The net-baryon number and Polyakov loop susceptibilities and their scaling at the deconfinement critical point

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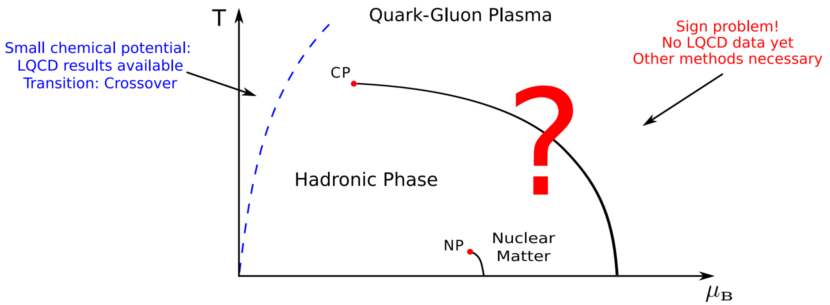


QCD phase diagram

- T, μ_B $\rightarrow$ most relevant experimentally
- Other directions possible $\rightarrow$ magnetic field, N_f, quark mass

QCD phase structure depends on quark masses

- Analytic crossover at physical quark masses at $\mu_B = 0$
- Large quark mass $\rightarrow$ Chiral symmetry not relevant
- Allows to focus on deconfinement

Pure gauge: deconfinement → 1st order

- Spontaneous Z_3 symmetry breaking
- Order parameter → Polyakov loop

$$\langle \ell \rangle = \langle \frac{1}{3} \mathrm{tr} \mathcal{P}\, e^{ig \int_0^\beta A_4 d\tau} \rangle$$

$$\langle \ell \rangle = \exp\left(-F_Q/T\right)$$

QCD: deconfinement → crossover

- Z_3 symmetry → explicitly broken
- Polyakov loop → approximate order parameter

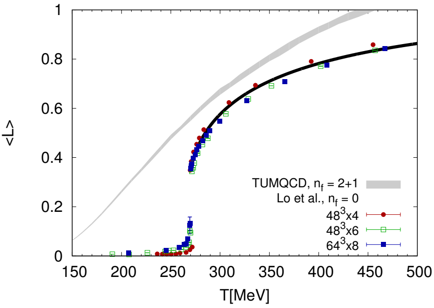
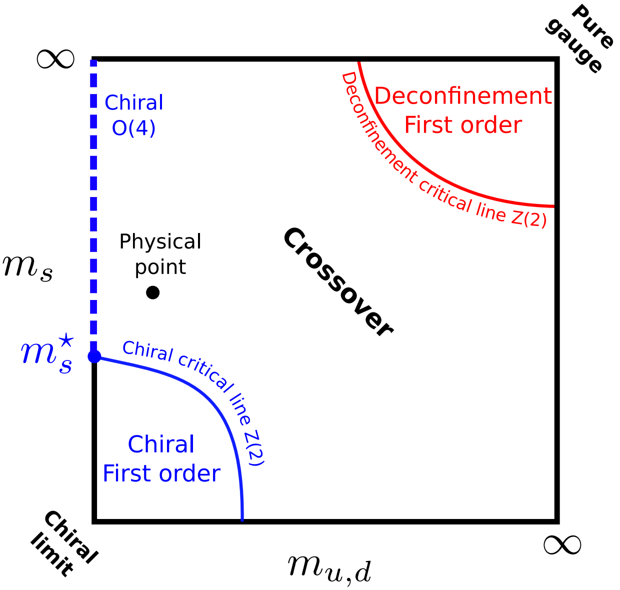


Figure: Polyakov loop in pure gauge[1] (colored points+black line) and in 2+1 QCD[2] (gray band)

Crossover → 1st order ⟹ Deconfinement critical point!

[1] *P. M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki, Phys. Rev. D* **88***, 014506 (2013)*

[2] *A. Bazavov, N. Brambilla, H.-T Ding, P. Petreczky, H. -P. Schadler, A. Vairo and J. H. Weber, Phys. Rev. D* **93***, 114502 (2016).*


Pure gauge
∞
Chiral O(4)
Deconfinement First order
Deconfinement critical line Z(2)
Physical point
Crossover
m_s
$m_s^\star$
Chiral critical line Z(2)
Chiral First order
Chiral limit
$m_{u,d}$
∞

Polyakov loop on the lattice

- Renormalization scheme dependence[1]
- Susceptibilities also affected[1]

Need for alternative probes of deconfinement!

This talk → Net-baryon number susceptibilities

$$\chi_n^B = \left. \frac{\partial^n (P/T^4)}{\partial(\mu_B/T)^n} \right|_{T=const.},$$

- Physical quark masses → Experimental searches for QCD CP
- Kurtosis → Sensitive to degrees of freedom

$$\frac{\chi_4^B}{\chi_2^B} = \begin{cases} 1, & T \ll T_c \\ \sim \frac{1}{9} & T \gg T_c \end{cases}$$

- Our work
  - Net-baryon number susceptibilities
  - Kurtosis as a probe of deconfinement of heavy quarks
  - Critical behavior of χ_n^B near the deconfinement CP

[1] *A. Bazavov, N. Brambilla, H.-T Ding, P. Petreczky, H. -P. Schadler, A. Vairo and J. H. Weber, Phys. Rev. D* **93**, *114502 (2016).*

QCD thermodynamics → Effective Polyakov loop model

$$U(\ell, \bar{\ell}) = U_G(\ell, \bar{\ell}) + U_Q(\ell, \bar{\ell}) \,,$$

Pure gluon contribution[1]

- Z_3 symmetry and its spontaneous breaking
- Captures pure $SU(3)$ EoS, Polyakov loop expectation value and its susceptibilities

Quark-gluon interaction

- Explicit Z_3 symmetry breaking
- 1-loop fermion determinant in A_4 background

Mean field approximation → Solve gap equations

$$\frac{\partial U}{\partial \ell} = 0 \qquad \frac{\partial U}{\partial \bar{\ell}} = 0$$

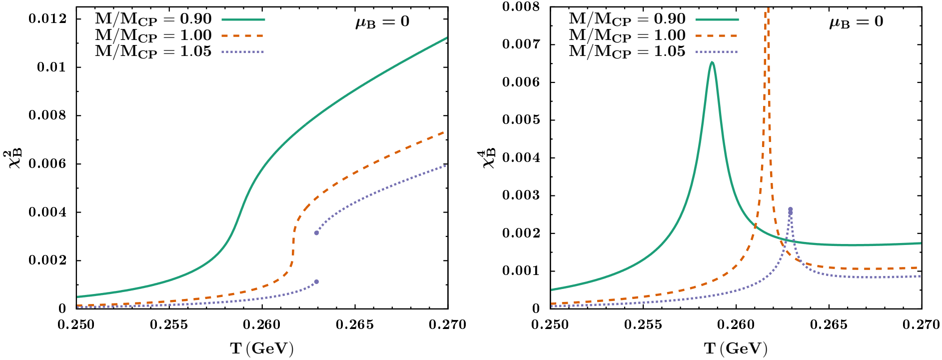
[1] *P. M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki, Phys. Rev. D* **88**, *014506 (2013)*

This talk → Net-baryon number susceptibilities

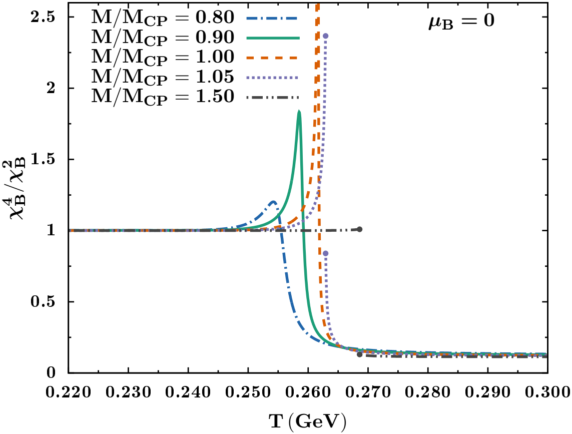
$$\chi_n^B = \left.\frac{\partial^n (P/T^4)}{\partial(\mu_B/T)^n}\right|_{T=const.},$$

Pressure → Potential evaluated with solutions of gap equations

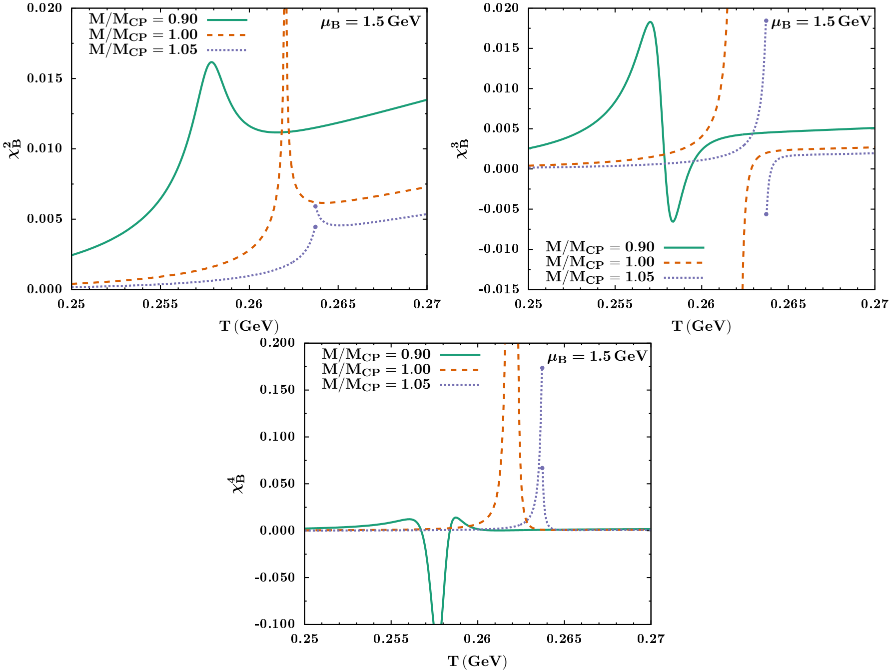
$$P = -U(T, \mu_B, \ell(T,\mu_B), \bar{\ell}(T,\mu_B))$$



$\mu_B = 0$: $M_{CP} = 1.352\,\mathrm{GeV},\ T_{CP} = 0.2617\,\mathrm{GeV}$



$\mu_B = 0$: $M_{CP} = 1.352\,\mathrm{GeV}$, $T_{CP} = 0.2617\,\mathrm{GeV}$



$\mu_B = 1.5\,\mathrm{GeV}$: $M_{CP} = 1.789\,\mathrm{GeV}$, $T_{CP} = 0.2621\,\mathrm{GeV}$

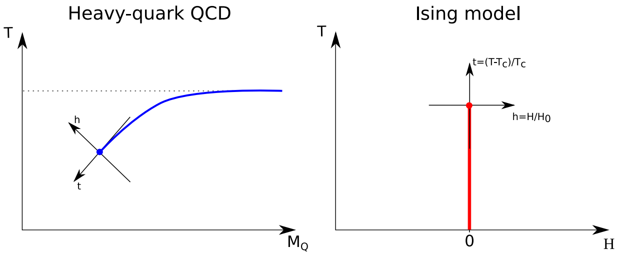
Landau-type potential

- Single critical mode: $\ell = \ell_{CP} + A\,\varphi$, $\bar{\ell} = \bar{\ell}_{CP} + \bar{A}\,\varphi$

$$\frac{U(\varphi, t, h)}{T^4} = u_0 + \frac{1}{2} u_2 t \varphi^2 + \frac{1}{4} u_4 \varphi^4 - h\varphi \,,$$

- Mapping between Ising model parameters and T, μ_B, M

$$\begin{aligned} t(T, \mu_B, M) &= A\frac{T - T_c}{T_c} + B\frac{\mu_B^2}{T_c^2} + C\frac{M - M_c}{M_c} \\ h(T, \mu_B, M) &= D\frac{T - T_c}{T_c} + E\frac{\mu_B^2}{T_c^2} + F\frac{M - M_c}{M_c} \end{aligned}$$



Order parameter and its susceptibility near the critical point

$$\varphi|_{h=0} \sim t^{\beta} \quad \varphi|_{t=0} \sim h^{1/\delta} \quad \chi|_{h=0} \sim |t|^{-\gamma}$$

Mean field critical exponents

$$\beta = \frac{1}{2}, \ \delta = 3, \ \gamma = 1$$

Critical exponents → depend on the route CP is approached[1]

Asymptotically tangential

- Stronger divergence
- Mean field

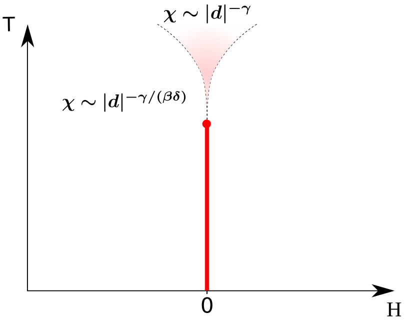
$$\varphi \sim t^{1/2} \quad \chi \sim |t|^{-1}$$

Other directions

- Mean field

$$\varphi \sim d^{1/3} \quad \chi \sim d^{-2/3}$$

- d – distance to CP



[1] R. Griffiths and J. Wheeler, Phys. Rev. A **2** (1970); Y. Hatta and T. Ikeda, Phys. Rev. D **67** (2003)

Determination of critical exponents for χ_B^n

$$\chi_n^B = \left.\frac{\partial^n (P/T^4)}{\partial(\mu_B/T)^n}\right|_{T=const.} ,$$

- Use the Landau-type potential
- Extract leading-order singularities

MF Prediction for leading-order singularities of χ_n^B:

- $\mu_B = 0$:

$$\chi_2^B \to \text{finite}\,, \quad \chi_4^B|_{crit.} \propto \chi_\varphi \,,$$

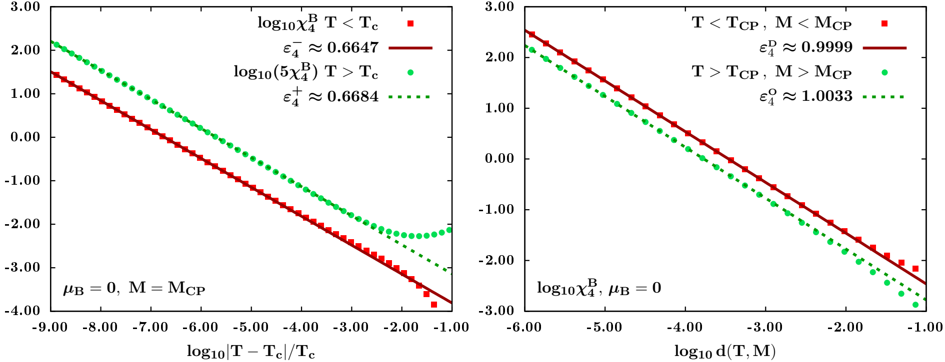
- $\mu_B > 0$:

$$\chi_2^B|_{crit.} \sim \chi_\varphi \,, \quad \chi_3^B|_{crit.} \sim \varphi\chi_\varphi^3, \quad \chi_4^B|_{crit.} \sim c_1\chi_\varphi^4$$

Numerical determination of critical exponents $\to$ linear fit

$$\log_{10} f = -\varepsilon_f \log_{10}[(T - T_{CP})/T_{CP}] + b \qquad \log_{10} f = -\varepsilon_f \log_{10} d(T, M) + b$$

$$d(T, M) = \sqrt{(T/T_{CP} - 1)^2 + (M/M_{CP} - 1)^2}$$



Determination of critical exponents for χ_B^n $\rightarrow$ utilize analytic formulas

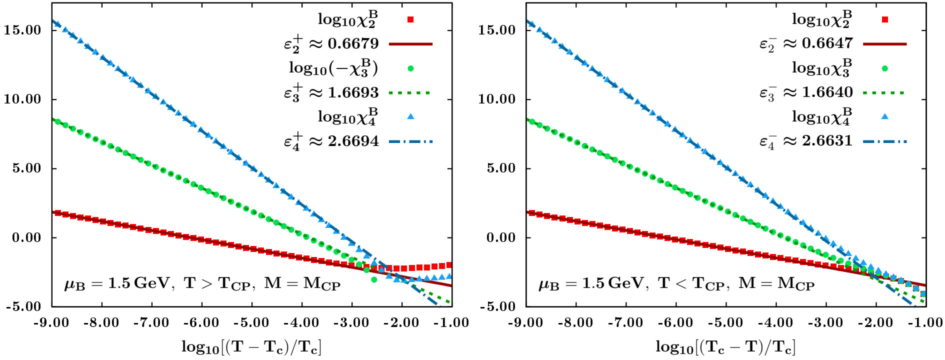
MF prediction at $\mu_B = 0$:

- Non-tangential direction ($M = M_{CP}$)

$$\chi_2^B \rightarrow \text{finite}, \qquad \chi_4^B \sim |t|^{-2/3}, \qquad t = (T - T_c)/T_c$$

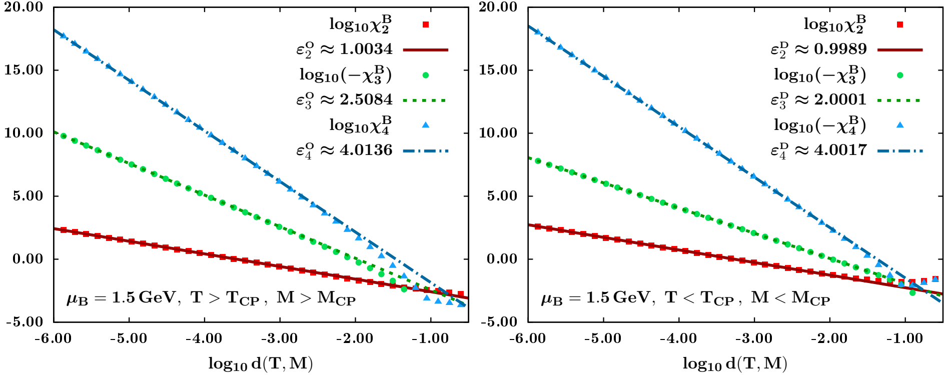
- Tangential direction

$$\chi_2^B \rightarrow \text{finite}, \qquad \chi_4^B \sim d^{-1}$$



MF predictions at $\mu_B \neq 0$, non-tangential direction ($M = M_{CP}$)

$$\chi_2^B \sim |t|^{-2/3} \qquad \chi_3^B \sim |t|^{-5/3} \qquad \chi_4^B \sim |t|^{-8/3}$$



MF predictions at $\mu_B \neq 0$, Tangential direction

$$\chi_2^B \sim |t|^{-1} \qquad \chi_3^B \sim |t|^{-5/2} \qquad \chi_4^B \sim |t|^{-4}$$

$\chi_3^B \sim \varphi\chi_\varphi^3 \to$ criticality from the subleading term in the disordered phase $(T < T_{CP},\ M < M_{CP})$

$$\chi_3^B|_{crit} \sim \chi_\varphi^2 \sim d^{-2}$$

Beyond MF approximation $\rightarrow$ scaling function approach, Z_2 universality class

$$f(T,M,\mu_B) = f_r(T,M,\mu_B) + f_s(T,M,\mu_B)\,,$$

with f_r the regular contribution, f_s – singular contribution

$$f_s = t^{2-\alpha}\psi\left(\frac{h}{t^{2-\alpha-\beta}}\right)$$

3D Ising model critical exponents: $\gamma \approx 1.2371$, $\beta \approx 0.3264$, $\alpha \approx 0.11$

Prediction for net-baryon number susceptibilities

- $\mu_B = 0$:

$$\chi_2^B \rightarrow \text{finite}\,, \quad \chi_4^B \sim \gamma\,,$$

- $\mu_B > 0$:

$$\chi_2^B \sim \gamma\,, \quad \chi_3^B \sim 2\gamma + \beta \approx 2.8006, \quad \chi_4 \sim 3\gamma + 2\beta \approx 4.3641$$

Conclusions

1. Net-baryon number susceptibilities $\rightarrow$ Alternative observables to study deconfinement CP
   - Sensitive probes of deconfinement also for heavy quarks
   - Kurtosis $\rightarrow$ step function as $M \rightarrow \infty$
   - Scaling properties near CP

2. Future prospects
   - Finite volume scaling
   - Going beyond the mean-field approximation

Thank You!

Appendix

Effective Polyakov loop model

$$U(\ell,\bar{\ell}) = U_G(\ell,\bar{\ell}) + U_Q(\ell,\bar{\ell}),$$

Pure gluon contribution[1]

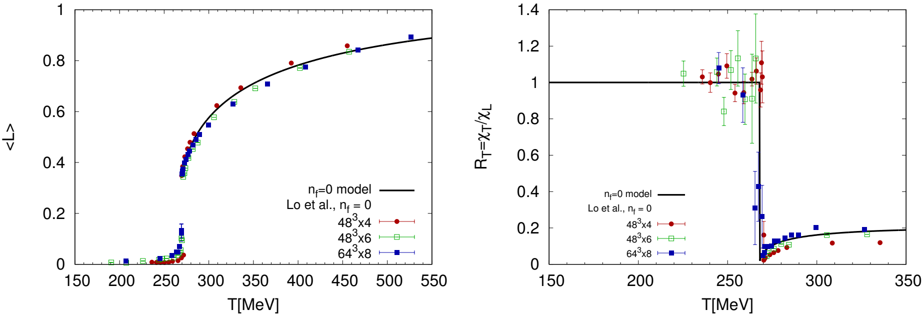
$$\begin{aligned}\frac{U_G(\ell,\bar{\ell})}{T^4} =& -\frac{1}{2}A(t)\ell\bar{\ell} + B(t)\ln(1 - 6\ell\bar{\ell} + 4(\ell^3 + \bar{\ell}^3) - 3(\ell\bar{\ell})^2) \\ &+ \frac{1}{2}C(t)(\ell^3 + \bar{\ell}^3) + D(t)(\ell\bar{\ell})^2,\end{aligned}$$

Coefficients matched to pure gauge equation of state, Polyakov loop expectation value and susceptibilities of real and imaginary parts

Quark-gluon interaction $\rightarrow$ 1-loop fermion determinant in A_4 background

$$\begin{aligned}U_Q = -2TN_f \int \frac{d^3q}{(2\pi)^3}\big[&\ln(1 + 3\ell e^{-\beta(E-\mu)} + 3\bar{\ell}e^{-2\beta(E-\mu)} + e^{-3\beta(E-\mu)}) + \\ &+ \ln(1 + 3\bar{\ell}e^{-\beta(E+\mu)} + 3\ell e^{-2\beta(E+\mu)} + e^{-3\beta(E+\mu)})\big]\end{aligned}$$

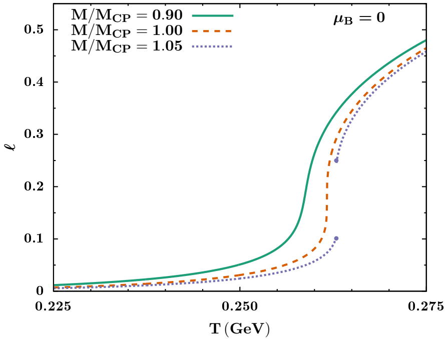
[1] *P. M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki, Phys. Rev. D* **88**, *014506 (2013)*



- Lattice → constraints on Polyakov loop susceptibilities
- Important for accurate description of fluctuations

Effect of dynamical quarks

- Break center symmetry explicitly
- Deconfinement CP
- $\mu_B \rightarrow$ enhances explicit Z_3 breaking strength
- Critical quark mass increases with μ_B



Present model ($N_f = 2$, degenerate quark mass):

- $\mu_B = 0$: $M_{CP} = 1.352\,\mathrm{GeV}$, $T_{CP} = 0.2617\,\mathrm{GeV}$
- $\mu_B = 1.5\,\mathrm{GeV}$: $M_{CP} = 1.789\,\mathrm{GeV}$, $T_{CP} = 0.2621\,\mathrm{GeV}$

Mean-field approach $\rightarrow$ gap equations

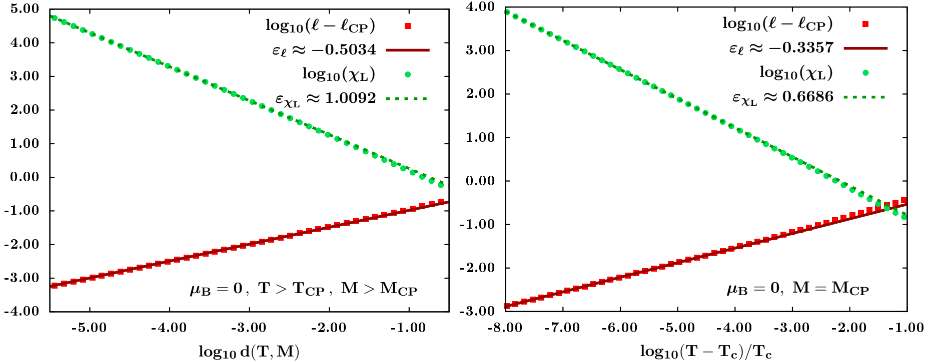
$$\frac{\partial U}{\partial \ell} = 0 \qquad \frac{\partial U}{\partial \bar{\ell}} = 0$$

Polyakov loop susceptibilities $\rightarrow$ inverse curvatures

$$\hat{\chi} = \begin{pmatrix} \dfrac{\partial^2 U}{\partial \ell \partial \bar{\ell}} & \dfrac{\partial^2 U}{\partial \ell \partial \bar{\ell}} \\ \dfrac{\partial^2 U}{\partial \ell \partial \bar{\ell}} & \dfrac{\partial^2 U}{\partial \ell \partial \bar{\ell}} \end{pmatrix}^{-1}$$

Relation to susceptibilities of real (χ_L) and imaginary parts (χ_T)

$$\chi_{L/T} = \frac{1}{2}\left[\chi_{\ell\bar{\ell}} \pm \frac{1}{2}\left(\chi_{\ell\ell} + \chi_{\bar{\ell}\bar{\ell}}\right)\right]$$



Numerical determination of critical exponents → linear fit

$$\log_{10} f = -\varepsilon_f \log_{10}[(T-T_{CP})/T_{CP}] + b \qquad \log_{10} f = -\varepsilon_f \log_{10} d(T,M) + b$$

Distance to CP: $d(T,M) = \sqrt{(T/T_{CP}-1)^2 + (M/M_{CP}-1)^2}$

Tangent line → $T = \alpha_{CP}(M - M_{CP}) + T_{CP}$

Results consistent with MF prediction

- $\chi \sim d^{-1}$, $\varphi \sim d^{1/2}$, $\chi \sim |t|^{-2/3}$, $\varphi \sim |t|^{1/3}$, Non-trivial connection between net-baryon and Polyakov loop susceptibilities

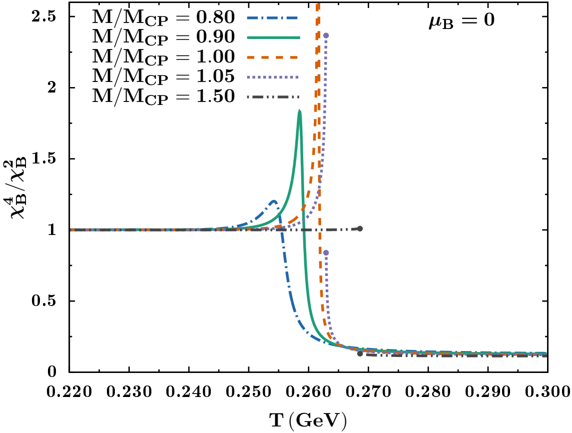
$$
\begin{aligned}
\chi_1^B &= p^{(1)}\,, \\
\chi_2^B &= p^{(2)} + \frac{\partial p^{(1)}}{\partial \phi_i}\hat{\chi}_{ij}\frac{\partial p^{(1)}}{\partial \phi_j}\,, \\
\chi_3^B &= \frac{\partial^3 p}{\partial\phi_k\partial\phi_l\partial\phi_m}\hat{\chi}_{ki}\hat{\chi}_{lj}\hat{\chi}_{mn}\frac{\partial p^{(1)}}{\partial \phi_i}\frac{\partial p^{(1)}}{\partial \phi_j}\frac{\partial p^{(1)}}{\partial \phi_m} + 3\frac{\partial^2 p^{(1)}}{\partial\phi_i\partial\phi_k}\hat{\chi}_{ij}\hat{\chi}_{kl}\frac{\partial p^{(1)}}{\partial \phi_j}\frac{\partial p^{(1)}}{\partial \phi_l}\ldots \\
\chi_4^B &= \frac{\partial^4 p}{\partial\phi_i\partial\phi_j\partial\phi_k\partial\phi_l}\hat{\chi}_{ip}\hat{\chi}_{jq}\hat{\chi}_{kr}\hat{\chi}_{ls}\frac{\partial p^{(1)}}{\partial \phi_p}\frac{\partial p^{(1)}}{\partial \phi_q}\frac{\partial p^{(1)}}{\partial \phi_r}\frac{\partial p^{(1)}}{\partial \phi_s} \\
&+ 12\frac{\partial^2 p^{(1)}}{\partial\phi_i\partial\phi_k}\hat{\chi}_{kl}\frac{\partial^3 p}{\partial\phi_l\partial\phi_m\partial\phi_r}\hat{\chi}_{ij}\hat{\chi}_{mn}\hat{\chi}_{rs}\frac{\partial p^{(1)}}{\partial \phi_j}\frac{\partial p^{(1)}}{\partial \phi_n}\frac{\partial p^{(1)}}{\partial \phi_s} \\
&+ 3\frac{\partial^3 p}{\partial\phi_k\partial\phi_l\partial\phi_m}\hat{\chi}_{mp}\frac{\partial^3 p}{\partial\phi_p\partial\phi_q\partial\phi_r}\hat{\chi}_{ki}\hat{\chi}_{lj}\hat{\chi}_{qn}\hat{\chi}_{rs}\frac{\partial p^{(1)}}{\partial \phi_i}\frac{\partial p^{(1)}}{\partial \phi_j}\frac{\partial p^{(1)}}{\partial \phi_n}\frac{\partial p^{(1)}}{\partial \phi_s} + \ldots
\end{aligned}
$$

with $\phi = (\ell, \bar{\ell})$, $p^{(n)} = \left.\frac{\partial^n P/T^4}{\partial(\mu_B/T)^n}\right|_{\ell=\bar{\ell}=const.}$, $\hat{\chi} = \begin{pmatrix} \frac{\partial^2 U/T^4}{\partial\ell^2} & \frac{\partial^2 U/T^4}{\partial\ell\partial\bar{\ell}} \\ \frac{\partial^2 U/T^4}{\partial\ell\partial\bar{\ell}} & \frac{\partial^2 U/T^4}{\partial\bar{\ell}^2} \end{pmatrix}^{-1}$

Explicit expressions for the Landau-type potential

$$
\begin{aligned}
p^{(1)} &= -\frac{u_2 B \mu_B}{T_c^2}\varphi^2 + \frac{2E\mu_B}{T_c^2}\varphi\,, \\
p^{(2)} &= -\frac{u_2 B}{T_c^2}\varphi^2 + \frac{2E}{T_c^2}\varphi\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial p^{(1)}}{\partial \varphi} &= -\frac{2u_2 B \mu_B}{T_c^2}\varphi + \frac{2E\mu_B}{T_c^2}\,, \\
\frac{\partial^2 p^{(1)}}{\partial \varphi^2} &= -\frac{2u_2 B \mu_B}{T_c^2} \\
\frac{\partial p^{(2)}}{\partial \varphi} &= -\frac{2u_2 B}{T_c^2}\varphi + \frac{2E}{T_c^2} \\
\frac{\partial^2 p^{(2)}}{\partial \varphi^2} &= -\frac{2u_2 B}{T_c^2} \\
\frac{\partial^3 p}{\partial \varphi^3} &= -6u_4\varphi \\
\frac{\partial^4 p}{\partial \varphi^4} &= -6u_4\,.
\end{aligned}
$$



Polyakov loop → Statistical confinement

$$f(E,\ell,\bar{\ell},\mu) = \begin{cases} \dfrac{1}{1+e^{3\beta(E-\mu_B/3)}} \xrightarrow{M/T\gg 1} e^{-3\beta\mu}\,, & \ell=\bar{\ell}=0\,, \quad \text{baryon-like} \\ \dfrac{1}{1+e^{\beta(E-\mu_B/3)}} \xrightarrow{M/T\gg 1} e^{-\beta\mu_B/3}\,, & \ell=\bar{\ell}=1\,, \quad \text{quark-like} \end{cases}$$