SPIN HYDRODYNAMICS

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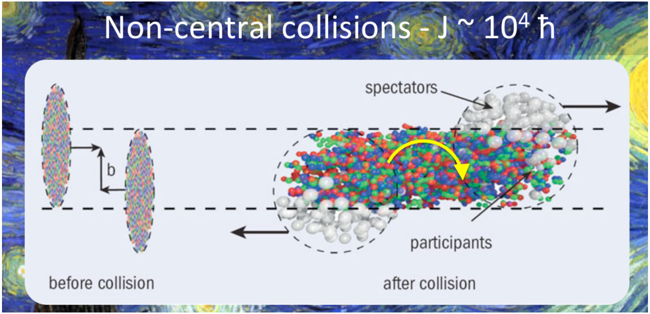
PART 1: PHYSICS MOTIVATION

Non-central heavy-ion collisions create fireballs with large global angular momenta, some part of the angular momentum can be transferred from the orbital to the spin part

$$\boldsymbol{J}_{\rm init} = \boldsymbol{L}_{\rm init} = \boldsymbol{L}_{\rm final} + \boldsymbol{S}_{\rm final}$$



(Michael Lisa, talk „Strangeness in Quark Matter 2016")

Warning: large angular momentum does not mean large angle of rotation!

$\Delta t = 1$ fm/c = 3×10^{-24} s, $\quad \Delta\phi = \Delta t\, \omega_{\max} = 27 \times 10^{-24} \times 10^{21} = 2.7 \times 10^{-2}$

Orbital and spin part of total angular momentum

Total angular momentum tensor $J^{\mu,\lambda\nu}$ has orbital $L^{\mu,\lambda\nu}$ and spin $S^{\mu,\lambda\nu}$ parts
orbital part is expressed by the energy-momentum tensor (stress-energy tensor), the spin part is called the spin tensor

$$J^{\mu,\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda} + S^{\mu,\lambda\nu} \equiv L^{\mu,\lambda\nu} + S^{\mu,\lambda\nu}$$

$$\partial_\mu J^{\mu,\lambda\nu} = T^{\lambda\nu} - T^{\nu\lambda} + \partial_\mu S^{\mu,\lambda\nu} = 0, \qquad \partial_\mu S^{\mu,\lambda\nu} = T^{\nu\lambda} - T^{\lambda\nu}$$

Antisymmetry in the last two indices:

$$J^{\mu,\lambda\nu} = -J^{\mu,\nu\lambda}, \quad L^{\mu,\lambda\nu} = -L^{\mu,\nu\lambda}, \quad S^{\mu,\lambda\nu} = -S^{\mu,\nu\lambda},$$

Warning: different forms of these tensors are used in the literature **(pseudogauge freedom)**
to be discussed later in this talk

At present, the most successful phenomenological description of the data has been achieved by Francesco Becattini and collaborators, based on the Zubarev formalism for non-equilibrium systems

F. Becattini, L. Tinti, V. Chandra, I. Del Zanna, E. Grossi, M. Buzzegoli, G. Inghirami, A. Palermo, I. Karpenko, S. Singh, ...

the central quantity (the starting point) in this case is a rank-2 antisymmetric tensor, called the thermal vorticity

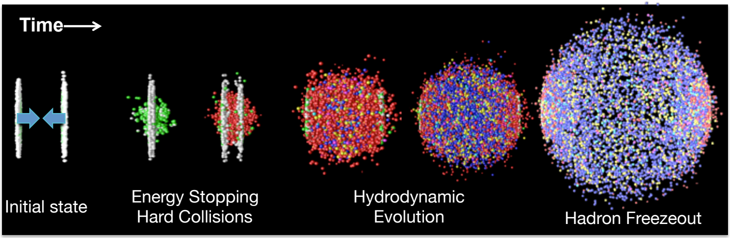
$$\omega_{\mu\nu} = \varpi_{\mu\nu} = -1/2\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right) \qquad \beta^\mu = u^\mu/T, \quad \beta = 1/T$$

u^μ – hydrodynamic flow vector, T – temperature

The formalism presented in this lecture starts from different principles,
eventually thermal vorticity appears on the way

PART 2: STANDARD RELATIVISTIC HYDRODYNAMICS

RELATIVISTIC HYDRODYNAMICS FORMS THE BASIC INGREDIENT OF THE STANDARD MODEL OF HEAVY-ION COLLISIONS



T. K. Nayak, Lepton-Photon 2011 Conference

data on spin polarization suggest that spin should be included in the hydrodynamic framework

Perfect fluid hydrodynamics

PERFECT-FLUID HYDRODYNAMICS = local equilibrium + conservation laws

one usually includes energy, linear momentum, baryon number, ...

T (temperature), u^μ (three independent components of flow), μ (baryon chemical potential)
ε (energy density), P (pressure), n (baryon density), σ (entropy density), $\xi = \mu/T$

$$T^{\mu\nu} = [\varepsilon(T,\mu) + P(T,\mu)]\, u^\mu u^\nu - P(T,\mu) g^{\mu\nu} \tag{1}$$

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu N^\mu = \partial_\mu (n u^\mu) = 0 \qquad \text{(4+1 eqs.)} \tag{2}$$

five equations for five unknown functions

dissipation does not appear

$$\partial_\mu S^\mu = \partial_\mu (\sigma u^\mu) = 0 \qquad \text{(1 eq.)} \tag{3}$$

entropy conservation follows from the energy-momentum conservation and the form of the energy-momentum tensor
Euler's equation (says that four-acceleration is caused by the pressure gradient)

$$u^\lambda \partial_\lambda u^\mu \equiv a^\mu = \frac{(g^{\mu\nu} - u^\mu u^\nu)}{\varepsilon + P} \partial_\nu P \equiv \frac{1}{\varepsilon + P} \Delta^{\mu\nu} \partial_\nu P \qquad \text{(3 eqs.)} \tag{4}$$

PART 3: CLASSICAL APPROACH TO SPIN

Classical treatment of spin – internal angular momentum

A particle can be characterised by the internal angular momentum tensor $s^{\alpha\beta}$
M. Matthison, Acta Phys. Polon. 6 (1937) 163

Neue Mechanik materieller Systeme

Nowa mechanika systemów materialnych

Von MYRON MATHISSON, Warschau

(Eingegangen am 8. September 1937)

Mathisson with Pauli
Copenhagen 1937

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta, \quad s \cdot p = 0, \quad s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta} p_\beta s_{\gamma\delta} \tag{5}$$

A straightforward generalization of the phase-space distribution function $f(x, \boldsymbol{p})$ is a spin dependent distribution $f(x, \boldsymbol{p}, s)$
WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

$$\int dS \ldots = \frac{m}{\pi \mathfrak{s}} \int d^4 s \, \delta(s \cdot s + \mathfrak{s}^2) \, \delta(p \cdot s) \ldots \qquad \mathfrak{s}^2 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4} \tag{6}$$

$$\int dS = \frac{m}{\pi \mathfrak{s}} \int d^4 s \, \delta(s \cdot s + \mathfrak{s}^2) \, \delta(p \cdot s) = 2 \tag{7}$$

from now on, we consider spin 1/2 only, extension to spin 1 presented by Valeriya Mykhaylova

Local equilibrium distribution without spin: Maxwell-Jüttner distribution

Maxwell distribution

$$f_{\rm eq}(\boldsymbol{v}) = \left[\frac{m}{2\pi k_B T}\right]^{3/2} \exp\left[-\frac{m\boldsymbol{v}^2}{2k_B T}\right] \tag{8}$$

Maxwell-Jüttner distribution (natural units, Boltzmann statistics)

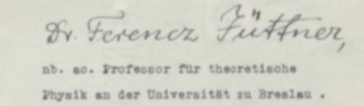
$$f_{\rm eq}(\boldsymbol{p}) = \exp\left[-\frac{\sqrt{m^2+\boldsymbol{p}^2}}{T}\right] \to \exp\left[-\frac{\sqrt{m^2+\boldsymbol{p}^2}-\mu}{T}\right] \tag{9}$$

$f(x,\boldsymbol{p})$ phase space distribution for unpolarized systems, Lorentz scalar

$$f_{\rm eq}(x,p) = 2\exp\left[-\frac{p^\mu u_\mu(x)-\mu(x)}{T(x)}\right] = 2\exp\left[-p^\mu \beta_\mu(x) + \xi(x)\right] \tag{10}$$

$\xi = \mu/T$ ratio of the baryon chemical potential and temperature, $\beta^\mu = u^\mu/T$ ratio of the hydrodynamic flow and temperature, 2 - spin degeneracy

$u'^\mu = (1,0,0,0)$ **in the local fluid rest frame (LRF)**



Local equilibrium function with spin, macroscopic currents

WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

spin conserving equilibrium distribution functions for particles and antiparticles

$$f^{\pm}_{\rm eq}(x,p,s) = \exp\left(\pm\xi(x) - p\cdot\beta(x) + \tfrac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right) \tag{11}$$

macroscopic currents

baryon current

$$N^{\lambda}_{\rm eq} = \int dP \int dS \; p^{\lambda}\, 1 \left[f^{+}_{\rm eq}(x,p,s) - f^{-}_{\rm eq}(x,p,s)\right] \tag{12}$$

energy-momentum tensor

$$T^{\lambda\mu}_{\rm eq} = \int dP \int dS \; p^{\lambda} p^{\mu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right] \tag{13}$$

spin tensor

$$S^{\lambda,\mu\nu}_{\rm eq} = \int dP \int dS \; p^{\lambda}\, s^{\mu\nu} \left[f^{+}_{\rm eq}(x,p,s) + f^{-}_{\rm eq}(x,p,s)\right] \tag{14}$$

Classical treatment of spin – spin hydrodynamics

construction of the equilibrium function implies the conservation laws

$$\partial_\mu N^\mu(x) = 0, \quad \partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}(x) = 0 \tag{15}$$

these are 11 equations for 11 Lagrange multipliers

$$\partial_\mu N^\mu[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(1 eq.)} \tag{16}$$

$$\partial_\mu T^{\mu\nu}[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(4 eqs.)} \tag{17}$$

$$\partial_\lambda S^{\lambda,\mu\nu}[\xi(x), \beta_\alpha(x), \omega_{\alpha\beta}(x)] = 0 \qquad \text{(6 eqs.)} \tag{18}$$

PERFECT-SPIN HYDRODYNAMICS

WF, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C97 (2018) 041901

CLASSICAL SPIN ≡ SPIN OPERATOR EXPECTATION VALUE

classical spin should be undestood as the expectation value of the spin polarization operator

VOLUME 2, NUMBER 10 PHYSICAL REVIEW LETTERS MAY 15, 1959

PRECESSION OF THE POLARIZATION OF PARTICLES MOVING IN A HOMOGENEOUS ELECTROMAGNETIC FIELD*

V. Bargmann
Princeton University, Princeton, New Jersey
Louis Michel
Ecole Polytechnique, Paris, France
and
V. L. Telegdi
University of Chicago, Chicago, Illinois
(Received April 27, 1959)

The problem of the precession of the "spin" of a particle moving in a homogeneous electromagnetic field – a problem which has recently acquired considerable experimental interest – has already been investigated for spin $\frac{1}{2}$ particles in some particular cases.[1] In the literature the results were derived by explicit use of the Dirac equation, with the occasional inclusion of a Pauli term to account for an anomalous magnetic moment. On the other hand, following a remark of Bloch[2] in connection with the nonrelativistic case, the expectation value of the vector operator representing the "spin" will necessarily follow the same time dependence as one would obtain from a classical equation of motion. To solve the pro-

customary equation of motion

$$d\vec{s}/d\tau = (ge/2m)(\vec{s}\times\vec{H}), \qquad (R) \qquad (3)$$

where $\vec{H}$, e, and m have their standard meanings, while the gyromagnetic ratio g is defined by this very equation. While s^0 vanishes by hypothesis in any instantaneous rest-frame, $ds^0/d\tau$ need not. In fact, (2) implies

$$ds^0/d\tau = \vec{s}\cdot(d\vec{v}/d\tau), \qquad (R) \qquad (4)$$

for such frames. In general, $du/d\tau = f/m$ (where f = four-force), while in a homogenous external electromagnetic field specified by $F = -(\vec{E}, \vec{H})$

$$du/d\tau = (e/m)F\cdot u. \qquad (5)$$

see also J.D. Jackson "Classical electrodynamics"

CAN SPIN-ORBIT INTERACTION BE NEGLECTED?

1. **ATOMIC PHYSICS CONTEXT:**

   $\boldsymbol{L}$ means rotation, for a charged fluid this generates $\boldsymbol{B}$, magnetic fields couples to magnetic moments $\boldsymbol{\mu}$ that are connected to spin $\boldsymbol{S}$, we effectively get $\boldsymbol{L} \cdot \boldsymbol{S}$ coupling,
   this effect (conserving entropy) can be easily included in our framework
   S. Bhadury, WF, A. Jaiswal, A. Kumar, R. Ryblewski, PRL129 (2022) 192301

2. **DIRAC EQUATION LESSON:**

   only $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is conserved, property reflected in asymmetric energy-momentum tensor $T^{\mu\nu}_{\rm can} \neq T^{\nu\mu}_{\rm can}$
   and non-zero divergence $\partial_\lambda S^{\lambda,\mu\nu}_{\rm can} = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can}$
   tensors defined with classical spin do not correspond to the canonical forms
   WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

3. **MICROSCOPIC SPIN-ORBIT INTERACTION:**

   collisions/decays of particles with spin, transfer between $\boldsymbol{L}$ and $\boldsymbol{S}$, for example $\rho^0 \to \pi^+\pi^-$
   such processes produce entropy, can be included if we extend the perfect-fluid picture, discussed later

perfect spin hydrodynamics = spin conservation

dissipative spin hydrodynamics = transfer between $\boldsymbol{S}$ and $\boldsymbol{L}$ possible

Joseph Kapusta: strange quark spin or helicity is unchanged from the time they are created to the time they hadronize
J. Kapusta, E. Rrapaj, S. Rudaz, PRC101 (2020) 024907

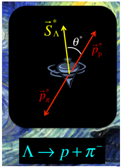
PART 3: QUANTUM APPROACH TO SPIN

Quantum spin description: spin vs. spinor density matrix

standard scalar functions $f(x,p)$ are generalized to **2×2 Hermitian matrices** in spin space for each value of the space-time position x and four-momentum p, the sign ± distinguishes particles from antiparticles, $\boldsymbol{\sigma}$ - Pauli matrices

$\zeta^{\pm}_{*} = 0$ no polarization, $\zeta^{\pm}_{*} = 1$ pure state, $0 < |\zeta^{\pm}_{*}| < 1$ mixed state, **asterisk denotes the particle rest frame (PRF)**

$$f^{\pm}_{rs}(x,p) = f^{\pm}_{0}(x,p)\left[\delta_{rs} + \boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})\cdot\boldsymbol{\sigma}_{rs}\right], \quad 0 \le |\zeta^{\pm}_{*}| \le 1 \qquad (19)$$



$\boldsymbol{\zeta}^{\pm}_{*}(x,\boldsymbol{p})$ can be interpreted as a spatial part of the polarization four-vector $\zeta^{\pm\mu}_{*}(x,\boldsymbol{p})$ with a vanishing zeroth component

$$\zeta^{\pm\mu}_{*} = \left(0, \boldsymbol{\zeta}^{\pm}_{*}\right) \qquad (20)$$

in the LAB frame – $\zeta^{\pm\mu}_{*}$ boosted with the velocity defined by the particle velocity

$$\zeta^{\mu}_{\pm} = \Lambda^{\mu}{}_{\nu}(\boldsymbol{v}_p)\zeta^{\nu}_{\pm *} = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}^{\pm}_{*}}{m}, \boldsymbol{\zeta}^{\pm}_{*} + \frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}^{\pm}_{*}}{m(E_{\rm p}+m)}\boldsymbol{p}\right), \qquad \zeta^{\mu}_{\pm}p_{\mu} = 0 \qquad (21)$$

transition to **4×4 spinor density matrices** $X^{\pm}$

$$f^{+}_{rs}(x,p) = \bar{u}_r(p)X^{+}u_s(p), \qquad f^{-}_{rs}(x,p) = -\bar{v}_s(p)X^{-}v_r(p) \qquad (22)$$

$u_s(p)$ and $v_r(p)$ – Dirac bispinors

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32

$\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ is the Dirac spin operator

$$X^{\pm}(x,p) = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right] \tag{23}$$

$\omega_{\mu\nu} = \Omega_{\mu\nu}/T$ ratio of the tensor spin chemical potential and temperature, altogether we have 11 Lagrange multipliers that control the conservation of the baryon number (1), energy (1), linear momentum (3), angular momentum (3), and Lorentz boost vectors (3)

$\omega_{\mu\nu}$ can be represented by electric- and magneticlike three-vectors (spin polarization tensor) in analogy to $F_{\mu\nu}$

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix} \tag{24}$$

this form of $X^{\pm}$ leads to the expected 2 by 2 form, Eq. (19), only for $|\boldsymbol{e}_*| << 1$ and $|\boldsymbol{b}_*| << 1$

WF, B. Friman, A. Jaiswal, E. Speranza, R. Ryblewski, Phys. Rev. D97 (2018) 116017

Quantum spin description: local equilibrium - revised formula

S. Bhadury, Z. Drogosz, WF, S. K. Kar, V. Mykhaylova, **arXiv:2505.02657**

with a spacelike four-vector a^μ ($a^2 < 0$)

$$X^\pm = \exp\left(\pm\xi - \beta_\mu p^\mu + \gamma_5 \not{a}\right) = \exp\left(\pm\xi - \beta_\mu p^\mu\right)\cosh\sqrt{-a^2}\left[1 + \frac{\gamma_5 \not{a}}{\sqrt{-a^2}}\tanh\sqrt{-a^2}\right] \tag{25}$$

transition to 2 by 2 spin density matrices uses identities

$$\bar{u}_r(p)\gamma_5 \not{\zeta} u_s(p) = 2m\,\zeta_* \cdot \sigma_{rs}, \qquad \bar{v}_s(p)\gamma_5 \not{\zeta} v_r(p) = -2m\,\zeta_* \cdot \sigma_{rs}, \tag{26}$$

which leads to identification and correct normalization

$$\zeta^\mu = \frac{a^\mu}{\sqrt{-a^2}}\tanh\sqrt{-a^2}, \qquad -\zeta^2 = \zeta_*^2 = \tanh^2\sqrt{\boldsymbol{a}_*^2} \le 1 \tag{27}$$

to get the agreement with the previous definition and classical description for $|\boldsymbol{e}_*| << 1$ and $|\boldsymbol{b}_*| << 1$ we choose

$$a_\mu(x,p) = -\frac{1}{2m}\tilde{\omega}_{\mu\nu}(x)p^\nu \tag{28}$$

the particle rest frame (PRF) where $p^\mu = (m,0,0,0)$ we have

$$\boldsymbol{a}_* = -\frac{\boldsymbol{b}_*}{2} \tag{29}$$

similarity to MHD - polarization effects determined by the magnetic(like) component of the spin polarization tensor

Quantum spin description: macroscopic currents

S. R. de Groot, W. A. van Leeuwen, Ch. G. van Weert, **Relativistic kinetic theory**, North-Holland Publishing Company 1980

baryon current: $$N^\lambda(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^\lambda\, 1 \left[f^+_{rr}(x,p) - f^-_{rr}(x,p) \right] \tag{30}$$

energy-momentum tensor: $$T^{\lambda\mu}(x) = \sum_{r=1}^{2} \int \mathrm{d}P\, p^\lambda p^\mu \left[f^+_{rr}(x,p) + f^-_{rr}(x,p) \right] \tag{31}$$

spin tensor: $$S^{\lambda,\mu\nu}(x) = \frac{1}{2} \sum_{r,s=1}^{2} \int \mathrm{d}P\, p^\lambda \left[\sigma^{+\mu\nu}_{sr}(p) f^+_{rs}(x,p) + \sigma^{-\mu\nu}_{sr}(p) f^-_{rs}(x,p) \right] \tag{32}$$

where $\sigma^{+\mu\nu}_{sr}(p) = 1/(2m)\, \bar{u}_s(p) \sigma^{\mu\nu} u_r(p)$ and $\sigma^{-\mu\nu}_{sr}(p) = 1/(2m)\, \bar{v}_r(p) \sigma^{\mu\nu} v_s(p)$, with $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$

these forms of currents are commonly known as the GLW versions (GLW pseudogauge)
for free Dirac equation (relativistic gas) these tensors are conserved

$$\partial_\mu N^\mu(x) = 0, \quad \partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}(x) = 0 \tag{33}$$

up to second order in ω averaging over classical spin equivalent to performing traces over spinor indices

$$s^\mu \quad \longleftrightarrow \quad \frac{1}{2}\gamma_5 \gamma^\mu \tag{34}$$

see also the talk by Zbigniew Drogosz

PART 3: PSEUDOGAUGE FREEDOM

vs. PSEUDOGAUGE DEPENDENCE

BLASPHEMY #3 (GLW form): ENERGY-MOMENTUM TENSOR SHOULD BE USED IN THE BELINFANTE-ROSENFELD FORM, SINCE THIS FORM APPEARS IN EINSTEIN'S EQUATIONS

Pseudo-gauge transformation (QCD language in the context of the proton spin puzzle: adding boundary terms)

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu} + \Phi^{\nu,\mu\lambda} + \Phi^{\mu,\nu\lambda} \right) \tag{35}$$

$$S'^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\rho Z^{\mu\nu,\lambda\rho} \tag{36}$$

Canonical forms (directly obtained from Noether's Theorem): asymmetric energy-momentum tensor, spin tensor directly expressed by axial current (couples to weak interactions)

Belinfante-Rosenberg version, $\Phi^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}, Z^{\mu\nu,\lambda\rho} = 0$, (couples to classical gravity); spin tensor appears in modified theories of gravity, couples to torsion

de Groot, van Leuveen, van Weert (GLW) forms: symmetric energy-momentum tensor and conserved spin tensor

Hilgevoord and Wouthuysen (HW) choice: symmetric energy-momentum tensor and conserved spin tensor

there is ongoing discussion if the physics is or is not pseudogauge dependent F. Hehl, Rept. Math. Phys. 9 (1976) 55

Sidney Coleman's old answer: ...we have an infinite family of possible definitions of the local current...some textbooks try to avoid this point, or nervously rub one foot across the other leg and natter about the best definition or the optimum definition...and the right answer is, of course, there's nothing to natter about, there's nothing to be disturbed about...it is something to be pleased about. If we have many objects that satisfy desirable general criteria, then that's better than having just one... the more freedom you have, the better. It's like being passed a plate of cookies and someone starts arguing about which is the best cookie. They're all edible!

WF, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709

GLW → canonical: superpotential defined as $\Phi^{\lambda,\mu\nu}_{\rm can} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW}$, then we have

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu}_{\rm can}$$

and

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can}\right)$$

canonical → Belinfante-Rosenfeld: superpotential defined as $\Phi_{\rm Bel} = S^{\lambda,\mu\nu}_{\rm can}$

$$S^{\lambda,\mu\nu}_{\rm Bel} = 0, \quad T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda \left(S^{\lambda,\mu\nu}_{\rm can} - S^{\mu,\lambda\nu}_{\rm can} - S^{\nu,\lambda\mu}_{\rm can}\right),$$

canonical spin tensor is the "right" observable – connected with spin operator

S. Dey, WF, A. Jaiswal, R. Ryblewski, Phys. Lett. B843 (2023) 137994

PART 4: THERMODYNAMICS WITH SPIN

A(nother) piece of physics history in Kraków

J. Weyssenhoff, A. Raabe, Acta Phys. Polon. 9 (1947) 7
first paper published after WWII in Acta Physica Polonica

$S^{\alpha\beta}$ – spin density tensor

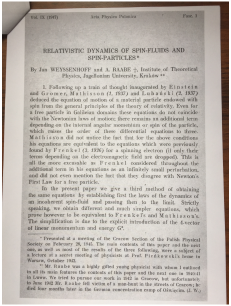
$S^{\lambda,\mu\nu} = u^{\lambda} S^{\mu\nu}$ – spin tensor

in analogy to

n – (baryon) number density

$N^{\lambda} = u^{\lambda} n$ – (baryon) number current

modeling of spinning particles in mind rather than thermodynamics or hydrodynamics



Vol. IX (1947) Acta Physica Polonica Fasc. 1

RELATIVISTIC DYNAMICS OF SPIN-FLUIDS AND SPIN-PARTICLES *

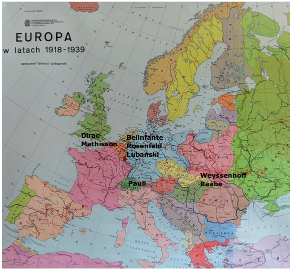
By Jan WEYSSENHOFF and A. RAABE †, Institute of Theoretical Physics, Jagellonian University, Kraków **

1. Following up a train of thought inaugurated by Einstein and Gromer, Mathisson (1, 1937) and Lubański (2, 1937) deduced the equation of motion of a material particle endowed with spin from the general principles of the theory of relativity. Even for a free particle in Galileian domains these equations do not coincide with the Newtonian laws of motion; there remains an additional term depending on the internal angular momentum or spin of the particle, which raises the order of these differential equations to three. Mathisson did not notice the fact that for the above conditions his equations are equivalent to the equations which were previously found by Frenkel (3, 1926) for a spinning electron (if only their terms depending on the electromagnetic field are dropped). This is all the more excusable as Frenkel considered throughout the additional term in his equations as an infinitely small perturbation, and did not even mention the fact that they disagree with Newton's First Law for a free particle.

In the present paper we give a third method of obtaining the same equations by establishing first the laws of the dynamics of an incoherent spin-fluid and passing then to the limit. Strictly speaking, we obtain different and much simpler equations, which prove however to be equivalent to Frenkel's and Mathisson's. The simplification is due to the explicit introduction of the 4-vector of linear monumentum and energy G^{α}.

* Presented at a meeting of the Cracow Section of the Polish Physical Society on February 28, 1945. The main contents of this paper and the next one, as well as most of the results of the three following, were a subject of a lecture at a secret meeting of physicists at Prof. Pieńkowski's home in Warsaw, October 1942.

** Mr. Raabe was a highly gifted young physicist with whom I outlined in all its main features the contents of this paper and the next one in 1940/41 in Lwów. We tried to pursue our work in 1942 in Cracow, but unfortunately in June 1942 Mr. Raabe fell victim of a man-hunt in the streets of Cracow; he died four months later in the German concentration camp of Oświęcim. (J. W.)



inclusion of spin, $\Omega_{\alpha\beta}$ - spin chemical potential

$S^{\alpha\beta}$ - spin density tensor introduced in J. Weyssenhoff, A. Raabe, Acta Phys. Polon. 9 (1947) 7

$$\varepsilon + P = T\sigma + \mu n + \tfrac{1}{2}\Omega_{\alpha\beta}S^{\alpha\beta} \tag{37}$$

$$d\varepsilon = Td\sigma + \mu dn + \tfrac{1}{2}\Omega_{\alpha\beta}dS^{\alpha\beta} \qquad dP = \sigma dT + nd\mu + \tfrac{1}{2}S^{\alpha\beta}d\Omega_{\alpha\beta} \tag{38}$$

multiplication of the above equations by the hydrodynamic flow vector u gives **the tensor (Israel-Stewart) form**
W. Israel, J.M.Stewart, Annals Phys. 118 (1979) 341 & Phys.Lett. A58 (1976) 213

$$S^{\mu}_{\rm eq} = P\beta^{\mu} - \xi N^{\mu}_{\rm eq} + \beta_{\lambda}T^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} \tag{39}$$

$$dS^{\mu}_{\rm eq} = -\xi dN^{\mu}_{\rm eq} + \beta_{\lambda}dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}dS^{\mu,\alpha\beta}_{\rm eq}, \quad d(P\beta^{\mu}) = N^{\mu}_{\rm eq}d\xi - T^{\lambda\mu}_{\rm eq}d\beta_{\lambda} + \tfrac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}d\omega_{\alpha\beta} \tag{40}$$

spin tensor

$$S^{\mu,\alpha\beta}_{\rm eq} = u^{\mu}S^{\alpha\beta}_{\rm eq} \tag{41}$$

analog to the perfect-fluid forms of $N^{\mu}_{\rm eq}$ and $T^{\lambda\mu}_{\rm eq}$, however, in kinetic theory we find

$$S^{\mu,\alpha\beta}_{\rm eq} = u^{\mu}S^{\alpha\beta}_{\rm eq} + \text{problem} \tag{42}$$

problem = term that is not proportional to u^{μ}

a solution stands behind the corner ....

WF, M. Hontarenko, PRL134 (2025) 082302
Z.. Drogosz, WF, M. Hontarenko, PRD 110 (2024) 096018
Boltzmann's definition of the entropy (*H*-function)

$$S^\mu = -\int dP\, dS\, p^\mu \left[f^+ \left(\ln f^+ - 1\right) + f^- \left(\ln f^- - 1\right)\right] \qquad \text{classical spin} \tag{43}$$

$$S^\mu = -\frac{1}{2}\int \mathrm{d}P\, p^\mu \left\{\mathrm{tr}_4 \left[X^+ \left(\ln X^+ - 1\right)\right] + \mathrm{tr}_4 \left[X^- \left(\ln X^- - 1\right)\right]\right\} \qquad \text{quantum spin} \tag{44}$$

Together with other kinetic-theory expressions, one obtains **tensor forms of thermodynamic relations** valid for any value of the spin polarization tensor ω

$$S^\mu_{\rm eq} = T^{\mu\alpha}_{\rm eq}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta}_{\rm eq} - \xi N^\mu_{\rm eq} + \mathcal{N}^\mu, \qquad \mathcal{N}^\mu \neq P\beta^\mu \tag{45}$$

$$dS^\mu_{\rm eq} = -\xi\, dN^\mu_{\rm eq} + \beta_\lambda\, dT^{\lambda\mu}_{\rm eq} - \tfrac{1}{2}\omega_{\alpha\beta}\, dS^{\mu,\alpha\beta}_{\rm eq} \qquad \text{first law of thermodynamics} \tag{46}$$

$$d\mathcal{N}^\mu = N^\mu_{\rm eq}\, d\xi - T^{\lambda\mu}_{\rm eq}\, d\beta_\lambda + \tfrac{1}{2}S^{\mu,\alpha\beta}_{\rm eq}\, d\omega_{\alpha\beta} \qquad \text{Gibbs-Duhem relations} \tag{47}$$

entropy conservation as a consequence of other conservation laws, very close similarity to MHD
M. Gedalin and I. Oiberman, Phys. Rev. E51 (1994) 4901

PART 5: GOING OFF EQUILIBRIUM

W. Israel, J.M.Stewart, Annals Phys. 118 (1979) 341 & Phys.Lett. A58 (1976) 213
here we use the IS method to construct the Navier-Stokes theory
replacement of the equilibrium currents by the general ones (equilibrium + non-equilibrium corrections)

$$S^\mu = T^{\mu\alpha}\beta_\alpha - \frac{1}{2}\omega_{\alpha\beta}S^{\mu,\alpha\beta} - \xi N^\mu + \mathcal{N}^\mu_{\rm eq} \quad (48)$$

Conservations laws, now for total angular momentum $J = L + S$

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu S^{\mu,\alpha\beta} = T^{\beta\alpha} - T^{\alpha\beta} \quad (49)$$

entropy production

$$\partial_\mu S^\mu \quad = \quad -\delta N^\mu \partial_\mu \xi + \delta T^{\mu\lambda}_s \partial_\mu \beta_\lambda + \delta T^{\mu\lambda}_a \left(\partial_\mu \beta_\lambda - \omega_{\lambda\mu}\right) - \frac{1}{2}\delta S^{\mu,\alpha\beta}\partial_\mu \omega_{\alpha\beta} \geq 0 \quad (50)$$

K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, Phys. Lett. B795, 100 (2019)
F. Becattini, A. Daher, Phys. Lett. B850, 138533 (2024)

$\delta T^{\mu\lambda}_s$ and $\delta T^{\mu\lambda}_a$, are symmetric and antisymmetric dissipative corrections to the energy-momentum tensor

second-order theory also developed
R. Biswas, A. Daher, A. Das, WF, R. Ryblewski, Phys. Rev. D108, 014024 (2023)

dissipative spin hydrodynamics = transfer between ***S*** and ***L*** possible

the second law of thermodynamics imposes constraints of non-equilibrium currents, they should be proportional to appropriate “gradients” multiplied by the kinetic coefficients, an example of the contribution is
K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, Phys. Lett. B795, 100 (2019)

$$\delta T_a^{\nu\mu} = d_a^{\mu} u^{\nu} - d_a^{\nu} u^{\mu} + \cdots, \qquad d_a^{\mu} = \lambda_a \beta^{-1} (\beta a^{\mu} + \beta^2 \nabla^{\mu} T - 2\omega^{\mu\nu} u_{\nu}), \qquad \lambda_a > 0 \tag{51}$$

$$\partial_{\lambda} S^{\lambda,\mu\nu} = \delta T_a^{\nu\mu} - \delta T_a^{\mu\nu} \tag{52}$$

"kinetic force/gradient"

$$\omega_{\lambda\mu} + \frac{1}{2} \left(\partial_{\lambda} \beta_{\mu} - \partial_{\mu} \beta_{\lambda} \right) \tag{53}$$

brings spin polarization tensor towards thermal vorticity

$$\omega_{\lambda\mu} \to \varpi_{\lambda\mu} = -\frac{1}{2} \left(\partial_{\lambda} \beta_{\mu} - \partial_{\mu} \beta_{\lambda} \right) \tag{54}$$

Z. Drogosz, WF, J. Witkowski, to appear soon
can we reach global equilibrium?

Generalized Tolman-Klein conditions define global equilibrium state

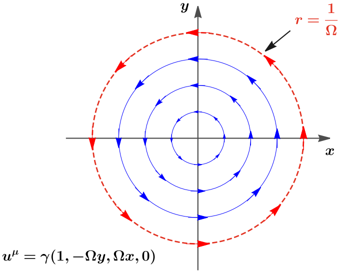
R. C. Tolman, Relativity, *Thermodynamics and Cosmology* (Oxford University Press, London, 1934)
O. Klein, Rev. Mod. Phys. 21 (1949) 531

$$\partial_\mu \xi = 0, \qquad \partial_{(\mu}\beta_{\lambda)} = 0, \qquad \omega_{\lambda\mu} = \partial_{[\mu}\beta_{\lambda]} = -\frac{1}{2}\left(\partial_\lambda\beta_\mu - \partial_\mu\beta_\lambda\right) \tag{55}$$

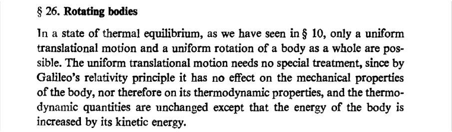
The middle equation, $\partial_\lambda\beta_\mu + \partial_\mu\beta_\lambda = 0$, is **the Killing equation** with a solution of the form

$$\beta^\mu = \beta^\mu_0 + \varpi^{\mu\nu}x_\nu, \qquad \varpi^{\mu\nu} = -\varpi^{\nu\mu} = \text{const}, \qquad \beta^\mu_0 = \text{const} \tag{56}$$

One possible solution: rigid rotation



L.D. Landau, E.M. Lifshitz , *Statistical Physics, Part 1* (Oxford Butterworth-Heinemann, 1980)



§ 26. **Rotating bodies**

In a state of thermal equilibrium, as we have seen in § 10, only a uniform translational motion and a uniform rotation of a body as a whole are possible. The uniform translational motion needs no special treatment, since by Galileo's relativity principle it has no effect on the mechanical properties of the body, nor therefore on its thermodynamic properties, and the thermodynamic quantities are unchanged except that the energy of the body is increased by its kinetic energy.

$T = T_0\gamma, \qquad \mu = \mu_0\gamma, \qquad \gamma = \frac{1}{\sqrt{1-\Omega^2 r^2}}, \qquad r < r_{\rm max} = \frac{1}{\Omega}$

a boundary needed to keep a system within a radius $r < r_{\rm max}$

PART 6: SPIN DYNAMICS WITH REALISTIC HYDRODYNAMIC BACKGROUND

Is there a place for perfect spin hydrodynamics in RHIC?

Sidney Coleman's QFT: as the collision energies decrease, all processes are dominated by s-wave scattering

S. K. Singh, R. Ryblewski, WF, Phys. Rev. C111 (2025) 024907

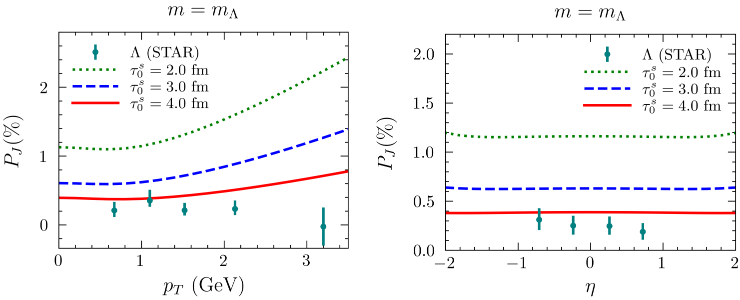
1. realistic 3D simulation of RHIC performed first, very good description of the rapidity distributions, transverse-momentum spectra, elliptic flow

   early stages, non-equilibrium processes, dissipation, **transfer between *L*** and ***S***

   late stages, spin approximately conserved, ***S*** ≈const
2. initialisation of the perfect spin hydrodynamics at the delayed proper time τ_0^S suitably chosen initial condition for the spin polarization tensor

$$\omega_{\mu\nu}(\tau_0^S, \eta, x, y) \tag{57}$$

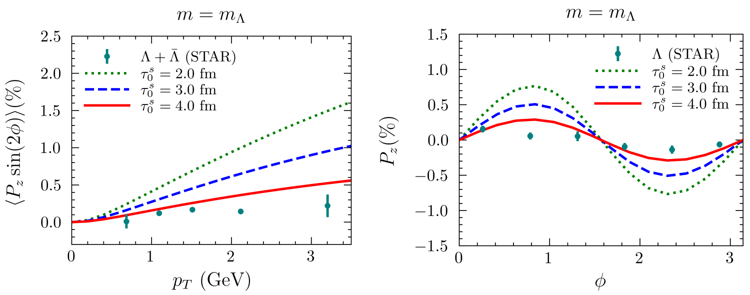
3. comparison with the data



Our numerical results for the component of Λ polarization along the orbital angular momentum direction for different initial time of spin evolution τ_0^s. Experimental data: STAR exp. at BNL, Au+Au collisions at $\sqrt{s_{\rm NN}}$= 200 GeV, c=20–60%

J. Adam et al. (STAR), Phys. Rev. C 98 (2018) 014910

J. Adam et al. (STAR), Phys. Rev. Lett. 123 (2019) 132301



Our numerical results for longitudinal Λ polarization for different initial time of spin evolution τ_0^s. Experimental data: STAR experiment at BNL, Au+Au collisions at $\sqrt{s_{\rm NN}}$= 200 GeV, c=20–60%
J. Adam et al. (STAR), Phys. Rev. C98 (2018) 014910
J. Adam et al. (STAR), Phys. Rev. Lett. 123 (2019) 132301

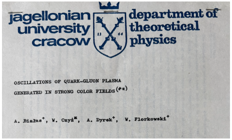
a more complete study describing the spin polarization generation
S. K. Singh, D. Wagner, Phys. Rev.C 112 (2025) 054902

Conclusions

1. The concept of perfect spin hydrodynamics has been proposed for a relativistic gas of spin 1/2 Dirac fermions. With a suitably chosen pseudo-gauge (GLW), a set of consistent hydrodynamic and thermodynamic relations is obtained. The correct expansion is twofold; in the magnitude of ω and gradients of hydrodynamic variables.
2. Two formulations (based on the classical-spin treatment, or using the spin density matrix) have been given and shown to be exactly consistent up to the second order in ω.
3. Extension to dissipative hydrodynamics has been formulated with the Israel-Stewart method. Dissipation introduces the genuine spin orbit interaction.
   ....
4. Generalizations to Fermi-Dirac and Bose-Einstein statistics have been made.
5. We are able to verify nonlinear causality and symmetric hyperbolicity of the equations of motion of perfect spin hydrodynamics constructed with our equilibrium functions, which ensures local well-posedness of the initial value problem and stability of the theory.

for more see the next talks by Zbigniew Drogosz and Valeriya Mykhaylova

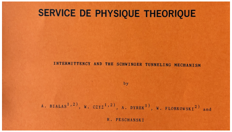
Many Thanks and best Wishes on the occasion of 90^{th} Birthday to Prof. Andrzej Białas!
Thank You for shaping my scientific life from 1983!


jagellonian university cracow

department of theoretical physics

OSCILLATIONS OF QUARK-GLUON PLASMA
GENERATED IN STRONG COLOR FIELDS(**)

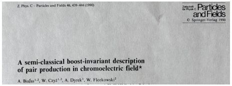
A. Białas+, W. Czyż*, A. Dyrek+, W. Florkowski+


SERVICE DE PHYSIQUE THEORIQUE

INTERMITTENCY AND THE SCHWINGER TUNNELING MECHANISM

by

A. BIALAS[1,2], W. CZYZ[1,2], A. DYREK[1], W. FLORKOWSKI[2] and
R. PESCHANSKI


Z. Phys. C – Particles and Fields 46, 439–444 (1990)

Zeitschrift für Physik C Particles and Fields
© Springer-Verlag 1990

A semi-classical boost-invariant description
of pair production in chromoelectric field*

A. Białas[1,2], W. Czyż[1,2], A. Dyrek[1], W. Florkowski[2]

Budapest 2004

BACK-UP SLIDES

LOCALITY VS. NON-LOCALITY

conservation of the total angular momentum of a particle implies conservation of spin if collisions are local (spacetime coordinate x^μ can be always set equal to 0)

$$j^{\alpha\beta} = l^{\alpha\beta} + s^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha + s^{\alpha\beta} \tag{58}$$

transfer between orbital and spin part is possible if collisions are non-local , this leads to dissipation and entropy production, **series of influential works by the Frankfurt group**
N. Weickgenannt, E. Speranza, Xin-Li Sheng, Q. Wang, D. Rischke, PRL127 (2021) 052301