Zakopane school 2026 in honour of Andrzej Bialas

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Dissipative correction to the momentum spectrum of particles emitted from a relativistic fluid

F.B., D. Roselli, X.L. Sheng arXiv:2512.24994

Deeply revised version to appear including Z. H. Zhang as co-author

CONTENT

An ambitious attempt to calculate non-equilibrium corrections to the momentum spectrum of particles with an *ab initio* calculation in a statistical QFT framework

RESULT

A critical revisitation of Kubo formulae: appearance of memory terms

Happy birthday Andrzej!

ISMD Cracow September 2003

SQM Cracow September 2011

Introduction

$$\varepsilon \frac{\mathrm{d}N}{\mathrm{d}^3\mathrm{k}} = \int_{\Sigma_D} \mathrm{d}\Sigma \cdot k \left(\frac{1}{\mathrm{e}^{\beta(x)\cdot p - \zeta q} \pm 1} + \delta f \right)$$

Non-equilibrium Cooper-Frye formula

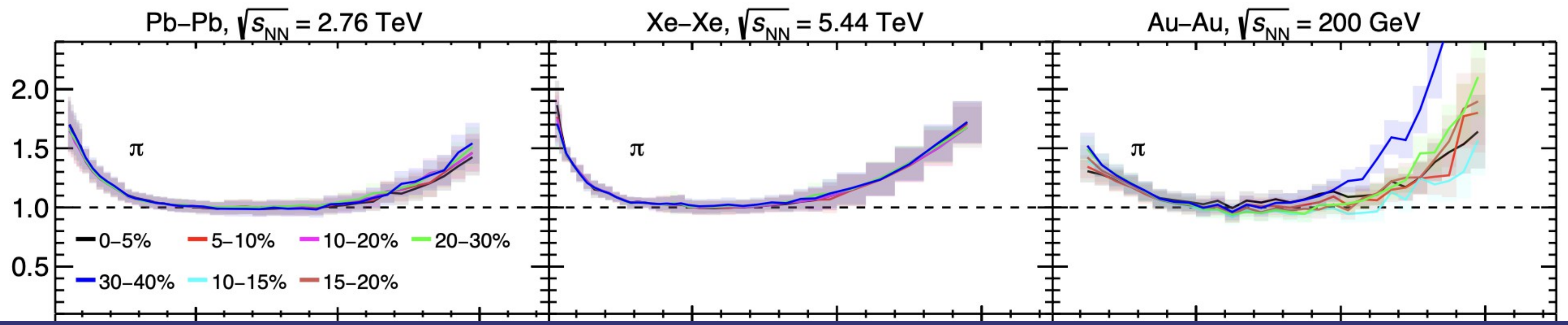
Dissipative correction from relativistic kinetic theory

(see e.g. P. and U. Romatschke, *Relativistic Fluid Dynamics In and Out of Equilibrium*, Cambridge University Press, 2019)

$$\delta f_i^{\mathrm{shear}} = f_i^{(0)}(1 \pm f_i^{(0)}) \frac{p_\mu p_\nu \pi^{\mu\nu}}{2T^2(e+P)}.$$

"Quadratic ansatz"

P. Lu et al. *Quantification of the low-pT pion excess in heavy-ion collisions at the LHC and top RHIC energy*, Nucl. Sci. Tech. 36 (2025) no.8, 142



Introduction (2)

Original motivation: the search of dissipative corrections to spin polarization

D. Wagner, N. Weickgenannt and E. Speranza, Phys. Rev. Res. 5 (2023), 013187
W.-B. Dong, Y.-L. Yin, X.-L. Sheng, S.-Z. Yang, and Q. Wang, , Phys. Rev. D 109 (2024) 056025
S. Z. Yang, X. Q. Xie, S. Pu, J. H. Gao and Q. Wang, Phys. Rev. D 112 (2025) no.9, 094040
M. Buzzegoli, JHEP 07 (2025), 255
Y. Li, S.Y.F. Liu, Phys. Rev. C 113 (2026), 034911
Z.-Y. Sun, Y.-Y. Li, and S. Y. F. Liu, arXiv:2503.13408.

PROGRAMME: An *ab initio* calculation within a statistical QFT framework

EXECUTION: Fields with spin too difficult → General interacting quantum scalar field.

OUTPUT: Surprising conclusions

Momentum spectrum: free fields

FREE SCALAR FIELD

$$\widehat{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3\mathrm{p}}{2\varepsilon_p} \left(\widehat{a}(p) e^{-\mathrm{i}p\cdot x} + \widehat{b}^\dagger(p) e^{\mathrm{i}p\cdot x} \right)$$

Wigner operator

$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4} \int \mathrm{d}^4 s \; \mathrm{e}^{-\mathrm{i}s\cdot k} : \widehat{\psi}^\dagger \left(x + \frac{s}{2} \right) \widehat{\psi} \left(x - \frac{s}{2} \right) :$$

Wigner function

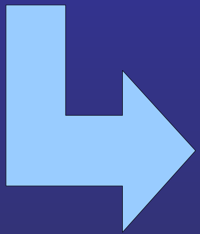
$$W(x,k) \equiv \mathrm{Tr} \left[\widehat{\rho}\, \widehat{W}(x,k) \right] ,$$

$$\widehat{W}(x,k) \equiv \widehat{W}^+(x,k) + \widehat{W}^-(x,k) + \widehat{W}_S(x,k) ,$$

$$\widehat{W}^\pm(x,k) \equiv \widehat{W}(x,k)\, \theta(k^2)\, \theta(\pm k^0) , \quad \widehat{W}_S \equiv \widehat{W}(x,k)\, \theta(-k^2)$$

Particle, antiparticle and mixed components

SPECTRUM



$$\frac{\mathrm{d}N_k}{\mathrm{d}^3\mathrm{k}} = \int \mathrm{d}k^0 \int_\Sigma \mathrm{d}\Sigma_\mu k^\mu \, W^+(x,k) = \frac{1}{2\varepsilon_{\mathbf{k}}} \langle \widehat{a}^\dagger(k) \widehat{a}(k) \rangle ,$$

Momentum spectrum: interacting fields

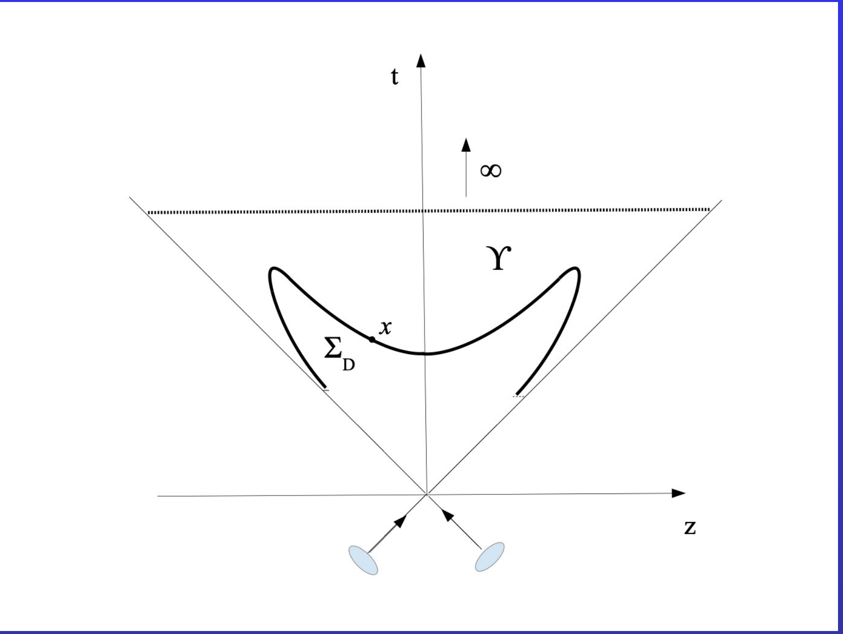
In a scattering experiment the free out-fields are (weak) limits of interacting fields

Yang-Feldman equation connecting interacting and out-field

$$\widehat{\phi}(x) = \widehat{\phi}_{\rm out}(x) - \int {\rm d}^4 y \, \Delta_{\rm adv}(x-y)\, \widehat{J}(y) \, ,$$

$$\lim_{t \to +\infty} \widehat{\phi}(x) = \widehat{\phi}_{\rm out}(x) \, .$$

$$(\Box + m^2)\, \widehat{\phi}(x) = \widehat{J}(x) \, .$$



Wigner function with interacting field

$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4} \int {\rm d}^4 s \, {\rm e}^{-{\rm i}s \cdot k} : \widehat{\phi}^\dagger \left(x + \frac{s}{2}\right) \widehat{\phi}\left(x - \frac{s}{2}\right) :$$

$$\frac{{\rm d}N_k}{{\rm d}^3{\rm k}} = \lim_{t \to +\infty} \int {\rm d}k^0 \int_\Sigma {\rm d}\Sigma_\mu k^\mu \, W^+(x,k) = \frac{1}{2\varepsilon} \langle \widehat{a}^\dagger_{\rm out}(k) \widehat{a}_{\rm out}(k) \rangle$$

= the quantum version of the Cooper-Frye formula

$$\frac{{\rm d}N_k}{{\rm d}^3{\rm k}} = \int {\rm d}k^0 \int_{\Sigma_D} {\rm d}\Sigma_\mu k^\mu \, W^+(x,k) + \int {\rm d}k^0 \int_\Upsilon {\rm d}^4 x \, k^\mu \partial_\mu W^+(x,k) \, ,$$

By using Gauss theorem

Fluid-decoupling hypersurface

Residual interaction (RKT)

Interacting fields (2)

General field expansion (formal)

$$\widehat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3\mathrm{p} \int_{-|\mathbf{p}|^2}^{+\infty} \mathrm{d}M^2 \, \frac{1}{2\sqrt{M^2+|\mathbf{p}|^2}} \frac{\varrho(p)}{2\pi} \left(\mathrm{e}^{-\mathrm{i}p\cdot x} \widehat{A}(M^2,\mathbf{p}) + \mathrm{e}^{\mathrm{i}p\cdot x} \widehat{B}^\dagger(M^2,\mathbf{p}) \right) ,$$

$$\widehat{A}(p) \equiv \frac{1}{(2\pi)^{3/2}\varrho(p)} \theta(p^0)\widehat{\phi}_F(p) , \qquad \widehat{B}^\dagger(p) \equiv \frac{1}{(2\pi)^{3/2}\varrho(p)} \theta(p^0)\widehat{\phi}_F(-p) ,$$

Spectral function in some state

$$\varrho(p) \equiv \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}p\cdot x} \left\langle \left[\widehat{\phi}(x), \widehat{\phi}^\dagger(0) \right] \right\rangle .$$

$$\varrho_{\mathrm{free}}(p) = 2\pi \, \mathrm{sign}(p^0)\delta(p^2-m^2) = 2\pi \, \mathrm{sign}(p^0)\delta(M^2-m^2) .$$

BEWARE! A and B do not fulfill free field commutation relations!

Wigner operator (particle part)

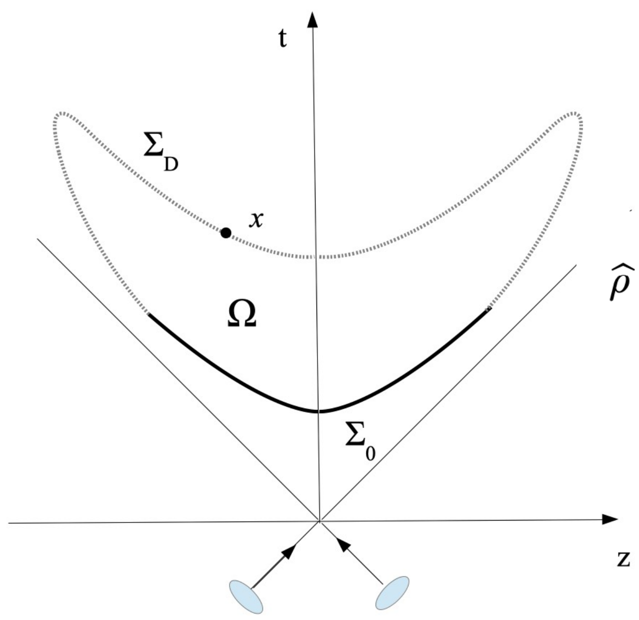
$$\widehat{W}^+(x,k) = \frac{2}{(2\pi)^5} \int \mathrm{d}^4q \, \mathrm{e}^{\mathrm{i}x\cdot q} \Big\{ \varrho(k_+)\varrho(k_-)\widehat{A}^\dagger(k_+)\widehat{A}(k_-)\theta(k_+^0)\theta(k_-^0) + \varrho(k_+)\varrho(-k_-) \left[\widehat{A}^\dagger(k_+)\widehat{B}^\dagger(-k_-) + \mathrm{h.c.} \right] \theta(k_+^0)\theta(-k_-^0) \Big\} .$$

$$k_\pm = k \pm q/2$$

Density operator

Quantum state (assumption): local equilibrium at the initial 3D Cauchy hypersurface

$$\widehat{\rho} = \widehat{\rho}_{\rm LE}(\tau_0) = \frac{1}{Z} \exp\left[- \int_{\Sigma_0} {\rm d}\Sigma_\mu(y) \left(\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \widehat{j}^\mu(y)\zeta(y) \right) \right] ,$$



In traditional calculations, the density operator is rewritten using the Gauss theorem:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma_D} {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu \right) + \int_\Omega {\rm d}^4 y \left(\widehat{T}^{\mu\nu}(y)\partial_\mu\beta_\nu(y) - \partial_\mu\zeta(y)\widehat{j}^\mu(y) \right) \right]$$

local equilibrium *dissipation*

Very good method if the system stays close to local thermodynamic equilibrium

$$\langle \widehat{O}(x) \rangle = {\rm Tr}(\widehat{\rho}\,\widehat{O}(x)) = {\rm Tr}(\widehat{\rho}_{\rm LE}\widehat{O}(x)) + \Delta O(x) = \langle \widehat{O}(x) \rangle_{\rm LE} + \Delta O(x)$$

Local equilibrium part *Dissipative part*

An equivalent method to calculate the non-equilibrium corrections

X. L. Sheng, F. B. and D. Roselli, arXiv:2509.14301

$$\widehat{\rho} = \widehat{\rho}_{\rm LE}(\tau_0) = \frac{1}{Z} \exp\left[-\int_{\Sigma_0} {\rm d}\Sigma_\mu(y) \left(\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \widehat{j}^\mu(y)\zeta(y) \right) \right] ,$$

Rewrite:

$$\beta_\nu(y) = \beta_\nu(x) + (\beta_\nu(y) - \beta_\nu(x)) = \beta_\nu(x) + \Delta\beta_\nu(y,x)$$

$$\zeta(y) = \zeta(x) + (\zeta(y) - \zeta(x)) = \zeta(x) + \Delta\zeta(y,x),$$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\beta(x)\cdot P + \zeta(x)\widehat{Q} - \int_{\Sigma_0} {\rm d}\Sigma_\mu(y) \left(\widehat{T}^{\mu\nu}(y)\Delta\beta_\nu(y,x) - \widehat{j}^\mu(y)\Delta\zeta(y,x) \right) \right] ,$$

Main term: global equilibrium at β(x), ζ(x)

Perturbation: Δβ, Δζ, assumed to be small enough

$$\langle \widehat{O}(x) \rangle \simeq \langle \widehat{O}(x) \rangle_{\rm GE} + \Delta O(x) ,$$

$$\widehat{\rho}_{\rm GE} = \frac{1}{Z_{\rm GE}} \exp\left[-\beta(x)\cdot\widehat{P} + \zeta(x)\widehat{Q} \right]$$

The Wigner function

$$\langle \widehat{W}^+(x,k)\rangle \simeq \langle \widehat{W}^+(x)\rangle_{\rm GE} + \Delta W^+(x,k) \ ,$$

Main term

$$\langle \widehat{W}^+(x,k)\rangle_{\rm GE} = \frac{2}{(2\pi)^4} n_{\rm B}(\beta(x)\cdot k)\, \varrho(k) \ ,$$

$$\frac{{\rm d}N_k}{{\rm d}^3{\rm k}} = \frac{2}{(2\pi)^4}\int_0^{+\infty} {\rm d}k^0\ \varrho(k) \int_{\Sigma_{\rm D}} {\rm d}\Sigma\cdot k\ n_B(\beta(x)\cdot k) \ ,$$

The well known Cooper-Frye formula (Statistical hadronization model)

Linear correction:

$$\Delta W^+(x,k) = \frac{2}{(2\pi)^5}\int_{\Sigma_0} {\rm d}\Sigma_\mu(y) \int {\rm d}^4 q \Bigg\{ \varrho(k_+)\varrho(k_-)\frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}{\rm e}^{{\rm i}q\cdot(x-y)}\ \theta(k_+^0)\theta(k_-^0)$$

$$\times \left(\left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{\mu\nu}(0)\right\rangle_{c,{\rm GE}} \Delta\beta_\nu(y,x) - \left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{j}^\mu(0)\right\rangle_{c,{\rm GE}} \Delta\zeta(y,x)\right)$$

$$+\ \varrho(k_+)\varrho(-k_-)\theta(k_+^0)\theta(-k_-^0)\frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}$$

$$\times \left({\rm e}^{{\rm i}q\cdot(x-y)} \left\langle \widehat{A}^\dagger(k_+)\widehat{B}^\dagger(-k_-),\ \widehat{T}^{\mu\nu}(0)\right\rangle_{c,{\rm GE}} \Delta\beta_\nu(y,x) - \left\langle \widehat{A}^\dagger(k_+)\widehat{B}^\dagger(-k_-),\ \widehat{j}^\mu(0)\right\rangle_{c,{\rm GE}} \Delta\zeta(y,x) + {\rm c.c}\right) \Bigg\}$$

Unknown correlators: they play a crucial role

Correlators

$$\Theta^{\mu\nu}(k,q,\beta) \equiv \varrho(k)\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\; \widehat{T}^{\mu\nu}(0)\rangle_{c,\mathrm{GE}} ,$$

Major constraints from energy-momentum conservation:

$$\Theta^{0\nu}(k,q^0,\mathbf{q}=0,\beta) = -\frac{\theta(k^0)}{(2\pi)^2\varrho(k)}\delta(q^0)\left[\frac{\partial}{\partial\beta_\nu(x)}\left(n_B(k)\varrho(k)\right)\right] ,$$

$$q_\mu\Theta^{\mu\nu}(k,q,\beta)=0\,, \qquad \forall\, k,\ q,\ \beta\,.$$

Ward identity

Resulting most general form:

$$\Theta^{\mu\nu}(k,q,\beta) = \int_{-\pi/2}^{\pi/2}\mathrm{d}\vartheta\sum_f \delta\left(q\cdot w(\vartheta)-f(S)\right)\Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)+\delta(q^2)\Gamma^{\mu\nu}_{q^2}(k,q,\beta)+\Xi^{\mu\nu}(k,q,\beta)\,.$$

S stands for scalars formed out of vectors *k, q,* β except linear combinations of *kq* and *qβ*

$$w^\mu(\vartheta) = \cos\vartheta\frac{k^\mu}{\sqrt{k^2}}+\sin\vartheta\frac{\beta^\mu}{\sqrt{\beta^2}}$$

Analiticity in *q*=0, delta distributions and Ward identities constrain their form

Example:

$$\begin{aligned}\Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta) = \Gamma_{1,f}(\vartheta;S)\Big\{&(q\cdot k)(q\cdot\beta)w^\mu(\vartheta)w^\nu(\vartheta)-f(S)\Big[(q\cdot\beta)\frac{\cos\vartheta}{\sqrt{k^2}}k^\mu k^\nu+(q\cdot k)\frac{\sin\vartheta}{\sqrt{\beta^2}}\beta^\mu\beta^\nu\Big]\Big\}\\ &+\Gamma_{2,f}(\vartheta;S)\left(q^\mu q^\nu-q^2g^{\mu\nu}\right)+\Gamma_{3,f}(\vartheta;S)\left[(q\cdot k)\left(k^\mu q^\nu+k^\nu q^\mu\right)-q^2k^\mu k^\nu-(q\cdot k)^2g^{\mu\nu}\right]\\ &+\Gamma_{4,f}(\vartheta;S)\left[(q\cdot\beta)\left(q^\mu\beta^\nu+q^\nu\beta^\mu\right)-q^2\beta^\mu\beta^\nu-(q\cdot\beta)^2g^{\mu\nu}\right] ,\end{aligned}$$

Γ_i ≡ thermal-gravitational form factors

Correlators (2)

Analiticity in q=0, delta distributions and Ward identities imply:

$$\Gamma^{\mu\nu}_{\vartheta,f} = \mathcal{O}(q^2) \;\text{ if } f \neq 0 \qquad \Gamma^{\mu\nu}_{q^2} = \mathcal{O}(q^2) \qquad \Xi^{\mu\nu} = \mathcal{O}(q^2)$$

Special case if $f(S)$=0

$$\begin{aligned}\Gamma^{\mu\nu}_{\vartheta}(k,q,\beta) = {} & \Gamma_1(\vartheta;S) w^\mu(\vartheta) w^\nu(\vartheta) + \Gamma_2(\vartheta;S)\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \\ & + \Gamma_3(\vartheta;S)\left[(q\cdot k)(k^\mu q^\nu + k^\nu q^\mu) - q^2 k^\mu k^\nu - (q\cdot k)^2 g^{\mu\nu}\right] \\ & + \Gamma_4(\vartheta;S)\left[(q\cdot\beta)(q^\mu \beta^\nu + q^\nu \beta^\mu) - q^2 \beta^\mu \beta^\nu - (q\cdot\beta)^2 g^{\mu\nu}\right]\end{aligned}$$

The only correlator surviving in the $q = 0$ limit

Energy-momentum conservation implies:

$$\frac{1}{\sqrt{k^2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \, \cos\vartheta \, \Gamma_1(\vartheta; k, 0, \beta) = \frac{\theta(k^0)}{(2\pi)^2} n_{\mathrm{B}}(k) \left[1 + n_{\mathrm{B}}(k) - \frac{\partial \log \varrho(k)}{\partial(k\cdot\beta)}\right] \, ,$$

$$\frac{1}{\sqrt{\beta^2}} \int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \, \sin\vartheta \, \Gamma_1(\vartheta; k, 0, \beta) = -\frac{2\theta(k^0)}{(2\pi)^2} n_{\mathrm{B}}(k) \frac{\partial \log \varrho(k)}{\partial \beta^2} \, .$$

Wigner-SET correlation function

The leading correction of the Wigner function can be also expressed in terms of a correlation function:

$$\Delta W^+(x,k)\Big|_{WT} = -\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\Delta\beta_\nu(y,x)\int_0^1 \mathrm{d}z\, \langle \widehat{W}^+(x,k), \mathrm{e}^{z\widehat{\mathcal{E}}_{\rm GE}}\widehat{T}^{\mu\nu}(y)\mathrm{e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\rangle_{c,\rm GE}$$
$$\equiv \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\Delta\beta_\nu(y,x)C^{\mu\nu}_{WT}(x-y)\ ,$$

By using Gauss theorem

$$\Delta W^+(x,k) = \int_{\Sigma_{\rm D}} \mathrm{d}\Sigma_\mu(y)\ \Delta\beta_\nu(y,x)C^{\mu\nu}_{WT}(x-y,k) + \int_\Omega \mathrm{d}^4y\ C^{\mu\nu}_{WT}(y-x,k)\frac{\partial}{\partial y^\mu}\Delta\beta_\nu(y,x)\ .$$

Hydrodynamic limit: separation of scales

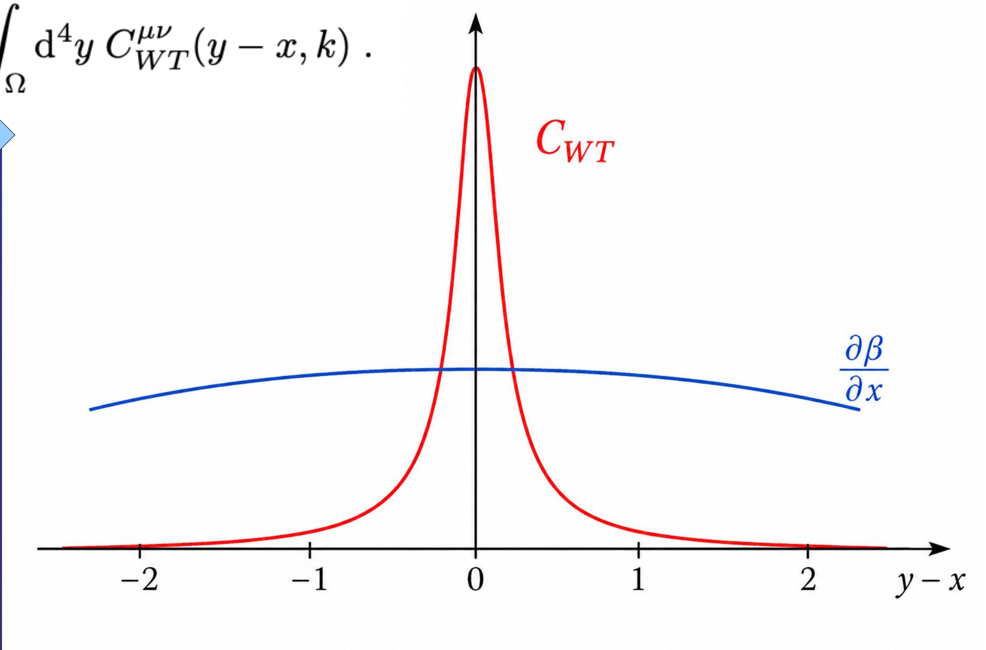
$$\Delta W^+(x,k) \simeq \frac{\partial\beta_\nu}{\partial x^\lambda}\int_{\Sigma_D} \mathrm{d}\Sigma_\mu(y)\ (y-x)^\lambda C^{\mu\nu}_{WT}(x-y,k) + \frac{\partial\beta_\nu}{\partial x^\mu}\int_\Omega \mathrm{d}^4y\ C^{\mu\nu}_{WT}(y-x,k)\ .$$

Local equilibrium correction

Dissipative part

Transport coefficients encoded in the "Kubo" formula

$$\int_\Omega \mathrm{d}^4y\ C^{\mu\nu}_{WT}(y-x,k)$$

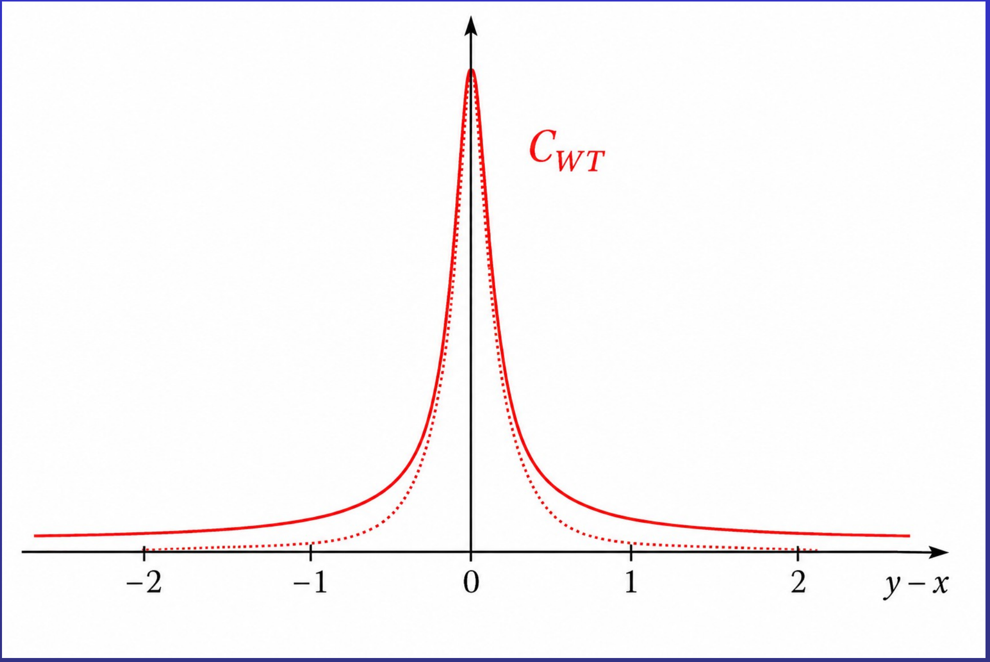


Correlation function (2)

if $\Omega = \mathbb{R}^3 \times [t_0, t]$

$$\int_\Omega \mathrm{d}^4 y \, C_{WT}^{\mu\nu}(y-x,k) = (t-t_0)(2\pi)^3 \varrho(k)\theta(k^0) \int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \, \frac{1}{|w^0(\vartheta)|} \Gamma_\vartheta^{\mu\nu}(k,0,\beta)$$

Surprising result: the Kubo formula diverges! The correlation function is not LOCAL enough



It is NOT possible to express the dissipative part of the Wigner function as a traditional gradient expansion with gradients evaluated at current time

Correlation function (3)

Correlators are basically the Fourier transform of the correlation function

$$C^{\mu\nu}_{WT}(x-y,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4q\, \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\mathrm{e}^{\mathrm{i}q\cdot(x-y)}\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\Theta^{\mu\nu}(k,q,\beta)$$

There are four contributing terms:

$$C^{\mu\nu}_{WT1}(x-y,k) = \frac{2}{(2\pi)^5}\int_{-\pi/2}^{\pi/2}\mathrm{d}\vartheta \int \mathrm{d}^4q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\delta(q\cdot w(\vartheta))\,\frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\Gamma^{\mu\nu}_{\vartheta}(k,q,\beta)$$

$$C^{\mu\nu}_{WT2}(x-y,k) = \frac{2}{(2\pi)^5}\sum_{f\neq 0}\int_{-\pi/2}^{\pi/2}\mathrm{d}\vartheta \int \mathrm{d}^4q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\delta\left(q\cdot w(\vartheta) - f(S)\right)\frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q} \times \Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)$$

$$C^{\mu\nu}_{WT3}(x-y,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\delta(q^2)\,\frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\,\Gamma^{\mu\nu}_{q^2}(k,q,\beta)$$

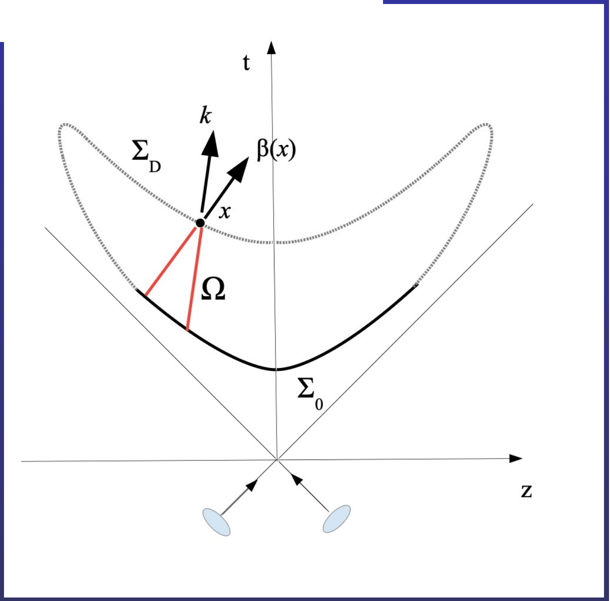
$$C^{\mu\nu}_{WT4}(x-y,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\,\frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\,\Xi^{\mu\nu}(k,q,\beta)$$

Constant over the world-lines

$$(x-y)^\mu = w^\mu(\vartheta)\tau$$

This is a long-distance contribution to the correlation function which is responsible for the divergence of the "Kubo" integral

$$\int_\Omega \mathrm{d}^4y\; C^{\mu\nu}_{WT}(y-x,k)$$



Wigner function non-equilibrium correction

$$\Delta W_i^+(x,k) = \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\ \left[C^{\mu\nu}_{WTi}(x-y,k)\Delta\beta_\nu(y,x) - C^{(\Xi)\mu\nu}_{Wj\,i}(x-y,k)\Delta\zeta(y,x)\right]$$
$$= \int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma_\mu(y)\ \left[C^{\mu\nu}_{WTi}(x-y,k)\Delta\beta_\nu(y,x) - C^{(\Xi)\mu\nu}_{Wj\,i}(x-y,k)\Delta\zeta(y,x)\right]$$
$$+ \int_\Omega \mathrm{d}^4 y\ \left[C^{\mu\nu}_{WTi}(x-y,k)\frac{\partial\beta_\nu}{\partial y^\mu} - C^{(\Xi)\mu\nu}_{Wj\,i}(x-y,k)\frac{\partial\zeta}{\partial y^\mu}\right] \qquad i=1,\ldots,4$$

We cannot take gradients out of the integrals as in the familiar Kubo viscosity formula (except i=4).
The non-equilibrium correction of the Wigner function depends on the entire history
(this is referred to as a memory effect), not just on gradients at freeze-out time.

Why ?

1) Wigner operator is not actually local

$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4}\int \mathrm{d}^4 s\, \mathrm{e}^{-\mathrm{i}s\cdot k} : \widehat{\psi}^{\dagger}\left(x+\frac{s}{2}\right)\widehat{\psi}\left(x-\frac{s}{2}\right):$$

$$\langle \widehat{W}^{+}(x,k)\widehat{T}^{\mu\nu}(y)\rangle \neq \langle \widehat{W}^{+}(x,k)\rangle\langle \widehat{T}^{\mu\nu}(y)\rangle \qquad (x-y)\ \text{large}$$

Correlation function does not decay rapidly enough

2) In the commonly used derivation of Kubo formulae of transport coefficients a *specific perturbation is turned on adiabatically*, but in the actual evolution process we deal with a convolution integral of the actual gradients

$$\Delta W^{+}(x,k) = \int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma_\mu(y)\, \Delta\beta_\nu(y,x) C^{\mu\nu}_{WT}(x-y,k) + \int_\Omega \mathrm{d}^4 y\, C^{\mu\nu}_{WT}(y-x,k)\frac{\partial}{\partial y^\mu}\Delta\beta_\nu(y,x)\,.$$

and it is thus not obvious that:

?

$$\Delta W^{+}(x,k) \simeq \frac{\partial\beta_\nu}{\partial x^\lambda}\int_{\Sigma_D} \mathrm{d}\Sigma_\mu(y)\,(y-x)^\lambda C^{\mu\nu}_{WT}(x-y,k) + \frac{\partial\beta_\nu}{\partial x^\mu}\int_\Omega \mathrm{d}^4 y\, C^{\mu\nu}_{WT}(y-x,k)\,.$$

Free field case

In the free field case, the correlation function can be calculated exactly

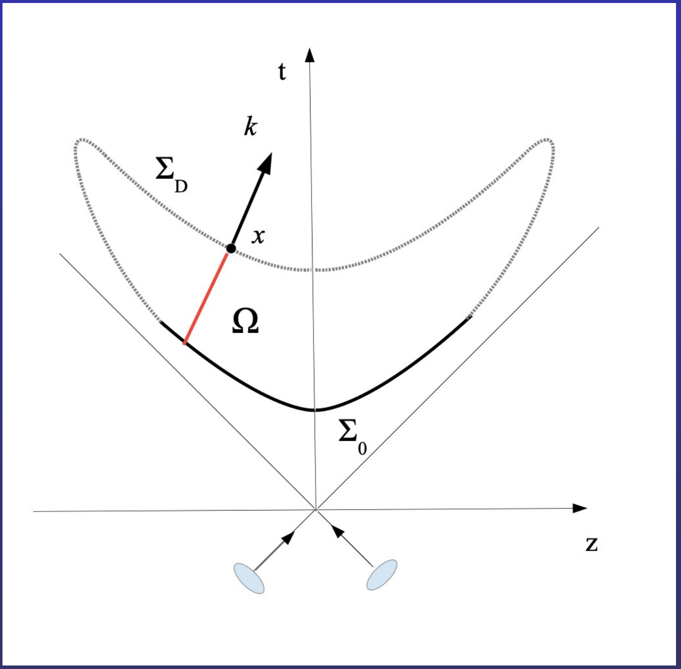
$$C_{WT}(x-y) = \int \mathrm{d}^4q \; \delta(k\cdot q) \mathrm{e}^{\mathrm{i}q\cdot(x-y)} G^{\mu\nu}(q) = -(2\pi)^3 \int_0^1 \mathrm{d}z \; \langle \widehat{W}^+(x,k), \mathrm{e}^{z\widehat{A}} \widehat{T}^{\mu\nu}(y) \mathrm{e}^{-z\widehat{A}} \rangle_{c,\mathrm{GE}} \, .$$

$$G^{\mu\nu}_{\mathrm{free}}(k,q,\beta) = \theta(k^0_+)\theta(k^0_-)\delta\left(k^2 + \frac{q^2}{4} - m^2\right) \frac{n_{\mathrm{B}}(k_+) + n_{\mathrm{B}}(k_-)}{(2\pi)^3 \beta(x)\cdot q} \left[k^\mu k^\nu - \frac{1}{4}\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\right] \, ,$$

The correlation function is constant over the worldline

$$\mathbf{y} = \mathbf{x} - \frac{\mathbf{k}}{k^0}(y^0 - x^0) \, ,$$

consistently with the *free streaming* from Σ_0 to Σ_D



It should not be surprising that for an interacting system a similar effect partially survives

Gradient expansion: is it possible?

$$C^{\mu\nu}_{WT1}(x-y,k) = \frac{2}{(2\pi)^5}\int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \int \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\delta(q\cdot w(\vartheta))\, \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\Gamma^{\mu\nu}_{\vartheta}(k,q,\beta)$$

$$C^{\mu\nu}_{WT2}(x-y,k) = \frac{2}{(2\pi)^5}\sum_{f\neq 0}\int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta \int \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\delta\left(q\cdot w(\vartheta) - f(S)\right) \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q} \times \Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)$$

$$C^{\mu\nu}_{WT3}(x-y,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\delta(q^2)\, \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\,\Gamma^{\mu\nu}_{q^2}(k,q,\beta)$$

$$C^{\mu\nu}_{WT4}(x-y,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k^0_+)\theta(k^0_-)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\,\Xi^{\mu\nu}(k,q,\beta)$$

$$\begin{aligned}\Delta W_i^+(x,k) &= \int_{\Sigma_0}\mathrm{d}\Sigma_\mu(y)\,\left[C^{\mu\nu}_{WTi}(x-y,k)\Delta\beta_\nu(y,x) - C^{(\Xi)\mu\nu}_{Wj\,i}(x-y,k)\Delta\zeta(y,x)\right] \\ &= \int_{\Sigma_D}\mathrm{d}\Sigma_\mu(y)\,\left[C^{\mu\nu}_{WTi}(x-y,k)\Delta\beta_\nu(y,x) - C^{(\Xi)\mu\nu}_{Wj\,i}(x-y,k)\Delta\zeta(y,x)\right] \\ &+ \int_\Omega \mathrm{d}^4y\,\left[C^{\mu\nu}_{WTi}(x-y,k)\frac{\partial\beta_\nu}{\partial y^\mu} - C^{(\Xi)\mu\nu}_{Wj\,i}(x-y,k)\frac{\partial\zeta}{\partial y^\mu}\right] \qquad i=1,\ldots,4\end{aligned}$$

$$\Delta W_1(x,k) = \sum_N a_N \partial^{(N)}\beta(\bar{y}(x))$$

$$\Delta W_2(x,k) = \sum_N a_N \partial^{(N)}\beta(\bar{y}(x))$$

→ Gradients on the initial hypersurface

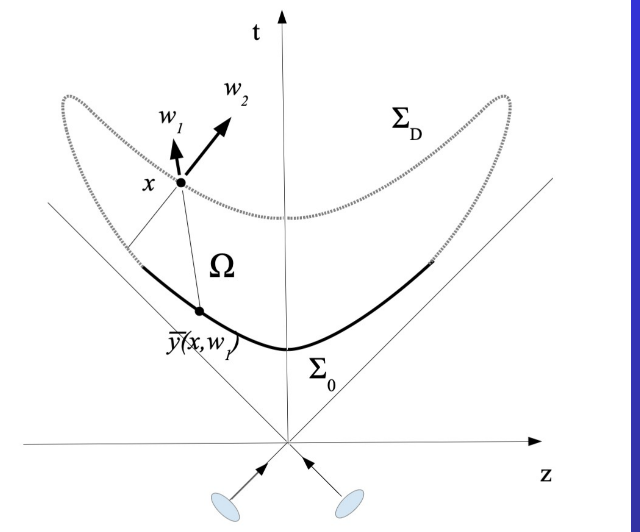
$$\Delta W_4(x,k) = \sum_N a_N \partial^{(N)}\beta(x)$$

→ usual gradient expansion

Leading correction to the spectrum

Correspondance between the order of the gradient N and the order of derivative in q in the coefficients a_N

$$\Gamma^{\mu\nu}_{\vartheta,f} = \mathcal{O}(q^2) \text{ if } f \neq 0 \qquad \Gamma^{\mu\nu}_{q^2} = \mathcal{O}(q^2) \qquad \Xi^{\mu\nu} = \mathcal{O}(q^2)$$



$$\Delta W(x,k)^+_1 = \mathcal{O}(\beta(\bar{y}(x)) - \beta(x))$$
$$\Delta W(x,k)^+_2 = \mathcal{O}(\partial^2 \beta(\bar{y}(x)))$$
$$\Delta W(x,k)^+_4 = \mathcal{O}(\partial^3 \beta(x))$$

The leading correction in the gradient expansion is 0^{th} order:

$$\Delta^{(0)} W^+(x,k) \simeq \frac{2\varrho(k)}{(2\pi)^2} \int_{-\pi/2}^{\pi/2} \mathrm{d}\vartheta\, \theta_\vartheta(x) \left\{ \Gamma_1(\vartheta; S_0)\left[w_\vartheta \cdot \beta(x) - w_\vartheta \cdot \beta(\bar{y}_\vartheta(x))\right] - \Upsilon_1(\vartheta; S_0)\left[\zeta(x) - \zeta(\bar{y}_\vartheta(x))\right] \right\}$$

$$\Delta \frac{\mathrm{d}N}{\mathrm{d}^3\mathrm{k}} \simeq \int \mathrm{d}k^0 \int_{\Sigma_D} \mathrm{d}\Sigma \cdot k\, \Delta^{(0)} W^+(x,k)$$

Summary

- The non-equilibrium corrections to the Wigner function cannot be expressed solely by means of an expansion in gradients evaluated at the current space-time point

- There are memory terms which cannot be captured by traditional relativistic kinetic theory

- The leading term in this expansion is proportional to the difference between the hydro-thermodynamic fields at decoupling and initial hypersurface

Outlook

- Implementation in hydro code? Applicable to study the pT pion excess ?

Approximation: linear response

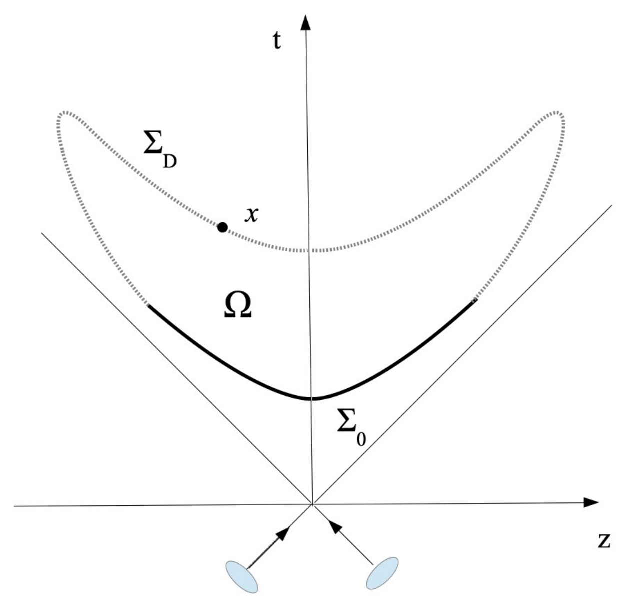
By expanding the density operator from GE, we can calculate the non-equilibrium correction of local operators at the linear order

$$\Delta O(x) = -\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y) \int_0^1 \mathrm{d}z \, \langle \widehat{O}(x), \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}} \left(\widehat{T}^{\mu\nu}(y)\Delta\beta_\nu(y,x) - \widehat{j}^\mu \Delta\zeta(y,x) \right) \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \rangle_{c,\mathrm{GE}} \,,$$

This expression is totally equivalent to the one obtained by separating local equilibrium and dissipative terms

$$\Delta O(x) = -\int_{\Sigma} \mathrm{d}\Sigma_\mu(y) \int_0^1 \mathrm{d}z \, \langle \widehat{O}(x), \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}} \left(\widehat{T}^{\mu\nu}(y)\Delta\beta_\nu(y,x) - \widehat{j}^\mu \Delta\zeta(y,x) \right) \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \rangle_{c,\mathrm{GE}}$$
$$+ \int_{\Omega} \mathrm{d}^4 y \int_0^1 \mathrm{d}z \, \langle \widehat{O}(x), \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}} \left(\widehat{T}^{\mu\nu}(y)\partial_\mu\beta_\nu(y) - \widehat{j}^\mu \partial_\mu\zeta(y) \right) \mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}} \rangle_{c,\mathrm{GE}} \,.$$

Linear response commutes with Gauss theorem



Comparison with stress-energy tensor

$$\Delta T^{\mu\nu}(x)_{\rm diss} = \int_\Omega {\rm d}^4 y \, C_{TT}^{\mu\nu\rho\sigma}(x-y)\partial_\rho\beta_\sigma(y) \simeq \partial_\rho\beta_\sigma(x)\int_\Omega {\rm d}^4 y \, C_{TT}^{\mu\nu\rho\sigma}(x-y)$$

Encodes transport coefficients:
Kubo formula of shear viscosity

$$C_{TT}^{\mu\nu\rho\sigma}(x-y) = \int_0^1 {\rm d}z \, \langle \widehat{T}^{\mu\nu}(x), {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\widehat{T}^{\rho\sigma}(y){\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\rangle_{c,{\rm GE}}$$

Write the correlation function as a Fourier transform

$$C_{TT}^{\mu\nu\rho\sigma}(x-y) = \int {\rm d}^4 q \, {\rm e}^{{\rm i}q\cdot(x-y)}\tilde{C}_{TT}^{\mu\nu\rho\sigma}(q)$$

Constraints from energy-momentum conservation:

$$\int {\rm d}^3 y \, C_{TT}^{\mu\nu 0\sigma}(x-y) = \langle \widehat{T}^{\mu\nu}(x), \widehat{P}^\sigma\rangle_{c,{\rm GE}} = -\frac{\partial}{\partial\beta_\sigma}\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE}$$

$$(2\pi)^3\int {\rm d}q^0 \, {\rm e}^{{\rm i}q^0(x^0-y^0)} \, \tilde{C}_{TT}^{\mu\nu 0\sigma}(q^0, \mathbf{q}=0) = -\frac{\partial}{\partial\beta_\sigma}\langle \widehat{T}^{\mu\nu}(x)\rangle_{\rm GE}$$

Ward identities $\quad q_{\mu_i}\tilde{C}_{TT}^{\mu_1\mu_2\mu_3\mu_4} = 0$

Comparison with stress-energy tensor (2)

$$C^{\mu\nu}_{WT}(x-y,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4 q\, \frac{\varrho(k_+)\varrho(k_-)}{\varrho(k)}\theta(k_+^0)\theta(k_-^0)\mathrm{e}^{\mathrm{i}q\cdot(x-y)}\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\Theta^{\mu\nu}(k,q,\beta)$$

$$\Theta^{0\nu}(k,q^0,\mathbf{q}=0,\beta) = -\frac{\theta(k^0)}{(2\pi)^2\varrho(k)}\delta(q^0)\left[\frac{\partial}{\partial\beta_\nu(x)}\left(n_B(k)\varrho(k)\right)\right],$$

$$(2\pi)^3\int \mathrm{d}q^0\, \mathrm{e}^{\mathrm{i}q^0(x^0-y^0)}\,\tilde{C}^{\mu\nu 0\sigma}_{TT}(q^0,\mathbf{q}=0) = -\frac{\partial}{\partial\beta_\sigma}\langle\widehat{T}^{\mu\nu}(x)\rangle_{\mathrm{GE}}$$

→ Fourier transform of C_{WT} is much more constrained Why?

Mathematically, the above constraint on Θ stems from:

$$\varrho(k_+)\varrho(k_-)\langle\widehat{A}^\dagger(k_+)\widehat{A}(k_-),\widehat{P}^\nu\rangle_{c,\mathrm{GE}} = -\frac{\partial}{\partial\beta_\nu(x)}\left(\varrho(k_+)\varrho(k_-)\langle\widehat{A}^\dagger(k_+)\widehat{A}(k_-)\rangle_{\mathrm{GE}}\right)$$

$$\langle\widehat{A}^\dagger(p)\widehat{A}(p')\rangle_{\mathrm{GE}} = \frac{2\pi}{\varrho^2(p)}\theta(p^0)\theta(p'^0)\delta^4(p-p')n_B(p)\varrho_{\mathrm{GE}}(p)$$