

# From $1/2$ to $1$ : foundations of perfect spin hydrodynamics

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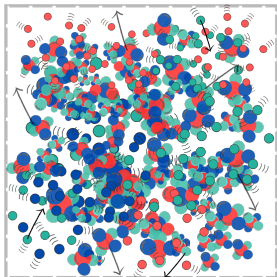
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*Based on Acta Phys.Polon.B 57 (2026) (arXiv:2602.00819),  
arXiv:2606.....*

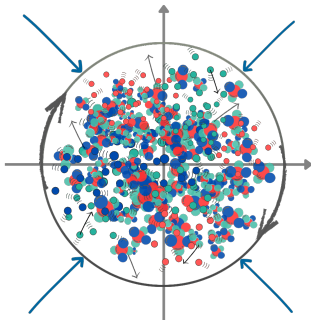


# Basic concepts in spin hydrodynamics

Perfect fluid, local eq., spin conservation  
(spin-orbit = dissipation,  $\omega^{\mu\nu} \rightarrow \varpi^{\mu\nu}$ )



Rigidly rotating cylinder,  
global equilibrium



The two approaches may converge at some point – **needs further studies**

[Bhadury, Drogosz, Florkowski, Kar, VM, arXiv:2505.02657; Becattini, Singh, EPJ C 85, 1338 (2025)]

# Spin in HIC

**Spin-1/2:** local and global spin polarizations (mainly of  $\Lambda$  hyperons).

**Spin-1:** Tensor polarizabilities / spin alignment of vector mesons.

Alignment: deviation of  $\rho_{00}$  coefficient from 1/3.

[Becattini et al., Int. J. Mod. Phys. E 33 (2024); Mohanty et al., Mod. Phys. Lett. A 36 (2021); Bhadury, Drogosz, Florkowski, VM, J. Sub-atomic Part. Cosmol. 4(2025)]

**Our goal:** A unified description of spin-1 and spin-1/2, allowing the construction of perfect spin hydro simultaneously incorporating both types of particles.

→ **Explore the capabilities of local-equilibrium spin density matrix and Wigner function.**

# Spin density matrix in adjoint representation $S$

Spin density matrix in the particle rest frame (PRF)

$$\rho_*^S = \frac{1}{3} \left[ 1 + \frac{3}{2} \mathcal{P}_*^i S^i + \sqrt{\frac{3}{2}} \mathcal{T}_*^{ij} (S^i S^j + S^j S^i) \right] \quad \text{with } (S^i)_{jk} = -i\epsilon_{ijk} \quad (1)$$

$$\begin{aligned} \rho_{rs*}^S &= \frac{1}{3} \left[ \delta_{rs} + \frac{3}{2} \mathcal{P}_*^i (S^i)_{rs} + \sqrt{\frac{3}{2}} \mathcal{T}_*^{ij} (S^i S^j + S^j S^i)_{rs} \right] \\ &= \frac{1}{3} \left[ \delta_{rs} - \frac{3i}{2} \mathcal{P}_*^i \epsilon_{irs} - \sqrt{6} \mathcal{T}_{rs*} \right] \end{aligned} \quad (2)$$

Spin polarization vector  $\mathcal{P}_*^i$  and tensor polarizabilities  $\mathcal{T}_*^{ij}$  are independent of the representation:

$$\mathcal{P}_*^i = \text{tr}_3 \left( \rho_*^A A^i \right), \quad A = J, S, T \quad (3)$$

$$\mathcal{T}_*^{ij} = \frac{1}{2} \sqrt{\frac{3}{2}} \left( \text{tr}_3 \left[ \rho_*^A (A^i A^j + A^j A^i) \right] - \frac{4}{3} \delta^{ij} \right) \quad (4)$$

# Spin-1 representations

J representation:  $\{|1, -1\rangle, |1, 0\rangle, |1, +1\rangle\}$

$$\begin{aligned} |e_1\rangle &= \frac{1}{\sqrt{2}}(|1, -1\rangle - |1, +1\rangle), & |e_2\rangle &= |1, 0\rangle, \\ |e_3\rangle &= \frac{i}{\sqrt{2}}(|1, -1\rangle + |1, +1\rangle) \end{aligned} \quad (5)$$

Pure state represented as a linear combination of  $|e_i\rangle$ :

$$\varepsilon_*^1 |e_1\rangle + \varepsilon_*^2 |e_2\rangle + \varepsilon_*^3 |e_3\rangle. \quad (6)$$

Useful basis  $\epsilon_{r*}^\mu$  ( $r = 1, 2, 3$ ) corresponding to **(5)**

$$\begin{aligned} \epsilon_{1*}^\mu &= (0, 1, 0, 0), \\ \epsilon_{2*}^\mu &= (0, 0, 1, 0), \\ \epsilon_{3*}^\mu &= (0, 0, 0, 1). \end{aligned} \quad (7)$$

# Lorentz covariant matrix

Polarization vectors  $\epsilon_r^\mu$  for particles with  $\mathbf{p}^\mu$  using canonical boost  $L^\mu_\nu(\mathbf{v}_p)$ :

$$p_*^\nu = (m, 0, 0, 0) \implies p^\mu = (E_p, \mathbf{p}), \quad E_p = \sqrt{\mathbf{p}^2 + m^2}$$

$$p^\mu = L^\mu_\nu(\mathbf{v}_p) p_*^\nu, \quad \epsilon_r^\mu = L^\mu_\nu(\mathbf{v}_p) \epsilon_{r*}^\nu. \quad (8)$$

More boosts:

$$\mathcal{P}^\mu = L^\mu_\nu(\mathbf{v}_p) \mathcal{P}_*^\nu, \quad \mathcal{T}^{\mu\nu} = L^\mu_\alpha(\mathbf{v}_p) L^\nu_\beta(\mathbf{v}_p) \mathcal{T}_*^{\alpha\beta} \quad (9)$$

$$\rho_{\mu\nu}(x, p) = \epsilon_\mu^r(p) \rho_{rs}^{*S}(x, p) \epsilon_\nu^S(p) \quad (10)$$

$$\rho_{rs*}^S = \frac{1}{3} \left[ \delta_{rs} - \frac{3i}{2} \mathcal{P}_*^i \epsilon_{irs} - \sqrt{6} \mathcal{T}_{rs*} \right] \quad (11)$$

# Spin density matrix in Lorentz space

For particles with momentum  $p$

$$\rho_{\mu\nu}(x, p) = -\frac{1}{3} \left[ \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right) + \frac{3i\epsilon_{\mu\nu\lambda\rho} \mathcal{P}^\lambda(x, p) p^\rho}{2m} + \sqrt{6} \mathcal{T}_{\mu\nu}(x, p) \right] \quad (12)$$

Equilibrium spin density matrix by [Xia, Li, Huang, Zhong Huang, PLB 817 (2021)]

$$\rho_{\text{eq}}^S = \frac{\exp[\boldsymbol{\alpha} \cdot \mathbf{S}]}{\text{tr}_3(\exp[\boldsymbol{\alpha} \cdot \mathbf{S}])} = \mathcal{Z}^{-1} \exp[\boldsymbol{\alpha} \cdot \mathbf{S}] \quad \boldsymbol{\alpha} - \text{angular velocity} \quad (13)$$

In our approach:

$$\boldsymbol{\alpha} = 2\mathbf{a}_*$$

$$\text{Spin potential:} \quad a_\mu = -\frac{1}{4m} \epsilon_{\mu\nu\rho\sigma} \omega^{\rho\sigma} p^\nu \quad (14)$$

$\omega^{\rho\sigma}$  - spin polarization, Lagrange multiplier enforcing spin conservation.

# Thermodynamics from the Wigner function

We connect Wigner function with the spin density matrix via

$$\mathcal{W}_{\mu\nu}(x, k) = - \int dP \delta^{(4)}(k - p) \underbrace{f_0(x, p) \mathcal{Z}(x, p) \rho_{\nu\mu}(x, p)}_{\chi_{\nu\mu}} \quad (15)$$

For relativistic gas of particles described by Proca field

[Weickgenannt, Wagner, Speranza, Phys. Rev. D 105, 116026 (2022)]:

$$T^{\mu\nu}(x) = \int d^4k k^\mu k^\nu \text{tr}[\mathcal{W}(x, k)] \quad (16)$$

$$S^{\lambda, \mu\nu}(x) = 2i \int d^4k k^\lambda \mathcal{W}^{[\mu\nu]}(x, k) \quad (17)$$

Klein-Gordon pseudogauge applied

(analogous expressions hold for spin-1/2 particles in GLW pseudogauge)

[Florkowski, Kumar, Ryblewski, Prog. Part. Nucl. Phys. 108, 103709 (2019)].

# Thermodynamics from the Wigner function

Boltzmann statistics:

$$\begin{aligned}\chi_{rs*}^B &= \exp(-\beta \cdot p + 2\mathbf{a}_* \cdot S)_{rs} \\ &= g_0^B \delta_{rs} + \frac{(\hat{\mathbf{a}}_* \cdot S)_{rs}}{2} (g_+^B - g_-^B) + \frac{((\hat{\mathbf{a}}_* \cdot S)^2)_{rs}}{2} (g_-^B - 2g_0^B + g_+^B)\end{aligned}\quad (18)$$

$$g_-^B = e^{-p \cdot \beta - 2\sqrt{-a^2}}, \quad g_0^B = e^{-p \cdot \beta}, \quad g_+^B = e^{-p \cdot \beta + 2\sqrt{-a^2}} \quad (19)$$

$ 1, -1\rangle$	$ 1, 0\rangle$	$ 1, 1\rangle$
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Bose-Einstein statistics:

$$\begin{aligned}\chi_{rs*}^{BE} &= (\exp(\beta \cdot p - 2\mathbf{a}_* \cdot S) - 1)_{rs}^{-1} \\ &= g_0^{BE} \delta_{rs} + \frac{(\hat{\mathbf{a}}_* \cdot S)_{rs}}{2} (g_+^{BE} - g_-^{BE}) + \frac{((\hat{\mathbf{a}}_* \cdot S)^2)_{rs}}{2} (g_-^{BE} - 2g_0^{BE} + g_+^{BE})\end{aligned}$$

$$g_-^{BE} = \frac{1}{e^{\beta \cdot p + 2\sqrt{-a^2}} - 1}, \quad g_0^{BE} = \frac{1}{e^{\beta \cdot p} - 1}, \quad g_+^{BE} = \frac{1}{e^{\beta \cdot p - 2\sqrt{-a^2}} - 1}$$

# Thermodynamics from the Wigner function

Starting with

$$T^{\mu\nu}(x) = \int d^4k k^\mu k^\nu \text{tr}[\mathcal{W}(x, k)] \quad (20)$$

$$S^{\lambda, \mu\nu}(x) = 2i \int d^4k k^\lambda \mathcal{W}^{[\mu\nu]}(x, k) \quad (21)$$

using our Wigner function and equilibrium spin density matrix we get

$$T^{\mu\nu}(x) = \int dP p^\mu p^\nu (g_- + g_0 + g_+) \quad (22)$$

$$S^{\lambda, \mu\nu}(x) = \frac{1}{m} \int dP p^\lambda \frac{g_+ - g_-}{\sqrt{-a^2}} \epsilon^{\mu\nu\alpha\beta} a_\alpha p_\beta \quad (23)$$

**for both Boltzmann and Bose-Einstein distributions.**

Conserved entropy:

$$dS^\mu = \beta_\lambda dT^{\mu\lambda} - \frac{1}{2} \omega_{\rho\sigma} dS^{\mu, \rho\sigma} \quad (24)$$

**Local equilibrium described by perfect spin hydrodynamics.**

# Polarization vector

Local-equilibrium spin density matrix

$$\rho_{\text{eq}}^S = \frac{\exp[\boldsymbol{\alpha} \cdot \mathbf{S}]}{\text{tr}_3(\exp[\boldsymbol{\alpha} \cdot \mathbf{S}])} = \mathcal{Z}^{-1} \exp[\boldsymbol{\alpha} \cdot \mathbf{S}] \quad (25)$$

**uniquely** determines **spin polarization vector** and tensor polarizabilities:

$$\mathcal{P}_* = \frac{2 \sinh(\alpha)}{2 \cosh \alpha + 1} \hat{\boldsymbol{\alpha}} \quad (26)$$

Exponential forms: 1 vs 1/2

$$\mathcal{P}_* = \frac{g_+ - g_-}{g_- + g_0 + g_+} \hat{\boldsymbol{\alpha}}, \quad \text{vs} \quad \mathcal{P}_*^{(1/2)} = \frac{1}{2} \frac{e^{\alpha/2} - e^{-\alpha/2}}{e^{\alpha/2} + e^{-\alpha/2}} \hat{\boldsymbol{\alpha}} \quad (27)$$

$\mathcal{P}_*$  is valid with both Boltzmann and Bose-Einstein distributions extended to spin space.

$e^\alpha$  quantifies the probability of having spin 1 oriented along  $\hat{\boldsymbol{\alpha}}$ .

$\boldsymbol{\alpha} = 2\mathbf{a}_* \sim$  angular velocity connected to spin potential

## Tensor polarizabilities

$$\mathcal{T}_*^{ij} = \frac{(3\hat{\alpha}^i \hat{\alpha}^j - \delta^{ij}) (\cosh \alpha - 1)}{\sqrt{6} (2 \cosh \alpha + 1)} = \frac{3\hat{\alpha}^i \hat{\alpha}^j - \delta^{ij}}{2\sqrt{6}} \frac{g_- - 2g_0 + g_+}{g_- + g_0 + g_+} \quad (28)$$

Central element of the spin density matrix defines the spin alignment

$$\mathcal{A} = \rho_{00} - \frac{1}{3} = -\sqrt{\frac{2}{3}} \mathcal{T}_*^{22} = -\frac{(3\hat{\alpha}_2^2 - 1) (\cosh \alpha - 1)}{3 (2 \cosh \alpha + 1)} \quad (29)$$

$$\alpha = 2\mathbf{a}_* = -\mathbf{b}_* = -\frac{1}{m} \left( E_p \mathbf{b} - \mathbf{p} \times \mathbf{e} - \frac{\mathbf{p} \cdot \mathbf{b}}{E_p + m} \mathbf{p} \right) \quad (30)$$

$$a_\mu = -\frac{1}{4m} \epsilon_{\mu\nu\rho\sigma} \omega^{\rho\sigma} p^\nu \quad (31)$$

electric- ( $\mathbf{e}$ ) and magnetic-like ( $\mathbf{b}$ ) components of spin polarization  $\omega^{\mu\nu}$

→ **Non-trivial alignment can occur in local equilibrium, no dissipation required**

More observables can be expressed by  $\mathcal{T}_*^{ij}$  rather than  $\mathcal{P}_*$

## Summary - We have determined

- Wigner function for spin-1 particles in local equilibrium
- Spin density matrix uniquely determining spin polarization vector and the tensor polarizabilities.  
→ connection of hyperon and vector-meson measurements.
- Relevant tensors satisfying general thermodynamic relations (independent of statistics and spin representation).
- Non-trivial spin alignment in the absence of dissipation.
- Divergence-type theory with nonlinearly causal and stable dynamical equations.
- Thermodynamic tensors with quantum and classical spin treatments coincide up to second order in spin polarization.

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