial dimension.

Simple hypotheses

1-d, two-particle collision driven by Schroedinger equation, initial state product of two independent Gaussian wave packets, entanglement measured from one-body matrices.

Consider a toy non relativistic Hamilltonian for two particles in one dimension, with unit mass and $\hbar = 1$ with p_i the momentum and x_i the position of particle i, $i = 1, 2$, in a potential v.
It reads,

$$H = P^2/4 + H_{rel}, \quad H_{rel} = p^2 + v(|x|).$$

$P = p_1 + p_2$ is the center-of-mass (cm) momentum,
$X = (x_1 + x_2)/2$ is the cm momentum.
$p = (p_1 - p_2)/2$ and $x = x_1 - x_2$ define the relative motion degrees of freedom governed by the Hamiltonian H_{rel}.

Measure of entanglement

Let $\rho_{12}(t) \equiv \rho_{12}(x_1, x_1', x_2, x_2'; t)$ be the density matrix of the system, and $\rho_i(t) = Tr_i\, \rho_{12}(t)$, be the one-body density matrices.

If the particles evolve independently, then $\rho_{12}(t) = \rho_1(t)\, \rho_2(t)$ and

$$S_{12}(t) = -Tr_{12}\, \rho_{12}(t) log[\rho_{12}(t)]$$
$$S_i(t) = -Tr_i\, \rho_i(t) log[\rho_i(t)] \quad i = 1, 2$$
$$S_{12}(t) = S_1(t) + S_2(t).$$

If, on the contrary, the particles become correlated, then

$$S_{12}(t) < S_1(t) + S_2(t).$$

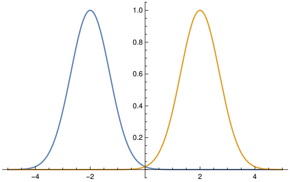
If initial states not correlated

$$\begin{aligned} S_{12}(t) &= 0 \\ S_1(t) = S_2(t) &> 0 \end{aligned}$$

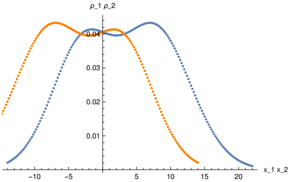
Toy model

$$\varphi_1(x_1;0) = \pi^{-\frac{1}{4}}\, e^{i\,k\,x_1}\, e^{-\frac{(x_1+s)^2}{2}}$$

$$\varphi_2(x_2;0) = \pi^{-\frac{1}{4}}\, e^{-i\,k\,x_2}\, e^{-\frac{(x_2-s)^2}{2}},$$

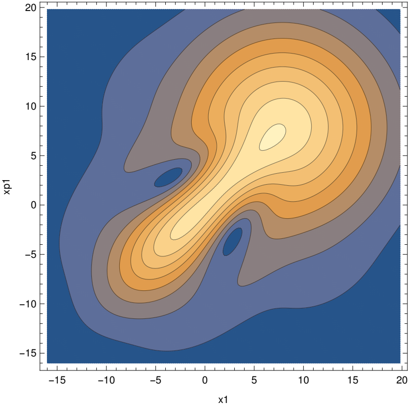


$t = 0$



$t = 8$

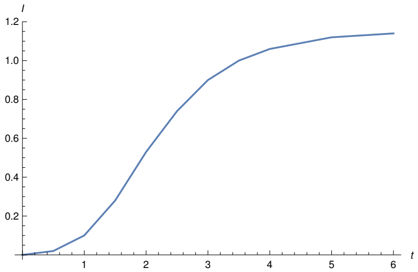
Scattered state density: $\rho_1(x_1, x_1')$



$$s = 2 \quad k = 2 \quad t = 8$$

Calculation of the Entropy $S_1(t)$

• Spectrum of $\rho_1(t)$: Projection on a harmonic oscillator basis, $\chi_m[x_1 - c(t)]$
•Shift towards the center $c(t)$ of the wave packet density $\rho_1(x_1, x_1; t)$,
• diagonalization of the matrix of coefficients $c_{mn}(t) = \int dx_1 dx_1' \chi_m[x_1 - c(t)]\, \rho_1(x_1, x_1'; t)\, \chi_n[x_1' - c(t)]$.



Saturation at enough time to switch the potential off : S_{matrix} value in a Quantum Mechanics Formalism