Lattice Flavourdynamics - 2025

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University of Southampton

Fundamental Interactions,
65 Years of the Cracow School
Zakopane, Poland
June 14th - 21st 2025

Outline

- I have two main aims in presenting these 2 lectures:
    1. To explain some of the theoretical issues in preparing lattice computations of hadronic effects $\Rightarrow$ to provide an understanding of which physical quantities can be computed and which cannot (yet?).
    2. To present some recent physics results, whose calculation requires new ideas, and future prospects for the computation of other phenomenologically important quantities.

Lecture 1

1. Introduction to the theoretical elements of lattice flavourdynamics from Euclidean correlation functions

   (This will form the core of the first lecture)

2. $K \to \pi\pi$ decays
3. The radiative decays $P \to \ell\bar{\nu}\gamma$

- Lecture 2 will be dedicated to the $b \to s$ FCNC decays $B \to K\ell^+\ell^-$ and $B_s \to \gamma\ell^+\ell^-$

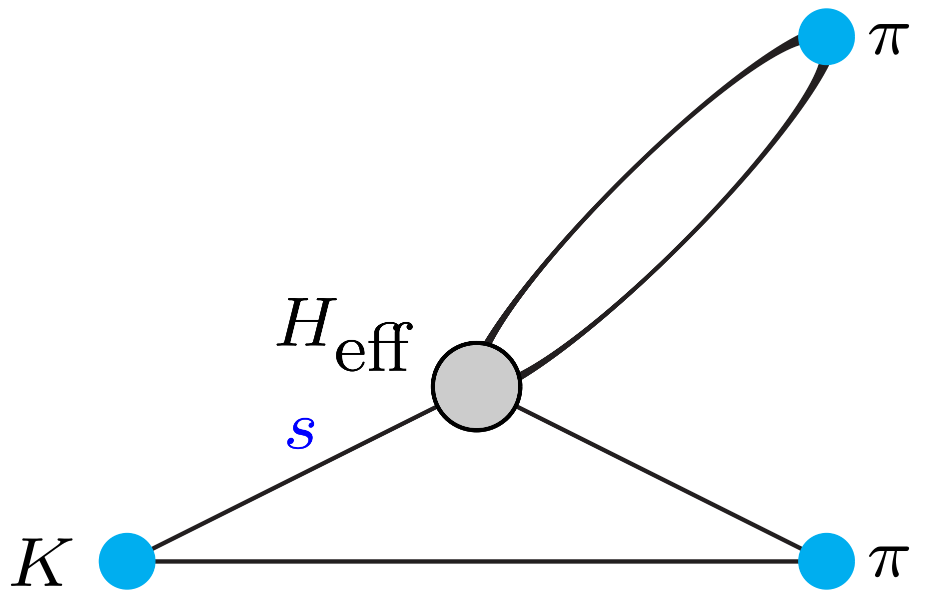
  (New theoretical issue is the Minkowski $\to$ Euclidean continuation)

1. Introduction - Precision Flavour Physics

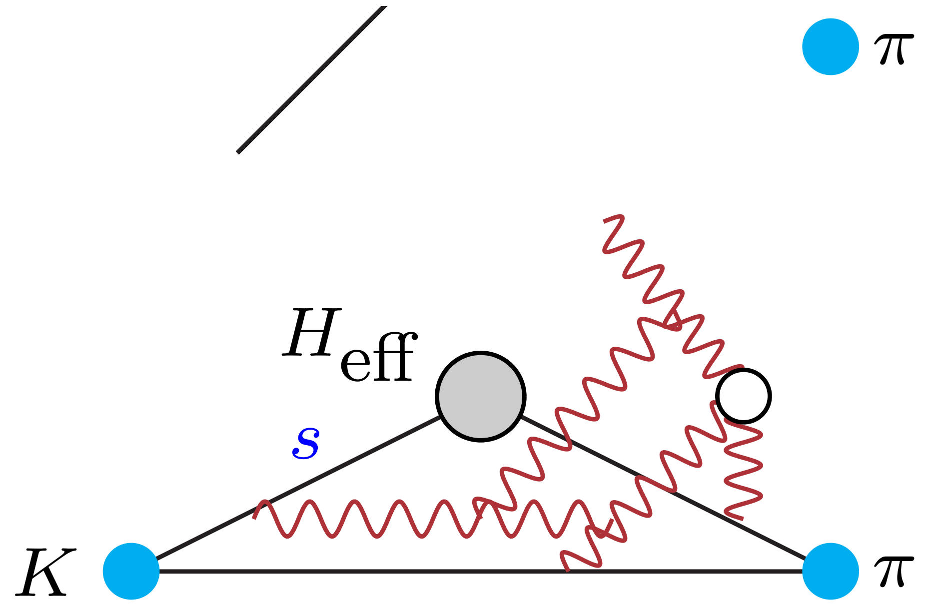
- Precision Flavour Physics, is a key approach, complementary to the large E_T searches at the LHC, in exploring the limits of the standard model and in searches for New Physics.
    - If the LHC experiments discover new elementary particles BSM, then precision flavour physics will be necessary to unravel the underlying framework.
    - The discovery potential of precision flavour physics would also not be underestimated. (In principle, the reach may be about two orders of magnitude deeper than the LHC!)

- Precision Flavour Physics requires control of hadronic effects for which Lattice QCD computations are essential.

- For illustration - a schematic diagram of $K \to \pi\pi$ decays:



means



Lattice Flavour Physics

- It is a pleasure to acknowledge the continuing collaboration with Guido Martinelli and colleagues in Rome123, which started in 1986 and which currently counts 88 joint publications.

- It is also a pleasure to acknowledge my continuing collaboration with the RBC/UKQCD collaborations on a number of the topics discussed in this talk.

THE KAON B-PARAMETER AND K–π AND K–$\pi\pi$ TRANSITION AMPLITUDES ON THE LATTICE

M.B. GAVELA[1]

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Received 22 January 1988

R_K and R_{K^*}

- Two of the most interesting recent tensions between theoretical predictions and experimental results have been in $R_{K^{(*)}}$ and in $(g-2)_\mu$.

$$R_H = \frac{\int dq^2 \, \frac{d\Gamma(B \to H\mu^+\mu^-)}{dq^2}}{\int dq^2 \, \frac{d\Gamma(B \to He^+e^-)}{dq^2}} \qquad \text{where } H = K, K^* \text{ and } q^2 = (p_{\ell^+} + p_{\ell^-})^2.$$

- LHCb Results:

$R_{K^+} = 0.745^{+0.090}_{-0.074} \pm 0.036$ for $1 < q^2 < 6\,\mathrm{GeV}^2$ arXiv:1406.6482

$R_{K^{*0}} = 0.66^{+0.11}_{-0.07} \pm 0.03$ for $0.045 < q^2 < 1.1\,\mathrm{GeV}^2$ arXiv:1705.05802

$R_{K^{*0}} = 0.69^{+0.11}_{-0.07} \pm 0.05$ for $1.1 < q^2 < 6\,\mathrm{GeV}^2$ arXiv:1705.05802

- For R_K and R_{K^*} hadronic uncertainties play almost no role.
- For R_K and the higher q^2 bin for R_{K^*}, the theoretical prediction is 1 to within 1 % or so.
- For the lower q^2 bin, $R_{K^*} = 0.906 \pm 0.028$, M.Bordone, G.Isidori, A.Pattori, arXiv:1605.07633
- Suggestion of violation of lepton flavour universality ⇒ 1000's of BSM papers with speculative explanations.

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- LHCb New Results: arXiv:2212.09153

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- Evidence for the violation of Lepton Flavour Universality has disappeared!

- A number of tensions in individual rates and angular distributions persist which need hadronic input $\Rightarrow$ Lattice computations (see later).
    - Tensions also exist in other processes such as $B \to K\nu\bar{\nu}$. Belle II, arXiv:2311.14647

$(g-2)_\mu$

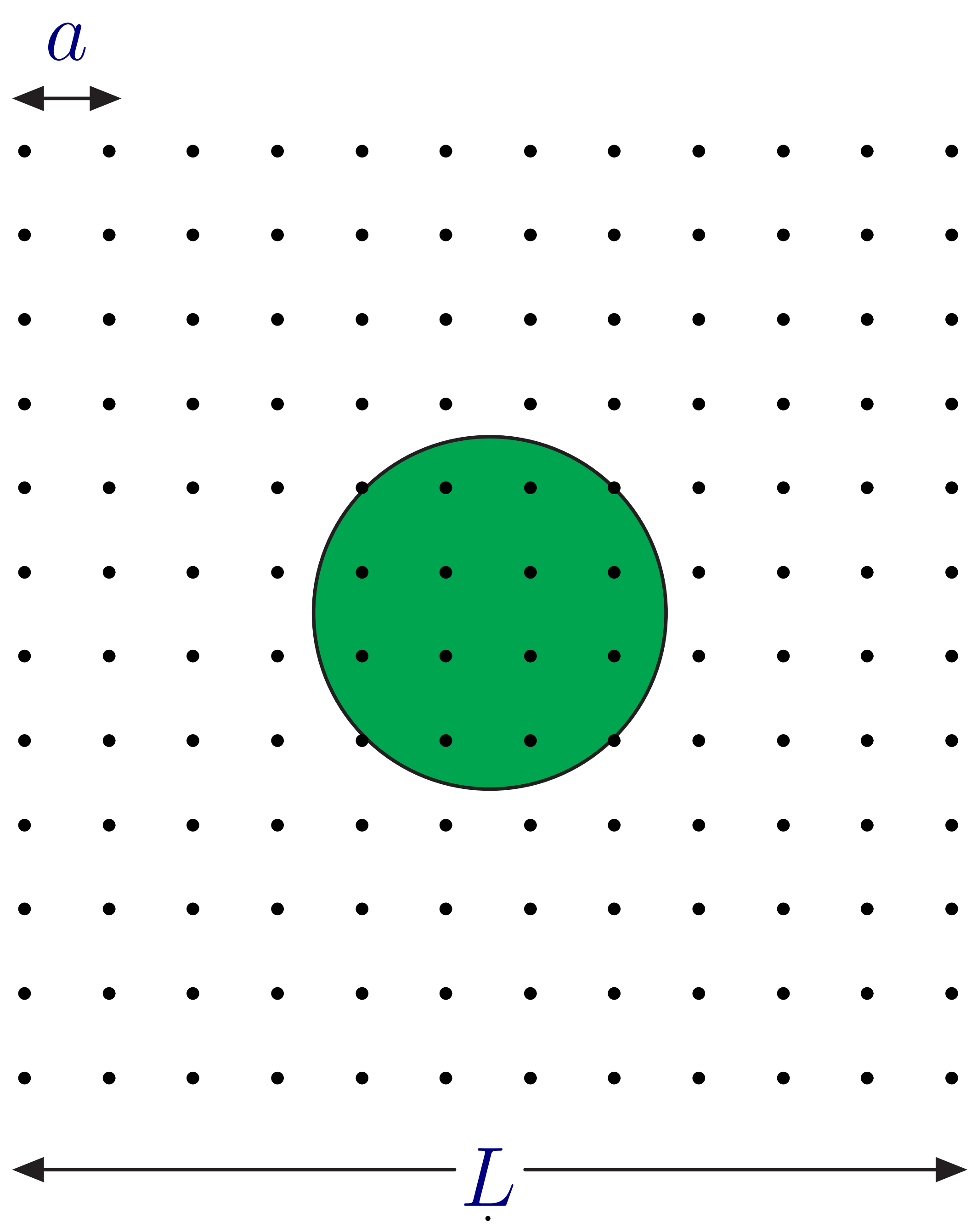
- The long standing tension between theory and experiment in $(g-2)_\mu$ also appears to be disappearing:

Abstract

We present the current Standard Model (SM) prediction for the muon anomalous magnetic moment, a_μ, updating the first White Paper (WP20) [1]. The pure QED and electroweak contributions have been further consolidated, while hadronic contributions continue to be responsible for the bulk of the uncertainty of the SM prediction. Significant progress has been achieved in the hadronic light-by-light scattering contribution using both the data-driven dispersive approach as well as lattice-QCD calculations, leading to a reduction of the uncertainty by almost a factor of two. The most important development since WP20 is the change in the estimate of the leading-order hadronic-vacuum-polarization (LO HVP) contribution. A new measurement of the $e^+e^- \to \pi^+\pi^-$ cross section by CMD-3 has increased the tensions among data-driven dispersive evaluations of the LO HVP contribution to a level that makes it impossible to combine the results in a meaningful way. At the same time, the attainable precision of lattice-QCD calculations has increased substantially and allows for a consolidated lattice-QCD average of the LO HVP contribution with a precision of about 0.9%. Adopting the latter in this update has resulted in a major upward shift of the total SM prediction, which now reads $a_\mu^{\rm SM} = 116\,592\,033(62) \times 10^{-11}$ (530 ppb). When compared against the current experimental average based on the E821 experiment and runs 1–3 of E989 at Fermilab, one finds $a_\mu^{\rm exp} - a_\mu^{\rm SM} = 26(66) \times 10^{-11}$, which implies that there is no tension between the SM and experiment at the current level of precision. The final precision of E989 is expected to be around 140 ppb, which is the target of future efforts by the Theory Initiative. The resolution of the tensions among data-driven dispersive evaluations of the LO HVP contribution will be a key element in this endeavor.

"The anomalous magnetic moment of the muon in the Standard Model: an update", white paper, arXiv:2505 .21476

Introduction (Cont.)



- Lattice QCD is a general *first-principles* technique used to compute non-perturbative QCD effects in a huge variety of applications.

- In principle the systematic errors are controllable, and can be progressively reduced.

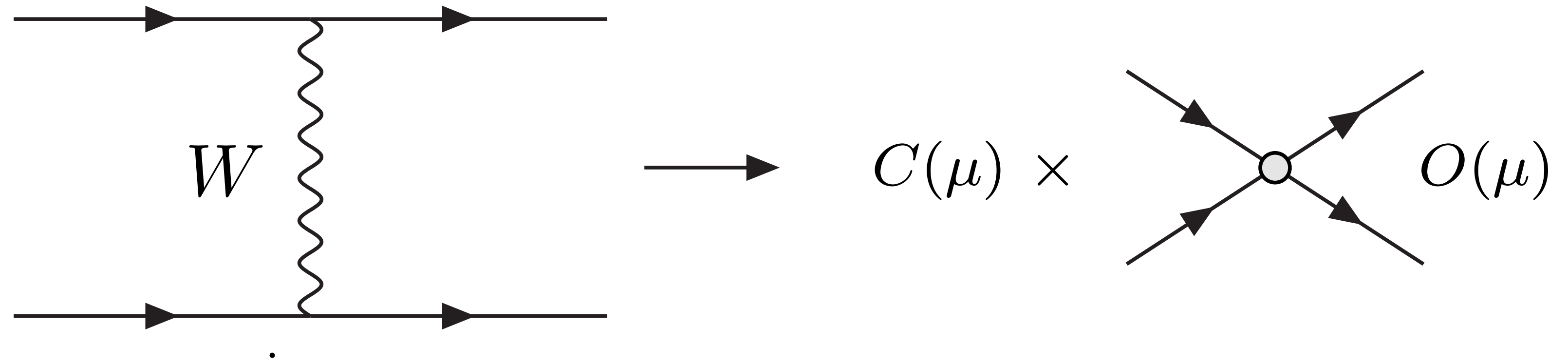
i) Continuum extrapolation $a \to 0$.

ii) Extrapolation to infinite-volume $L \to \infty$.

iii) Minkowski → Euclidean continuation.

- For some simple quantities in spectroscopy and flavour physics, the M→E continuation is not an issue, the discretisation and finite-volume effects are under control and results can be obtained with a precision at the sub-percent level.

- The lattice spacing a (typically $0.05 - 0.1$ fm) is far too large to allow for propagating W,Z - bosons ⇒ use the Operator Product Expansion.



$C(\mu)$ - perturbative

Matrix element of $O(\mu)$ non-perturbative

Some Recent Results from FLAG

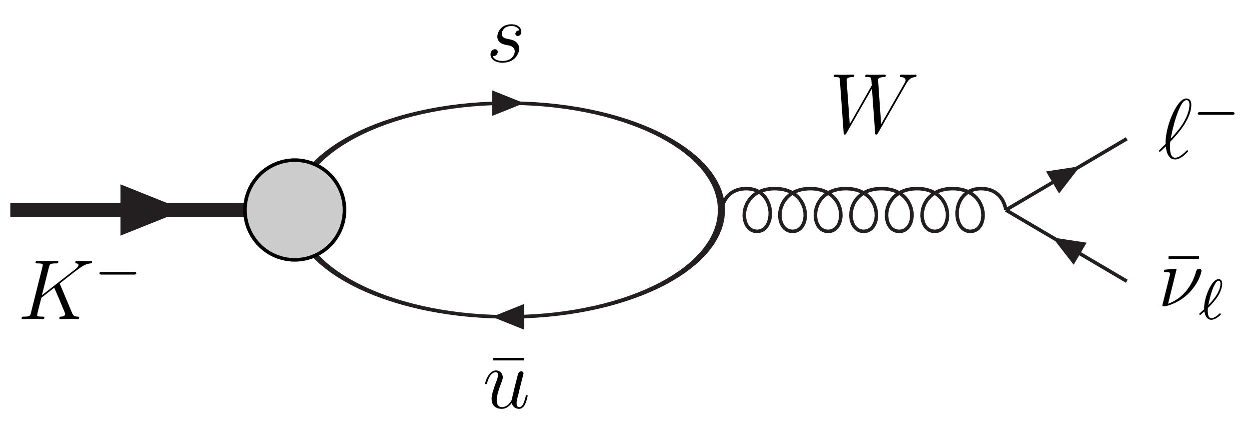
Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
m_{ud}[MeV]	4.1.1	3.427(51)	[7–9]	3.387(39)	[10–16]		
m_s[MeV]	4.1.1	93.46(58)	[7–9, 17, 18]	92.4(1.0)	[11–15, 19]		
m_s/m_{ud}	4.1.2	27.227(81)	[7, 8, 20, 21]	27.42(12)	[12–14, 19, 22]		
m_u[MeV]	4.1.3	2.14(8)	[9, 23]	2.27(9)	[24]		
m_d[MeV]	4.1.3	4.70(5)	[9, 23]	4.67(9)	[24]		
m_u/m_d	4.1.3	0.465(24)	[23, 25]	0.485(19)	[24]		
$\overline{m}_c$(3 GeV)[GeV]	4.2.2	0.989(10)	[7–9, 18, 26, 27]	0.991(6)	[15, 28–32]		
m_c/m_s	4.2.3	11.766(30)	[7–9, 18]	11.82(16)	[29, 33]		
$\overline{m}_b(\overline{m}_b)$[GeV]	4.3	4.200(14)	[9, 34–37]	4.171(20)	[15]		
$f_+(0)$	5.3	0.9698(17)	[38, 39]	0.9677(27)	[40, 41]		
$f_{K^\pm}/f_{\pi^\pm}$	5.3	1.1934(19)	[20, 42–45]	1.1917(37)	[12, 46–50]		
$f_{\pi^\pm}$[MeV]	5.6			130.2(8)	[12, 46, 47]		
$f_{K^\pm}$[MeV]	5.6	155.7(3)	[21, 42, 43]	155.7(7)	[12, 46, 47]		
Re(A_2)[GeV]	6.2			$1.50(4)(14) \times 10^{-8}$	[51]		
Im(A_2)[GeV]	6.2			$-8.34(1.03) \times 10^{-13}$	[51]		
$\hat{B}_K$	6.3	0.717(18)(16)	[52]	0.7533(91)	[12, 53–56]	0.727(22)(12)	[57]
B_2	6.4	0.46(1)(3)	[52]	0.488(15)	[55, 56, 58]	0.47(2)(1)	[57]
B_3	6.4	0.79(6)	[52]	0.757(27)	[55, 56, 58]	0.78(4)(2)	[57]
B_4	6.4	0.78(2)(4)	[52]	0.903(14)	[55, 56, 58]	0.76(2)(2)	[57]
B_5	6.4	0.49(3)(3)	[52]	0.691(14)	[55, 56, 58]	0.58(2)(2)	[57]

FLAG Review 2024, Y.Aoki et al., arXiv:2411.04268

- Lattice QCD results for some physical quantities are now so precise (sub percent) that QED corrections need to be included to make further progress.

- A good example is:

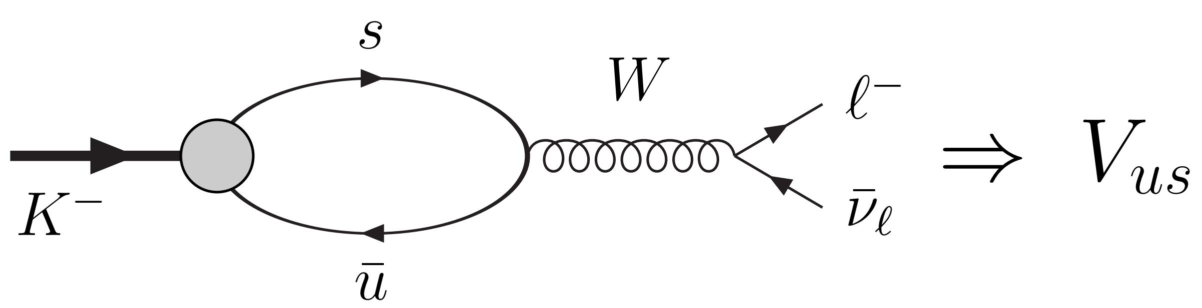
$$f_K = 155.7(3)\ \text{MeV}$$



$$\langle 0|A_\mu|K(p)\rangle = f_K p_\mu\,, \qquad \Gamma^{(0)} = \frac{G_F^2 |V_{us}|^2 f_K^2}{8\pi} m_K^3 r_\ell^2 \left(1 - r_\ell^2\right)^2 \qquad r_\ell = m_\ell/m_K$$

Well-studied quantities in lattice kaon physics

1. Leptonic decay constant f_K



$$\langle 0|A_\mu|K(p)\rangle = f_K p_\mu \,,$$

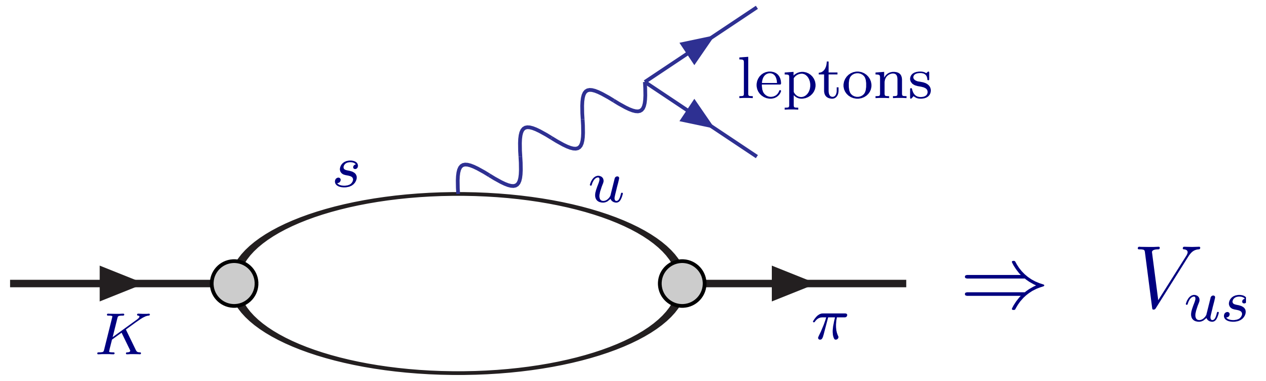
$$\Gamma^{(0)} = \frac{G_F^2|V_{us}|^2 f_K^2}{8\pi} m_K^3 r_\ell^2 \left(1 - r_\ell^2\right)^2$$

$$r_\ell = \frac{m_\ell}{m_K}$$

$f_K = 155.7(3)$ MeV

FLAG Review 2024, Y.Aoki et al., arXiv:2411.04268

2. $K_{\ell 3}$ decays



$$\langle \pi(p_\pi)|\bar{s}\gamma_\mu u|K(p_K)\rangle = f_0(q^2)\frac{m_K^2 - m_\pi^2}{q^2} q_\mu$$

$$+ f_+(q^2)\left[(p_\pi + p_K)_\mu - \frac{m_K^2 - m_\pi^2}{q^2} q_\mu\right]$$

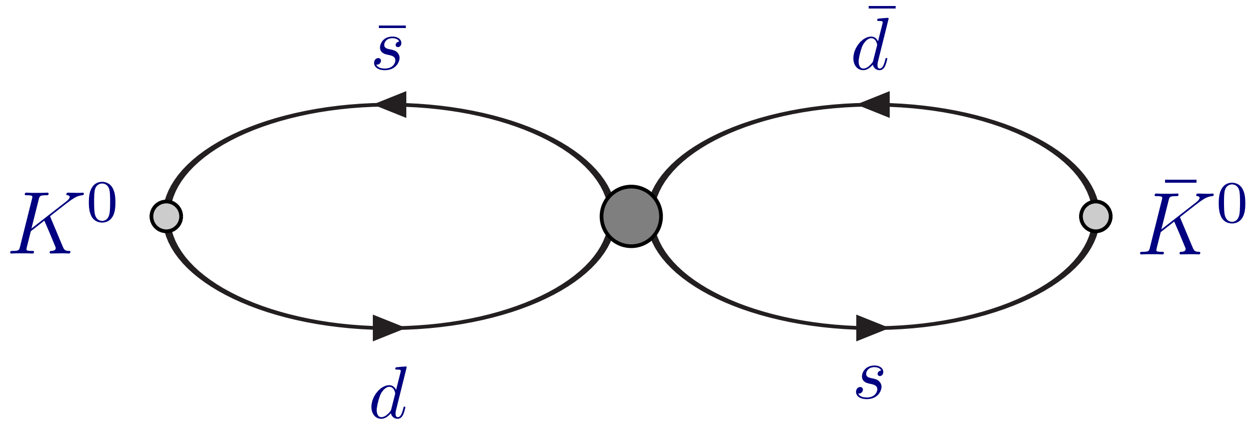
where $q = p_K - p_\pi$.

$f_0(0) = 0.9698(17)$

- Shape of form factor also computed.

FLAG Review 2024, Y.Aoki et al., arXiv:2411.04268 from ETM (arXiv:1602.04113) and FNAL/MILC (arXiv:1809.02827) collaborations.

3. K^0-$\bar{K}^0$ mixing



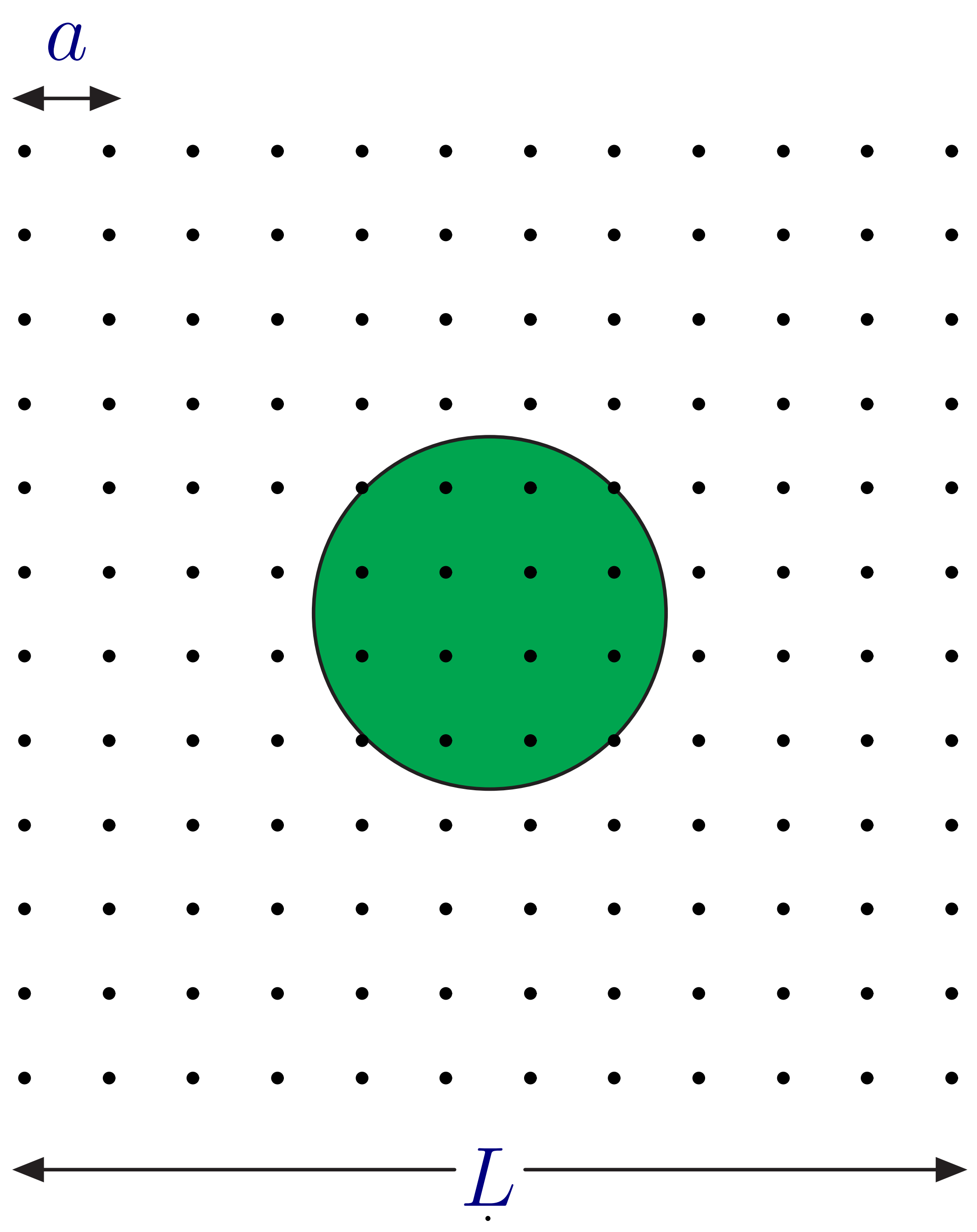
$$\langle \bar{K}^0|\bar{s}\gamma^\mu(1-\gamma^5)d\;\bar{s}\gamma_\mu(1-\gamma^5)d|K^0\rangle = \frac{8}{3} f_K^2 m_K^2 B_K(\mu)$$

$$\hat{B}_K \equiv \alpha_s(\mu)^{-\gamma_0/2\beta_0}\left(1 + O(\alpha_s(\mu))\right) B_K(\mu)$$

$\hat{B}_K = 0.717(18)(16)$

FLAG Review 2021, Y.Aoki et al., arXiv:2111.09849 from ETM (arXiv:1505.06639) collaboration.

Correlation Functions



- Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$< 0 \,|\, O(x_1, x_2, \cdots, x_n) \,|\, 0 > = \frac{1}{Z} \int [dA_\mu]\,[d\psi]\,[d\bar{\psi}]\, e^{-S}\, O(x_1, x_2, \cdots, x_n)$$

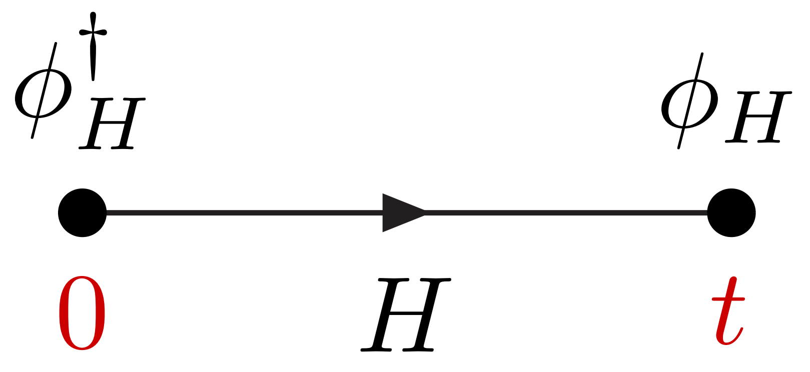
where $O(x_1, x_2, \cdots, x_n)$ is a multifocal operator composed of quark and gluon fields and Z is the partition function.

- The physics which can be studied depends on the multilocal operator $O(x_1, x_2, \cdots, x_n)$.

- The functional integral is performed by discretising Euclidean space-time and using Montecarlo integration.

- For the purposes of these lecture, I assume that a suitable discretisation has been chosen and that the correlation functions can be computed.

Correlation functions (cont.)



$$C_2(t) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}}\, \langle 0|\phi_H(t,\vec{x}\,)\,\phi_H^\dagger(0)\,|0\rangle \;= \sum_n \int d^3x\; e^{i\vec{p}\cdot\vec{x}} \langle 0|\phi_H(t,\vec{x}\,)|n\rangle\,\langle n|\phi_H^\dagger(0)|0\rangle$$

$$= \frac{1}{2E}\; e^{-iEt}\langle 0|\phi_H(0)|H(\vec{p}\,)\rangle\,\langle H(\vec{p}\,)|\phi_H^\dagger(0)|0\rangle + \cdots \qquad \left(E = \sqrt{m_H^2 + \vec{p}^{\,2}}\,\right)$$

$$\Rightarrow \frac{1}{2E}\; e^{-Et}\langle 0|\phi_H(0)|H(\vec{p}\,)\rangle\,\langle H(\vec{p}\,)|\phi_H^\dagger(0)|0\rangle + \cdots \qquad \text{in Euclidean space}$$

- H is the lightest state which can be created by $\phi_H^\dagger$ and the ellipses represent the contributions from heavier states.
- By fitting $C_2(t)$ as a function of t, both E (or m_H if $\vec{p} = 0$) and $|<0|\phi_H(0)|H>|$ can be determined.
- For example: if $\phi_H = \bar{s}\gamma^\mu\gamma^5 u$, then f_K can be evaluated, $|\langle 0|\bar{s}\gamma^\mu\gamma^5 u|K^+\rangle| = f_K p^\mu$.

Calibration

- $N_f = 2+1+1$ isosymmetric QCD has four parameters, the three independent quark masses, $m_u = m_d, m_s, m_c$, and the coupling constant g.
    - 4 predictions therefore have to be sacrificed to determine the "physical" values of these parameters.
    - For the coupling constant we use "dimensional transmutation" to trade $g(a)$ for the lattice spacing a.
    - For example, one might require that $m_{\pi^0}, m_{K^0}, m_{D^+}$ and m_Ω take their physical values.
        - Of course, we only know how well we have done a posteriori.
        - We now have considerable experience to be able to tune the parameters with good precision.
- Once these parameters have been tuned, the evaluation of all other quantities constitute "predictions".
- The quark masses determined in this way, are "bare" masses corresponding to the lattice discretisation being used. They need to be renormalised (see below).
- Given the precision we have seen in the FLAG table, the evaluation of isospin-breaking corrections, including electromagnetic contributions, is currently an important area of investigation.

Renormalisation

- $M_W \simeq 80\,\mathrm{GeV} \Rightarrow$ we rely on the operator product expansion. Schematically:

$$H_{\mathrm{eff}} = \sum_i C_i(M_W^2/\mu^2)\, O_i(\mu^2/p^2)$$

  - μ is the renormalisation scale and p represents various hadronic (non-perturbative) scales.
  - The Wilson coefficients are calculated in perturbation theory, generally in the $\overline{\mathrm{MS}}$ scheme (or similar), based on the dimensional regularisation of the ultra-violet divergences. The matrix elements of the operators O_i should therefore be evaluated in the same scheme.
  - However, the $\overline{\mathrm{MS}}$-scheme is only defined in perturbation theory and so the matrix elements of the O_i are not defined non-perturbatively.

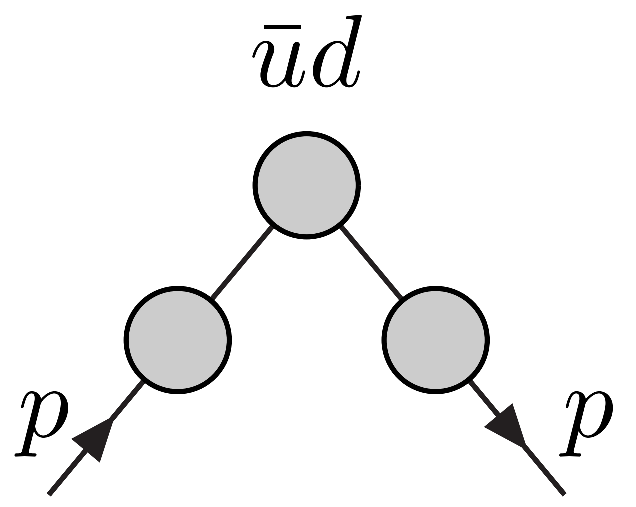
- Steps in the procedure:

  1) Compute the matrix elements non-perturbatively in the lattice regularisation, $\langle f\,|\,O_i^{\mathrm{Lat}}(a)\,|\,i\rangle$.
  2) Non-perturbatively compute the renormalisation constants relating the bare lattice operators with those in a scheme which can be defined non-perturbatively, $O_i^{\mathrm{NP}}(\mu) = Z_{ij}^{\mathrm{Lat}\to\mathrm{NP}}(a^2\mu^2)\, O_j^{\mathrm{Lat}}(a)$.
  3) Match $O_i^{\mathrm{NP}}(\mu)$ to $O_i^{\overline{\mathrm{MS}}}(\mu)$ using perturbation theory, $O_i^{\overline{\mathrm{MS}}}(\mu) = Z_{ij}^{\mathrm{NP}\to\overline{\mathrm{MS}}}\, O_j^{\mathrm{NP}}(\mu)$. Unavoidable!

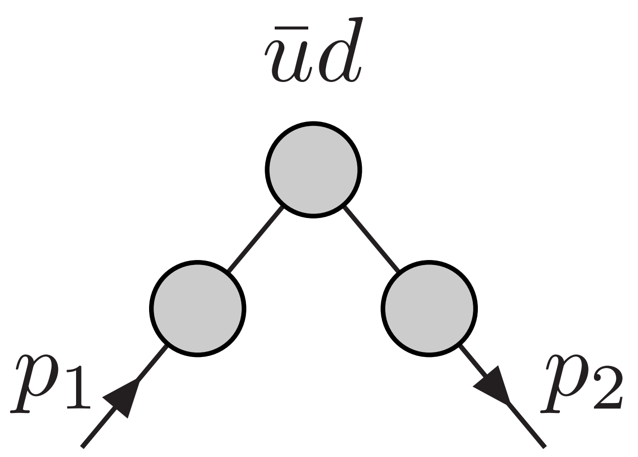
Renormalisation (cont.) - Non-perturbative renormalisation

"A general method for the non-perturbative renormalisation of Lattice Operators"

G.Martinelli, C.Pittori, CTS, M.Testa and A.Vladikas, hep-lat/9411010



- In this paper we demonstrate that non-perturbative renormalisation is possible and defined the NP=RI-MoM scheme.
  - We impose the condition that the matrix element, between quark states, at $p^2 = \mu^2$ in the Landau gauge, takes its tree-level value. This defines $(\bar{u}d)^{\mathrm{RI-MoM}}(\mu) = Z(\mu^2 a^2)(\bar{u}d)^{\mathrm{Lat}}(a)$.
  - RI stands for "Regularisation Independent" - Tautology

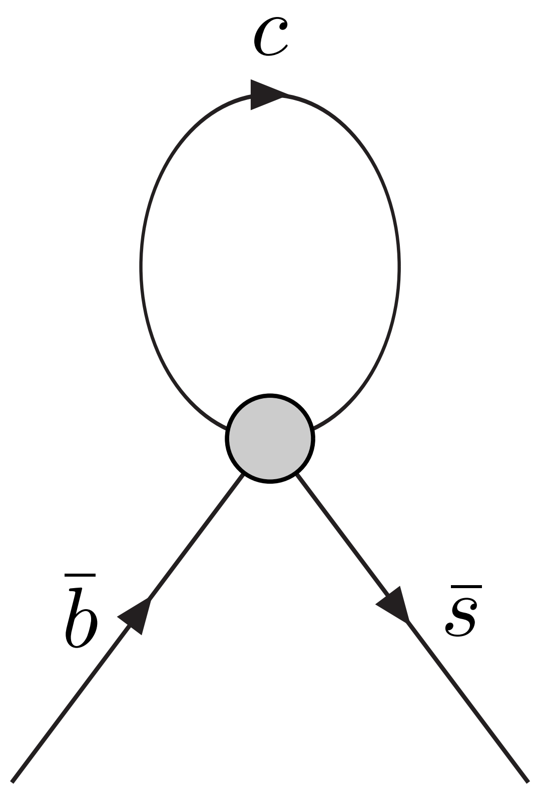


- In most applications it is more precise to chose different incoming and out going momenta, $p_1^2 = p_2^2 = (p_1 - p_2)^2 = \mu^2$. This defines RI-SMoM.

Y.Aoki, N.H.Christ, T.Izubuchi, CTS, C.Sturm and A.Soni , arXiv:0901.2599

- (Many technicalities skipped here.)

Renormalisation (cont.) - power divergences



- In non-perturbative renormalisation schemes, in addition to logarithmic ultra-violet divergences, power divergences appear.
    - In lattice computations the power divergences appear in the form $1/a^n$.
- Example: For FCNC $\bar{b} \to \bar{s}$ transitions, the following current-current operators contribute:

$$O_1 = (\bar{b}^i \gamma^\mu P_L c^j)(\bar{c}^j \gamma_\mu P_L s^i) \qquad O_2 = (\bar{b}^i \gamma^\mu P_L c^i)(\bar{c}^j \gamma_\mu P_L s^j)$$

where $P_L = (1 - \gamma^5)/2$.

- $O_{1,2}$ can mix with lower dimensional operators, such as the scalar and pseudoscalar densities, $(\bar{b}\, s)$ and $(\bar{b}\, \gamma^5\, s)$ respectively, with coefficients which diverge as $1/a^3$ (in practice the lattice symmetries result in a reduction in the degree of divergence).
- The power divergences must be subtracted non-perturbatively and, depending on the process being studied and the lattice discretisation being implemented, this requires some significant effort.

L.Maini, G.Martinelli, CTS, Nucl.Phys. B368 (1992) 281

- There are no power divergences in dimensional regularisation, but the non-perturbative effects show up as "renormalons", i.e. divergent perturbation series which are not Borel summable. G.Martinelli, CTS, hep-ph/960533

Finite-Volume Effects - One Dimensional Illustration

- Let $f(p^2)$ be a smooth function of p^2. For sufficiently large L we have

$$\frac{1}{L}\sum_n f(p_n^2) = \int \frac{dp}{2\pi}\, f(p^2)$$

where $p_n = n(2\pi/L)$.

- In actual lattice computations, the spacings between momenta are $O(100)$ MeV and so the relation does not hold locally.

- The starting point for studies of finite-volume corrections is the (exact) Poisson summation formula,

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} f(p_n^2) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, f(p^2) + \sum_{l\neq 0}\int_{-\infty}^{\infty} \frac{dp}{2\pi}\, f(p^2)\, e^{i\,(lpL)}\,.$$

- If $f(p^2)$ is a smooth function then the FV corrections are exponentially small, e.g. $\propto e^{-m_\pi L}$. Otherwise, if $f(p^2)$ has singularities then the corrections will be larger, perhaps power-like, $1/L^n$, or taking a more complicated form.
- For many simple applications, e.g. leptonic decay constants, semileptonic form factors, P^0 - $\bar{P}^0$ mixing, the finite volume corrections are exponentially small whereas for many others e.g. $K \to \pi\pi$ decays, QED effects, they are larger.

An Illustrative one-dimensional example

- A one-dimensional example of the key ingredient for the derivation of the Luscher quantisation formula for $\pi\pi$ states in a finite volume is:

$$\frac{1}{L}\sum_n \frac{f(p_n^2)-f(k^2)}{k^2-p_n^2} = \int \frac{dp}{2\pi}\,\frac{f(p^2)-f(k^2)}{k^2-p^2}\,,$$

up to exponentially small FV corrections.

- Rearranging the above equation we can write:

$$\frac{1}{L}\sum_n \frac{f(p_n^2)}{k^2-p_n^2} = \mathscr{P}\int \frac{dp}{(2\pi}\,\frac{f(p^2)}{k^2-p^2} + \frac{f(k^2)\,\cot\frac{kL}{2}}{2k}\,, \qquad (\,*\,)$$

where $\mathscr{P}$ stands for Principle Value.

- The size of the FV corrections depends on how close k^2 is to one of the allowed momenta p_n .
- One possibility (in principle) is to take the infinite-volume limit keeping $kl = (2n+1)\pi$ so that the cotangent term vanishes.
- Another is to chose volumes such that $k \to p_{n_0}$ for some chosen n_0

$$\frac{1}{L}\sum_{n\neq n_0} \frac{f(p_n^2)}{k^2-p_n^2} = \mathscr{P}\int \frac{dp}{(2\pi}\,\frac{f(p^2)}{k^2-p^2} + 2\,\frac{f'(k^2)}{L} - \frac{f(k^2)}{2Lk^2} \quad \Rightarrow \quad \text{power corrections}\,.$$

- In practice we use the 3 spatial dimensional generalisation of $(\,*\,)$.

Extending the range of Lattice Flavourdynamics

- In the past, most lattice computations in flavour physics have been of matrix elements of the form

$$\langle f \,|\, O(0) \,|\, i \rangle$$

where $|\, i \rangle$ is a single-hadron state, $|\, f \rangle$ is the vacuum or single-hadron state and $O(0)$ is a local composite operator.

- In recent years, together with my collaborators in Rome and in the RBC-UKQCD collaboration, we have been working to extend the range of physical processes for which the hadronic effects can be computed, including

- Matrix elements of bifocal operators: $\displaystyle\int d^4y \, \langle f \,|\, O_1(0)\, O_2(y) \,|\, i \rangle$. For example:

(i) Δm_K and long distance contributions to ϵ_K . Here O_1 and O_2 are both 4-quark weak operators.

N.H.Christ, T.Izubuchi, CTS, A.Soni and J.Yu , arXiv:1212.5931; Z.Bai, N.H.Christ, T.Izubuchi, CTS, A.Soni and J.Yu, arXiv:1406.0916
Z.Bai, N.H.Christ and CTS, EPJ WebConf. 175 (2018) 13017; Z.Bai, N.H.Chris, J.Karpie, CTS, A.Soni and B.Wang, arXiv:2309.01193

(ii) The rare kaon decays $K \to \pi \ell^+ \ell^-$ and $K \to \pi \nu \bar{\nu}$. Here O_1 and O_2 can both be weak operators $(K \to \pi \nu \bar{\nu})$ or a weak operator and an electromagnetic current $(K \to \pi \ell^+ \ell^-)$.

N.H.Christ, X.Feng, A.Portelli and CTS, arXiv:1507.03094, arXiv:1605.04442 + a series of numerical studies

- For these processes, the theoretical frameworks have been developed, exploratory numerical computations have been performed, but computations on the next generation of machines will have to be performed to achieve, precise, robust results.

2. K $\to$ $\pi\pi$ Decays

- For these decays $|\,f\,\rangle$ consists of two hadrons which interact in the finite volume.
- $K \to \pi\pi$ decays are a very important class of processes with a long and noble history.
  - It is in these decays that both indirect and direct CP-violation was discovered.
- Bose symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2 ,

$$_{I=2}\langle \pi\pi\,|\,H_W\,|\,K^0\rangle = A_2\,e^{i\delta_2}\,, \qquad _{I=0}\langle \pi\pi\,|\,H_W\,|\,K^0\rangle = A_0\,e^{i\delta_0}\,.$$

- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule ($\mathrm{Re}A_0/\mathrm{Re}A_2 \simeq 22.5$) and an understanding of the experimental value of ϵ'/ϵ, the parameter which was the first experimental evidence for direct CP-violation.
- See the following two RBC-UKQCD papers, which however represent the culmination of many years of preparatory work:

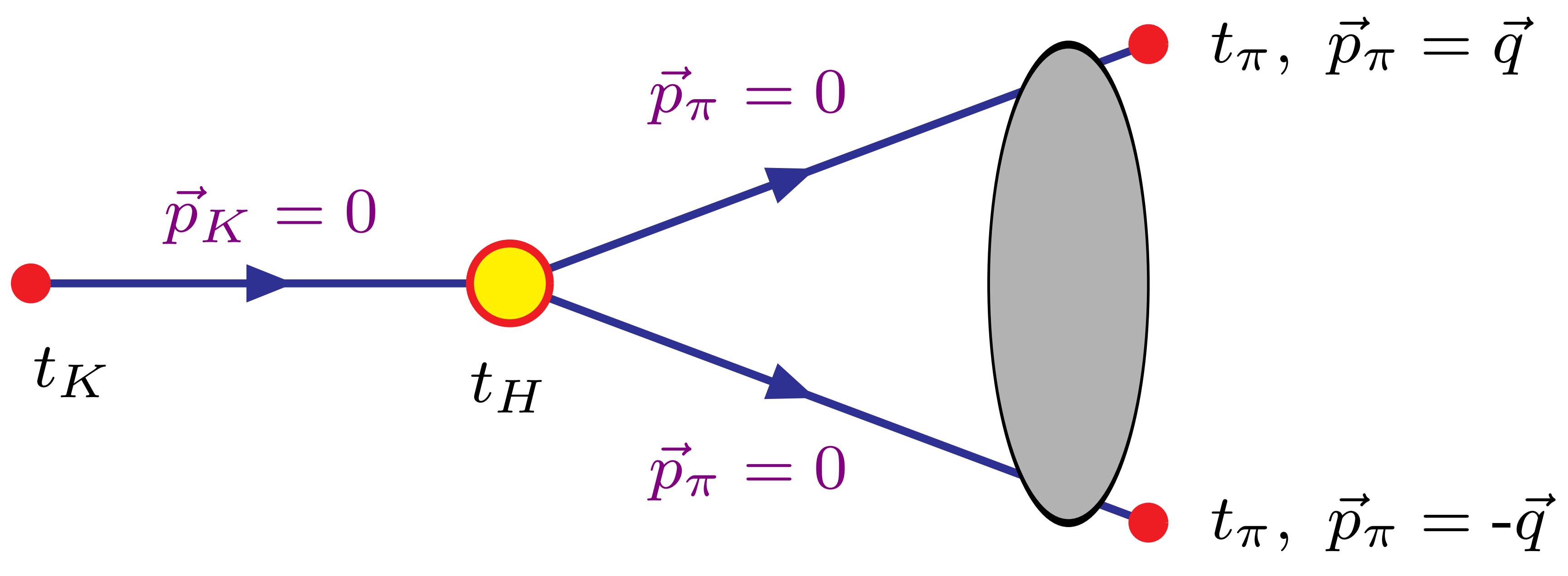
1. *"$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit"*
T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Janowski, C.Jung, C.Kelly, C.Lehner, A.Lytle, R.D.Mawhinney, CTS., A.Soni, H.Yin, and D.Zhang arXiv:1502.00263

2. *"Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay in the Standard Model"*
R.Abbott, T.Blum, P.A.Boyle, M.Bruno, N.H.Christ, D.Hoying, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, D.J.Murphy, CTS, A. Soni, M.Tomii and T.Wang, arXiv:2004.09440

(Building on RBC-UKQCD, Z.Bai et al. arXiv:1505.07863)

- Detailed references to earlier work can be found in these papers.

Why are the amplitudes difficult to compute?



- $K \to \pi\pi$ correlation function is dominated by the lightest intermediate state. L.Maiani and M.Testa, Phys.Lett. B245 (1990) 585
  - With periodic boundary conditions this is the $\pi\pi$ state with both pions at rest for A_2 and the vacuum state for A_0.
  - We have chosen to use anti periodic boundary conditions for the d-quark for A_2 and G-parity boundary conditions for A_0.
  - Work is in progress to compute the amplitudes with periodic boundary conditions with excited $\pi\pi$ states. M.Tomii, Lattice 2023
- Volume must be tuned to ensure $E_{\pi\pi} = m_K$.
  - Moreover, the s-wave $I = 0$ and $I = 2$ channels are attractive and repulsive respectively and so the two cases must be treated separately.
- Finite-volume effects are not exponentially small and must be corrected. L.Lellouch and M.Lüscher, hep-lat/00030023, C.J.D.Lin, G.Martinelli, CTS and M.Testa, hep-lat/0104006 C-h.Kim, CTS and S.Sharpe, hep-lat/0507006

Summary of our Results

- $\mathrm{Re}\,A_0 = 2.99\,(0.32)\,(0.59) \times 10^{-7}\,\mathrm{GeV}$ (Experiment $3.3201(18) \times 10^{-7}\,\mathrm{GeV}$);

  $\mathrm{Im}\,A_0 = -\,6.98\,(0.62)\,(1.44) \times 10^{-11}\,\mathrm{GeV}$.

- $\mathrm{Re}\,A_2 = 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\,\mathrm{GeV}$, (Experiment $1.4787(31) \times 10^{-8}\,\mathrm{GeV}$) ;

  $\mathrm{Im}\,A_2 = -\,6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\,\mathrm{GeV}$.

- We find $\dfrac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2} = 19.9 \pm 2.3 \pm 4.4$ in good agreement with the experimental result of 22.45(6) .

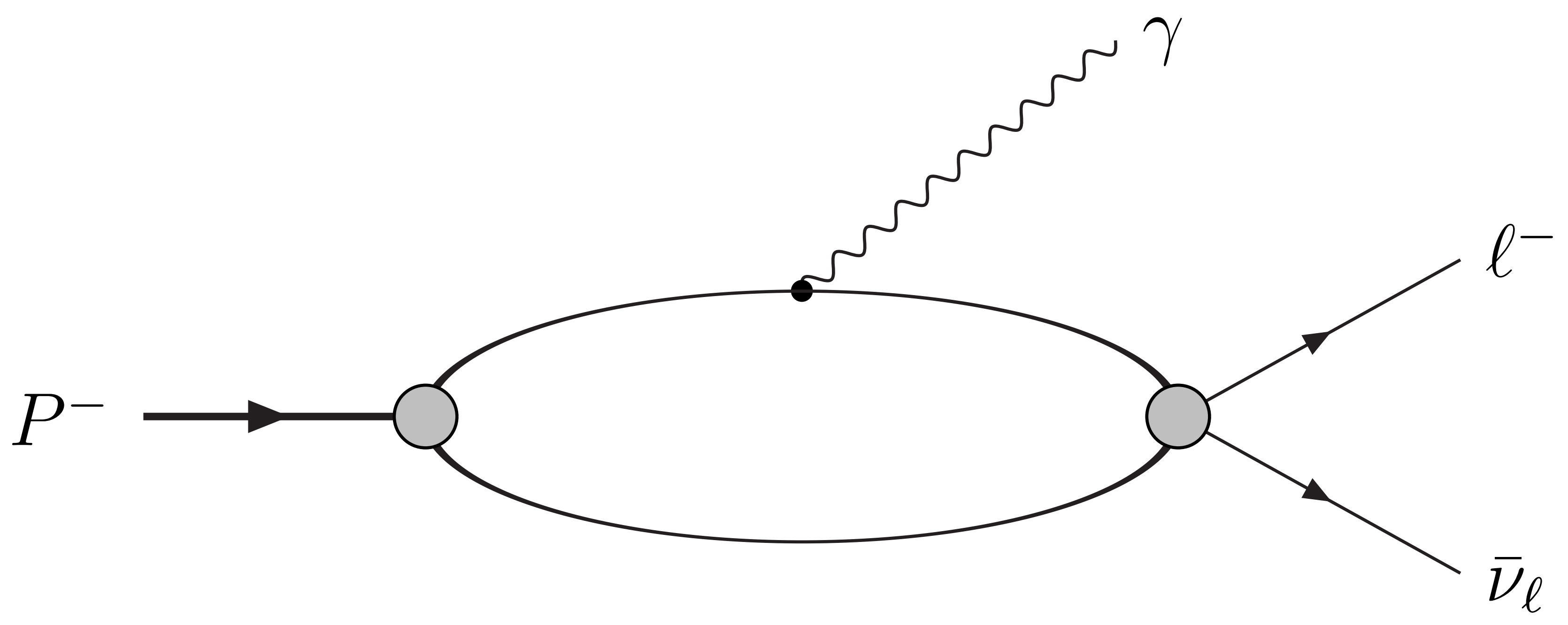
- Combining the result for $\mathrm{Im}\,A_0$ and $\mathrm{Im}\,A_2$ and using the experimental results for the real parts we obtain

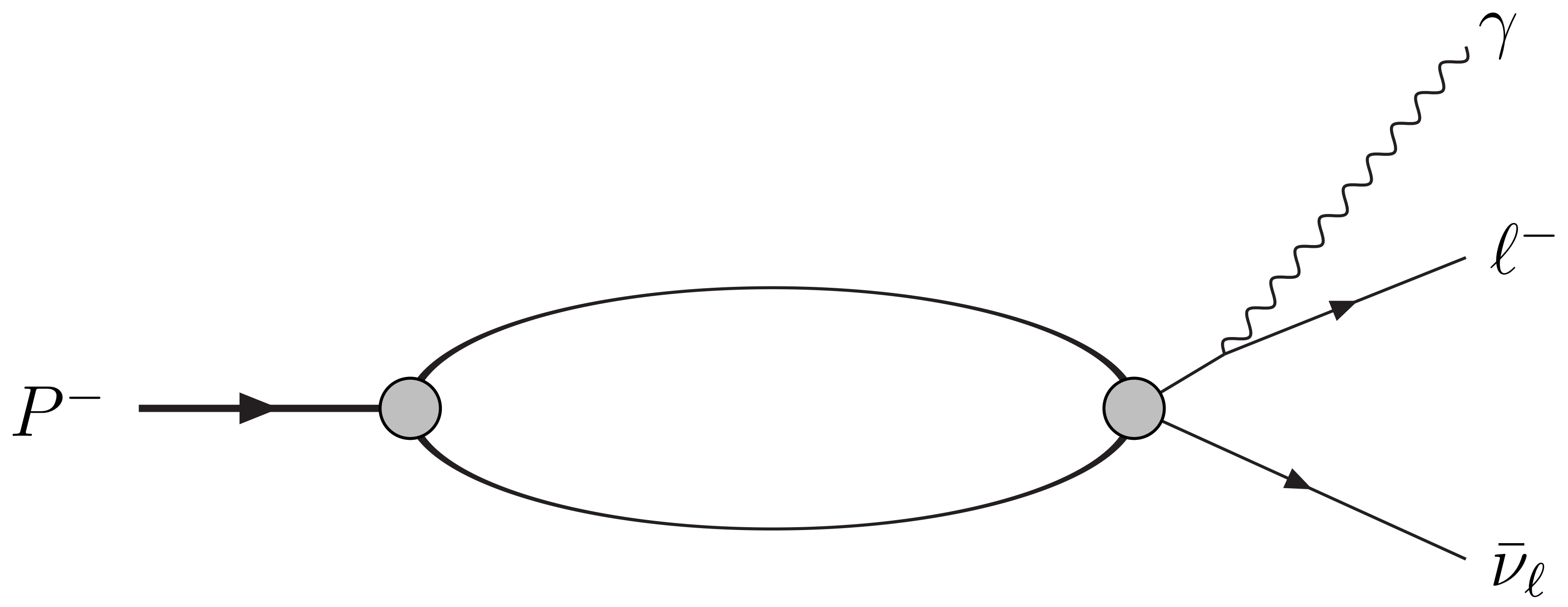
$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = 0.00217\,(26)_{\mathrm{stat}}\,(62)_{\mathrm{syst}}\,(50)_{\mathrm{IB}}\,.$$

  The result is consistent with the experimental value of 0.00166 (23) .

- The RBC/UKQCD Collaboration continues work to reduce the uncertainties. Important priority is to control the IB effects.

3. P $\rightarrow \ell\nu_\ell\gamma$ radiative decays - the form factors.





- Non-perturbative contribution to $P \rightarrow \ell\bar{\nu}_\ell\gamma$ is encoded in:

$$H_W^{\alpha r}(k,\mathbf{p}) = \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k,\mathbf{p}) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \mathrm{T}\ \langle 0 |\, j_W^\alpha(0)\, j_{\rm em}^\mu(y)\, | P(\mathbf{p}) \rangle$$

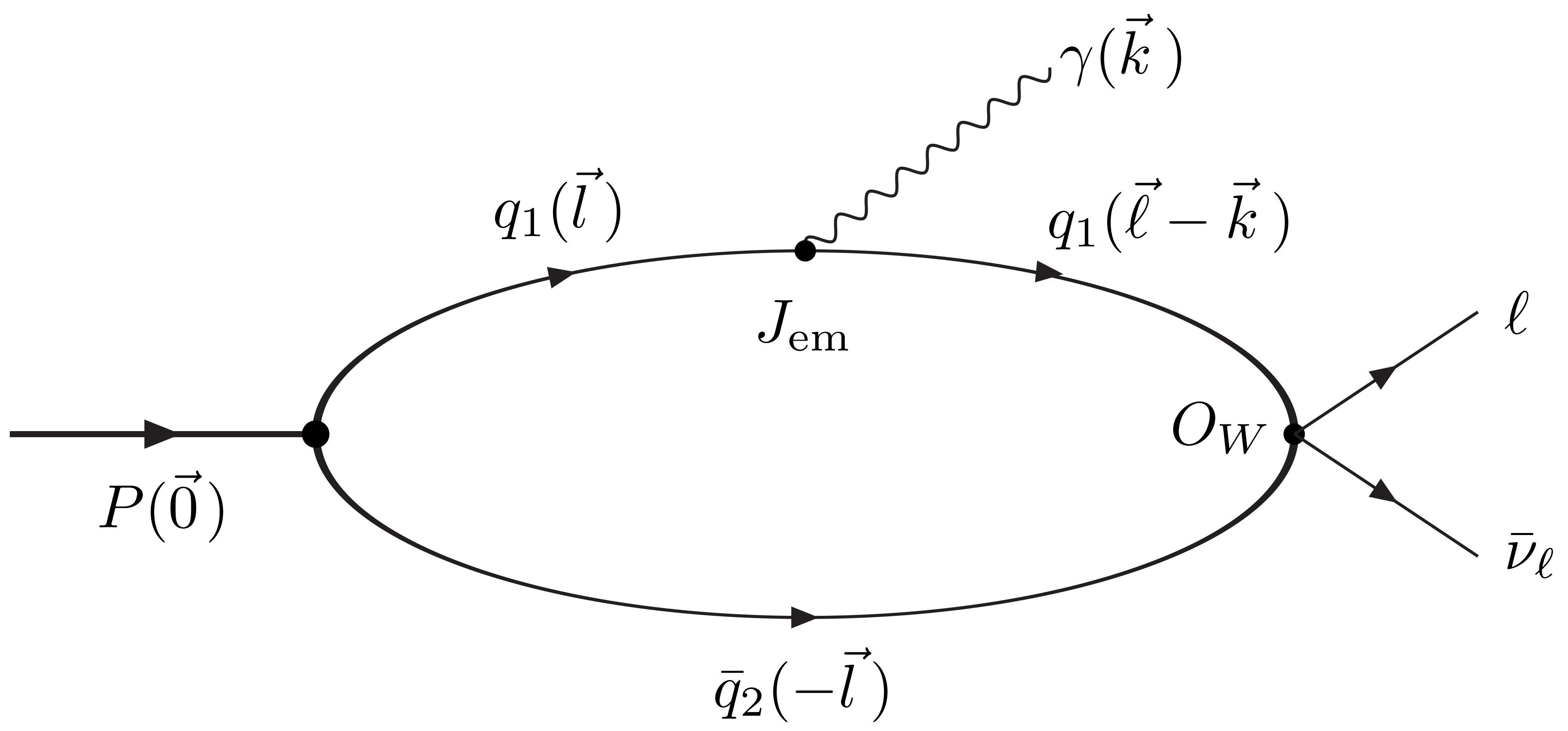
$$= \epsilon_\mu^r(k) \left\{ \frac{H_1}{m_K}\left[k^2 g^{\mu\alpha} - k^\mu k^\alpha\right] + \frac{H_2}{m_K}\frac{\left[(p\cdot k - k^2)k^\mu - k^2(p-k)^\mu\right](p-k)^\alpha}{(p-k)^2 - m_K^2} \right.$$

$$\left. - i\frac{F_V}{m_K}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_K}\left[(p\cdot k - k^2)g^{\mu\alpha} - (p-k)^\mu k^\alpha\right] + f_P\left[g^{\mu\alpha} - \frac{(2p-k)^\mu(p-k)^\alpha}{(p-k)^2 - m_K^2}\right] \right\}$$

- For decays into a real photon, $k^2 = 0$ and $\varepsilon \cdot k = 0$, only the decay constant f_P and the vector and axial form factors $F_V(x_\gamma)$ and $F_A(x_\gamma)$ are needed to specify the amplitude ($x_\gamma = 2p\cdot k/m_P^2\,,\ 0 < x_\gamma < 1 - m_\ell^2/m_P^2$).

- In phenomenology $F^\pm \equiv F_V \pm F_A$ are more natural combinations.

Minkowski → Euclidean Continuation



- We assume that P is the lightest particle with quantum numbers $q_1\bar{q}_2$.
- The decay $P \to |n, \gamma\rangle$, where $|n\rangle$ also has quantum numbers $q_1\bar{q}_2$, is therefore not possible.
- The states propagating between $J_{\rm em}$ and O_W can therefore not be on-shell.

- In this case the photon is real, and so there is also no on-shell state which can propagate between $O_W(t_W)$ and $J_{\rm em}(t_{\rm em})$ where $t_{\rm em} > t_W$.
- As expected, the Minkowski-Euclidean continuation is therefore straightforward.
- This is not the case in general when the emitted photon is virtual.

Computing the Form Factors

$$H_W^{\alpha r}(k,\mathbf{p}) = \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k,\mathbf{p}) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\; T\langle 0\,|\, j_W^\alpha(0)\, j_{\rm em}^\mu(y)\,|\, P(\mathbf{p})\rangle$$

- Euclidean Correlation Functions:

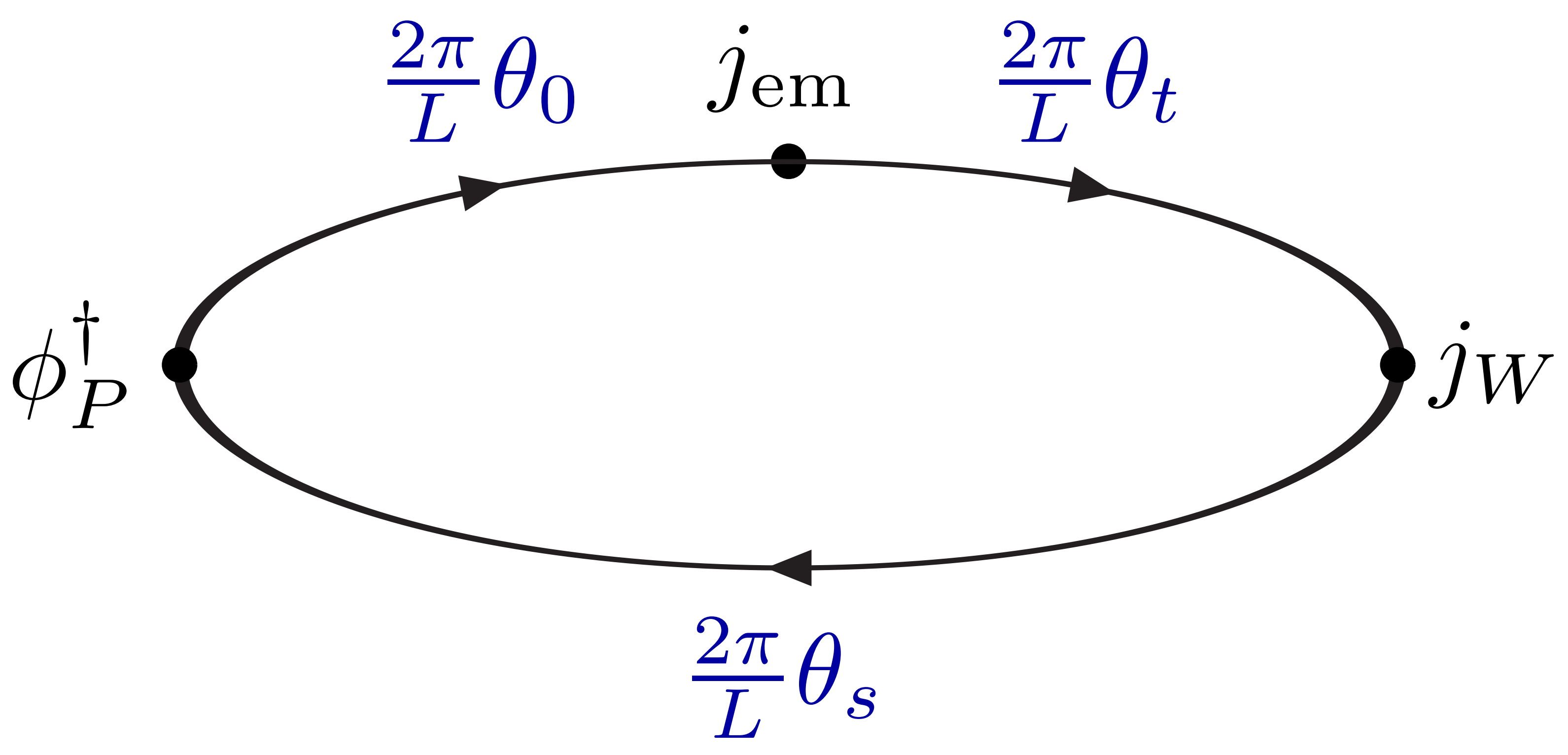
$$C_W^{\alpha r}(t;\mathbf{k},\mathbf{p}) = -\,i\epsilon_\mu^r(k)\int d^4y \int d^3x\; e^{t_y E_\gamma - i\mathbf{k}\cdot\mathbf{y}}\, e^{i\mathbf{p}\cdot\mathbf{x}}\; T\langle 0\,|\, j_W^\alpha(t,\mathbf{0})\, j_{\rm em}^\mu(y)\, \phi_P^\dagger(0,\mathbf{x})\,|\,0\rangle$$

- $H_W^{\alpha r}(k,\mathbf{p})$ can be obtained from the large t limit of the correlation function:

$$R_W^{\alpha r}(t;k,\mathbf{p}) \equiv \frac{2E}{e^{-(E-E_\gamma)t}\langle P(\mathbf{p})\,|\,\phi_P^\dagger(0)\,|\,0\rangle}\, C_W^{\alpha r}(t;k,\mathbf{p}) + \cdots$$

where $E = \sqrt{m_P^2 + \mathbf{p}^2}$.

Choice of Kinematics



- We use twisted boundary conditions to introduce momenta,

CTS & G.Villadoro, hep-lat/0411033

$$\mathbf{p}=\frac{2\pi}{L}(\theta_0-\theta_s)\,;\qquad \mathbf{k}=\frac{2\pi}{L}(\theta_0-\theta_t)\,,$$

with both $\mathbf{p}$ and $\mathbf{k}$ in the z direction

$$\mathbf{p}=(0,0,|\,\mathbf{p}\,|)\,;\qquad \mathbf{k}=(0,0,E_\gamma)\,.$$

- For the polarisation vectors we choose, $\epsilon^1_\mu=\left(0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right),\quad \epsilon^2_\mu=\left(0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right),\quad \epsilon^3_\mu=\epsilon^0_\mu=0\,.$

- With these choices

$$R_A(t)\equiv\frac{1}{2m_P}\sum_{r=1,2}\sum_{j=1,2}\frac{R^{jr}(t;k,\mathbf{p})}{\epsilon^r_j}\to x_\gamma\,F_A(x_\gamma)+\frac{2f_P}{m_P}\qquad R_V(t)\equiv\frac{m_P}{4}\sum_{r=1,2}\sum_{j=1,2}\frac{R^{jr}_V(t;k,\mathbf{p})}{i\left(E_\gamma\,\epsilon^{\mathbf{r}}\times\mathbf{p}-E\,\epsilon^{\mathbf{r}}\times\mathbf{k}\right)^j}\to F_V(x_\gamma)\,.$$

- Thus in principle the two form factors, F_V and F_A can be determined.

$\mathrm{P} \to \ell \nu_\ell \gamma$ radiative decays - the form factors

- We have computed $F_V(x_\gamma)$ and $F_A(x_\gamma)$ for $\pi, K, D_{(s)}$ mesons. A.Desiderio et al. arXiv:2006.05358
    - The computations were performed on 11 ETMC $N_f = 2+1+1$ ensembles with 0.062 fm < a <0.089 fm and 227 MeV< m_π <441 MeV and a range of volumes.
    - Computations are performed in the electroquenched approximation.

- Our data is fully consistent with a parametrisation of the form :

$$F^P_{A,V}(x_\gamma) = C^P_{A,V} + D^P_{A,V} x_\gamma \,.$$

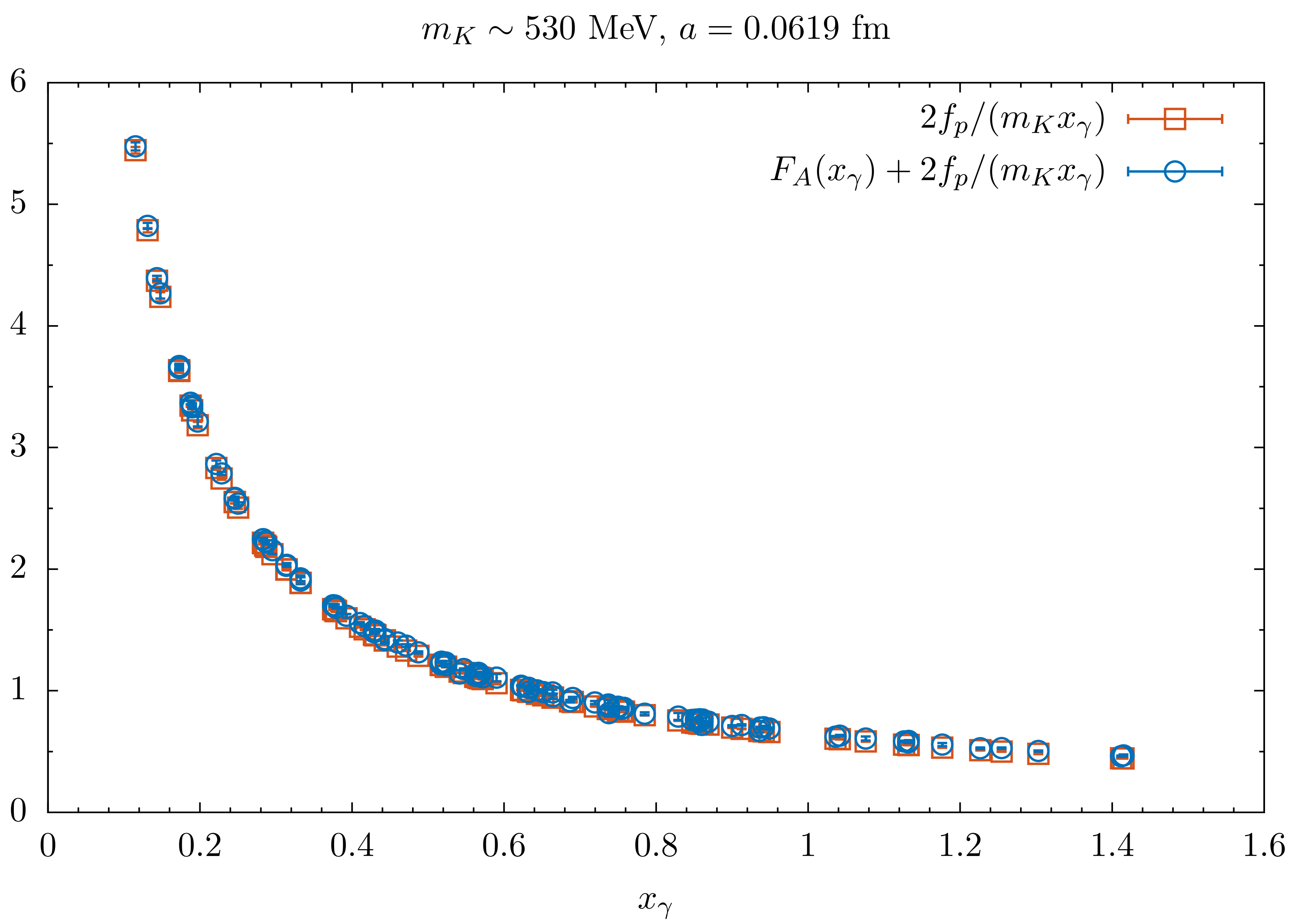
- Other parametrisation were also tried and presented.

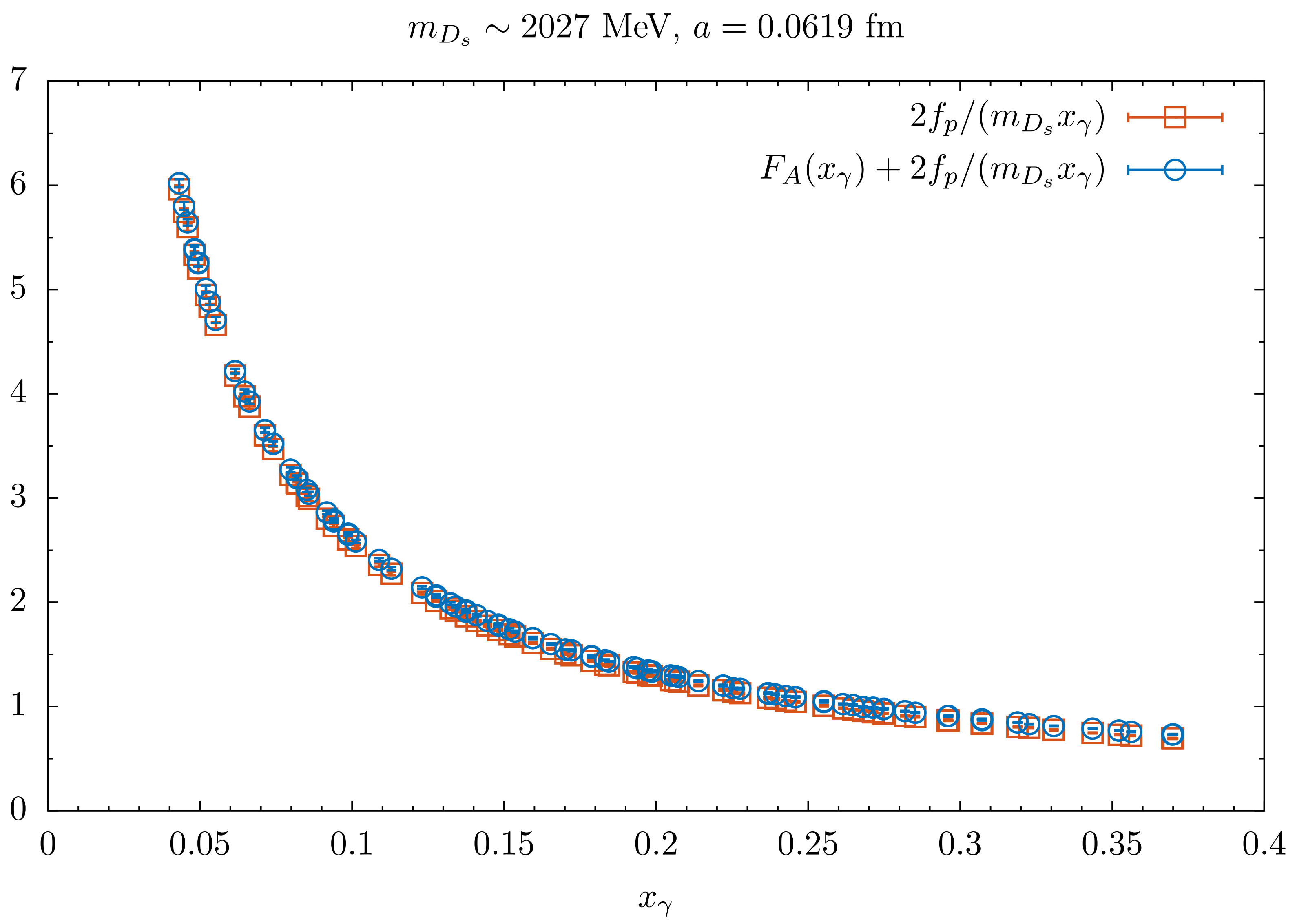
- Values of the parameters are presented in the paper.

- Below we compare our results to the experimental data and also to LO ChPT:

$$F_A(x_\gamma) = \frac{8m_P}{f_P}(L_9^r + L_{10}^r) \simeq \frac{8m_P}{f_P}(0.0017)\,, \qquad F_V(x_\gamma) = \frac{m_P}{4\pi^2 f_p}\,.$$

Non-perturbative subtractions of IR divergent discretisation effects





- The combination $F_A(x_\gamma) + 2f_p/(m_P x_\gamma)$ is dominated by $2f_p/(m_P x_\gamma)$, particularly at small x_γ.
- We rewrite the behaviour of the axial estimator to include discretisation effects

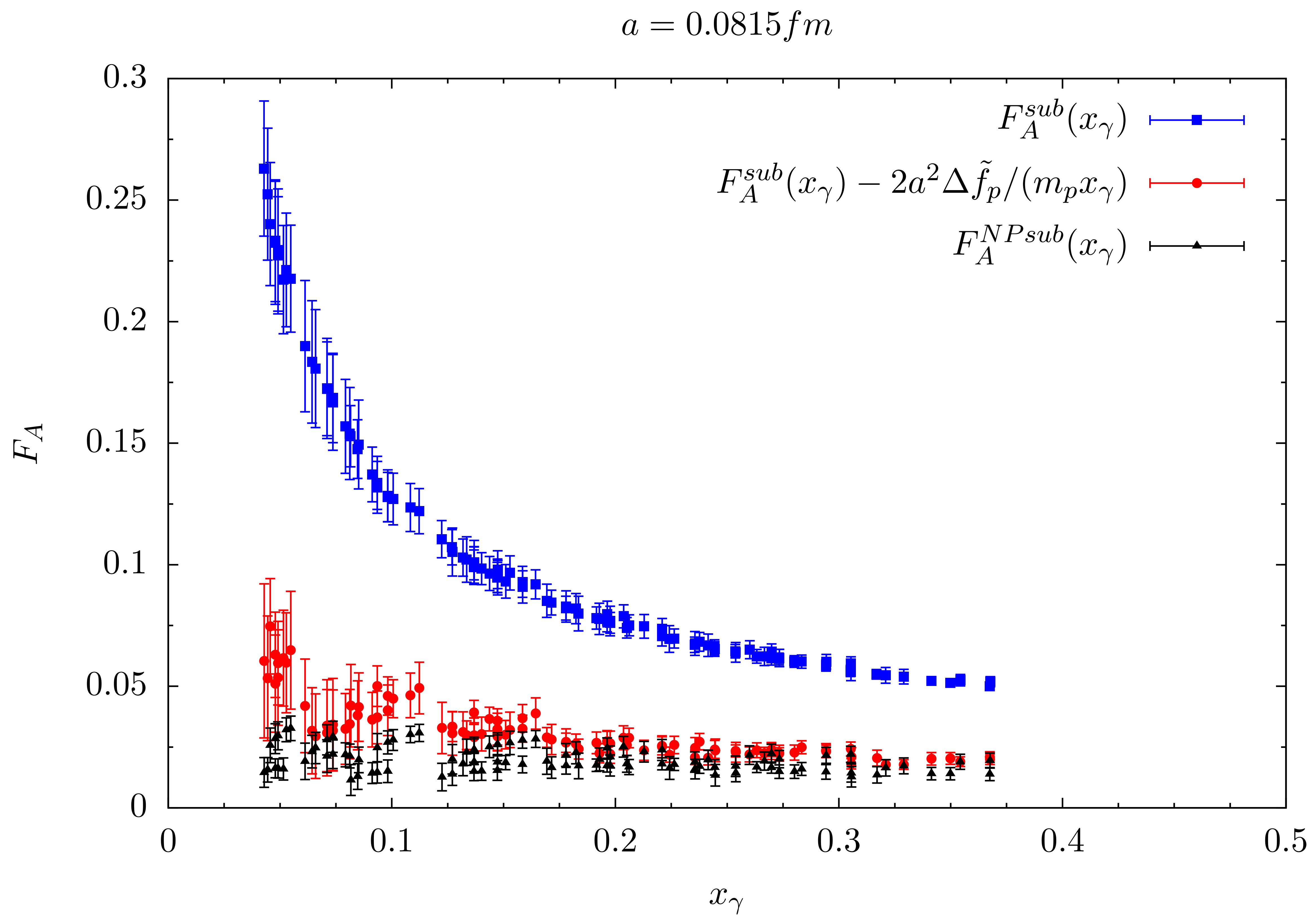
$$\frac{R_A(t)}{x_\gamma} \to \left[F_A(x_\gamma) + a^2 \Delta F_A(x_\gamma)\right] + \frac{2}{m_P x_\gamma}\left(f_P + a^2 \Delta f_P\right) + \cdots$$

- f_P obtained from two-point functions $\neq (f_P + a^2 \Delta f_P) \Rightarrow$ incomplete cancelation of the infrared divergent term.
- We introduce the modified estimator

$$\bar{R}_A(t) = e^{-tE_\gamma} \frac{\sum_{r=1,2} \sum_{j=1,2} \frac{R^{jr}(t;k,\mathbf{p})}{\epsilon_j^r}}{\sum_{r=1,2} \sum_{j=1,2} \frac{R^{jr}(t;0,\mathbf{p})}{\epsilon_j^r}} - 1$$

$$\frac{2f_P}{m_P x_\gamma} \bar{R}_A(t) \to F_A^{\mathrm{NPsub}}(x_\gamma) = F_A(x_\gamma) + O(a^2)\,.$$

Non-perturbative subtractions of IR divergent discretisation effects (cont.)



- Blue points - $F_A(x_\gamma)$ obtained by performing the subtraction using the value of f_P obtained from two-point correlation functions.

- Red Points - Discretisation effects in f_P fitted and subtracted.

- Black Points - $F_A^{\mathrm{NPsub}}(x_\gamma)$

- Illustrative example: $F_A(x_\gamma)$ for the D_s meson.

Comparison with Experimental Data

R.Frezzotti, M.Garofalo, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula and N.Tantalo, arXiv:2012.02120

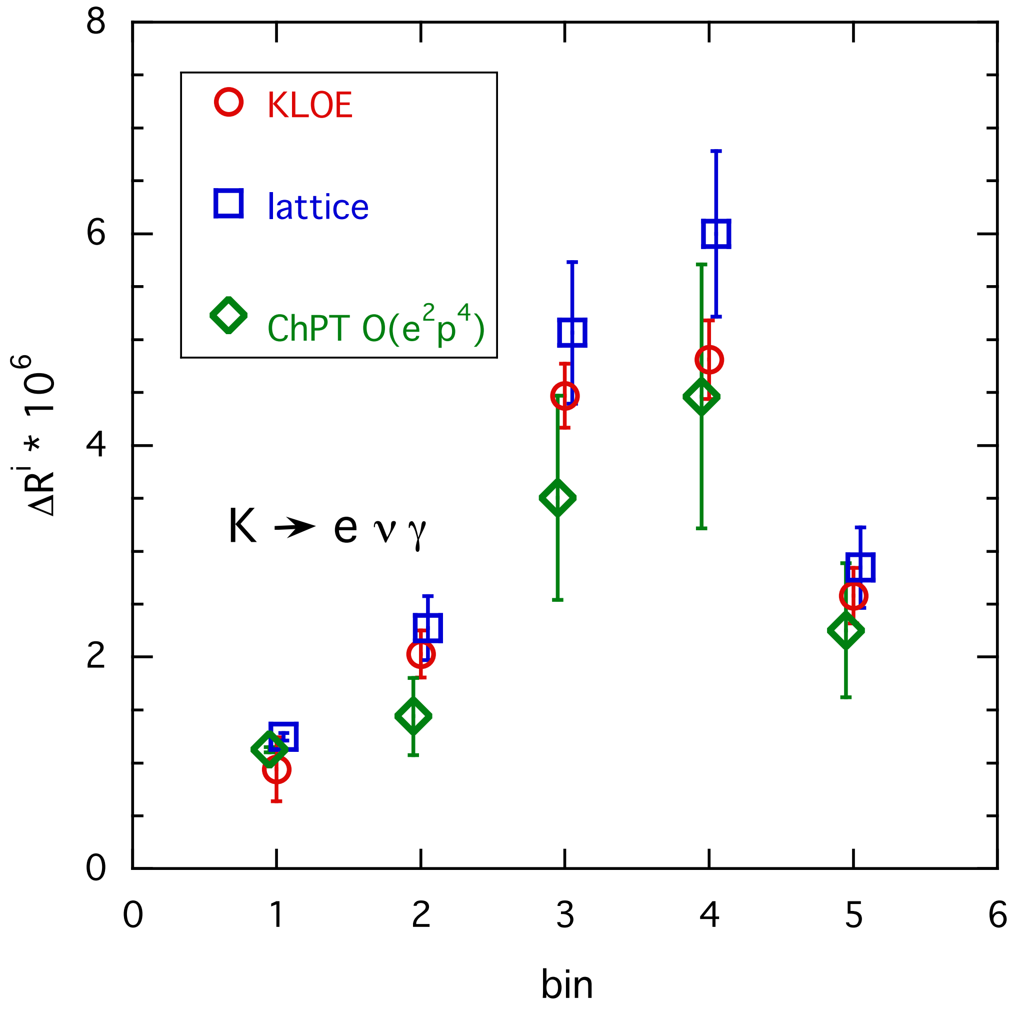
- $K \to e\nu_e\gamma$ KLOE, arXiv:0907.3594
  J-PARC E36, arXiv:2107.03583
  NA62, arXiv:2???.?????

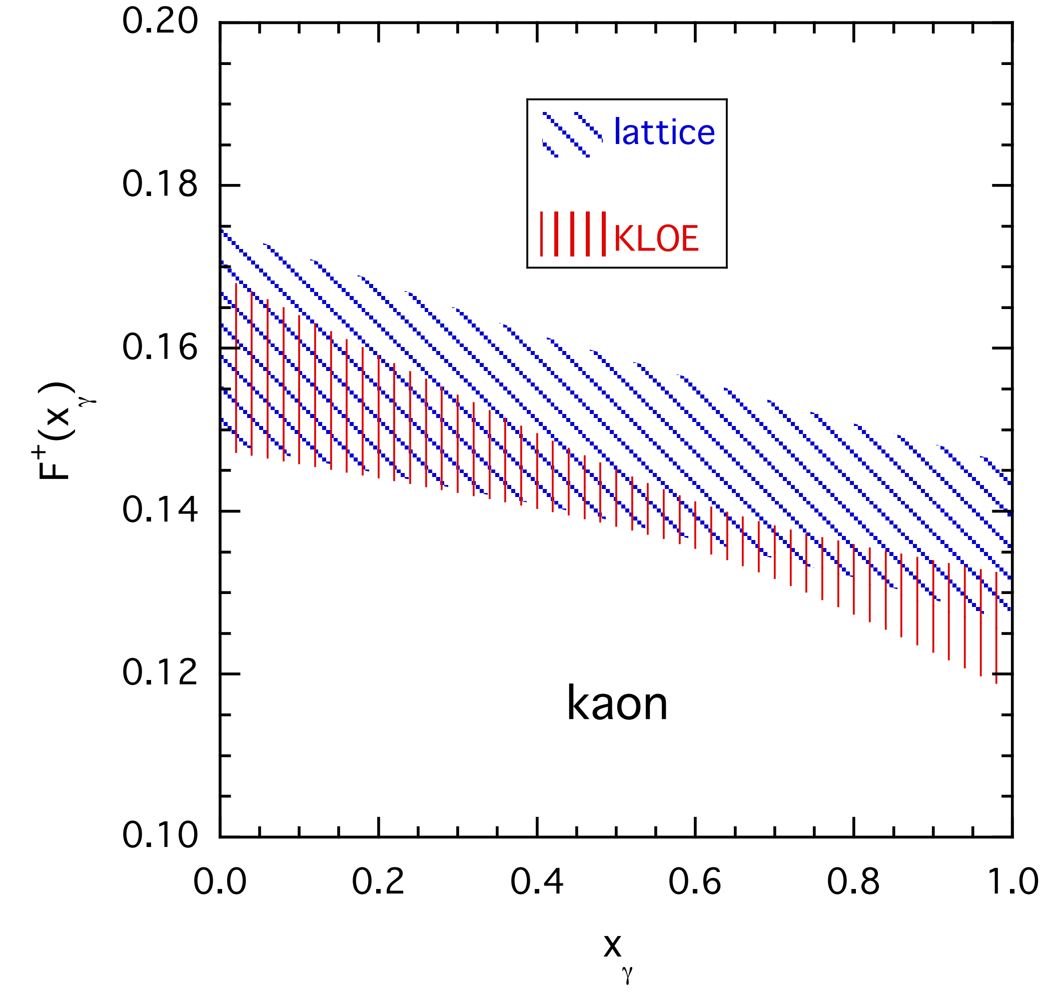
- $K \to \mu\nu_\mu\gamma$ E787@BNL AGS, hep-ex/0003019
  ISTRA+ @U-79 Protvino, arXiv:1005.3517
  OKA@U-79 Protvino, arXiv:1904.10078

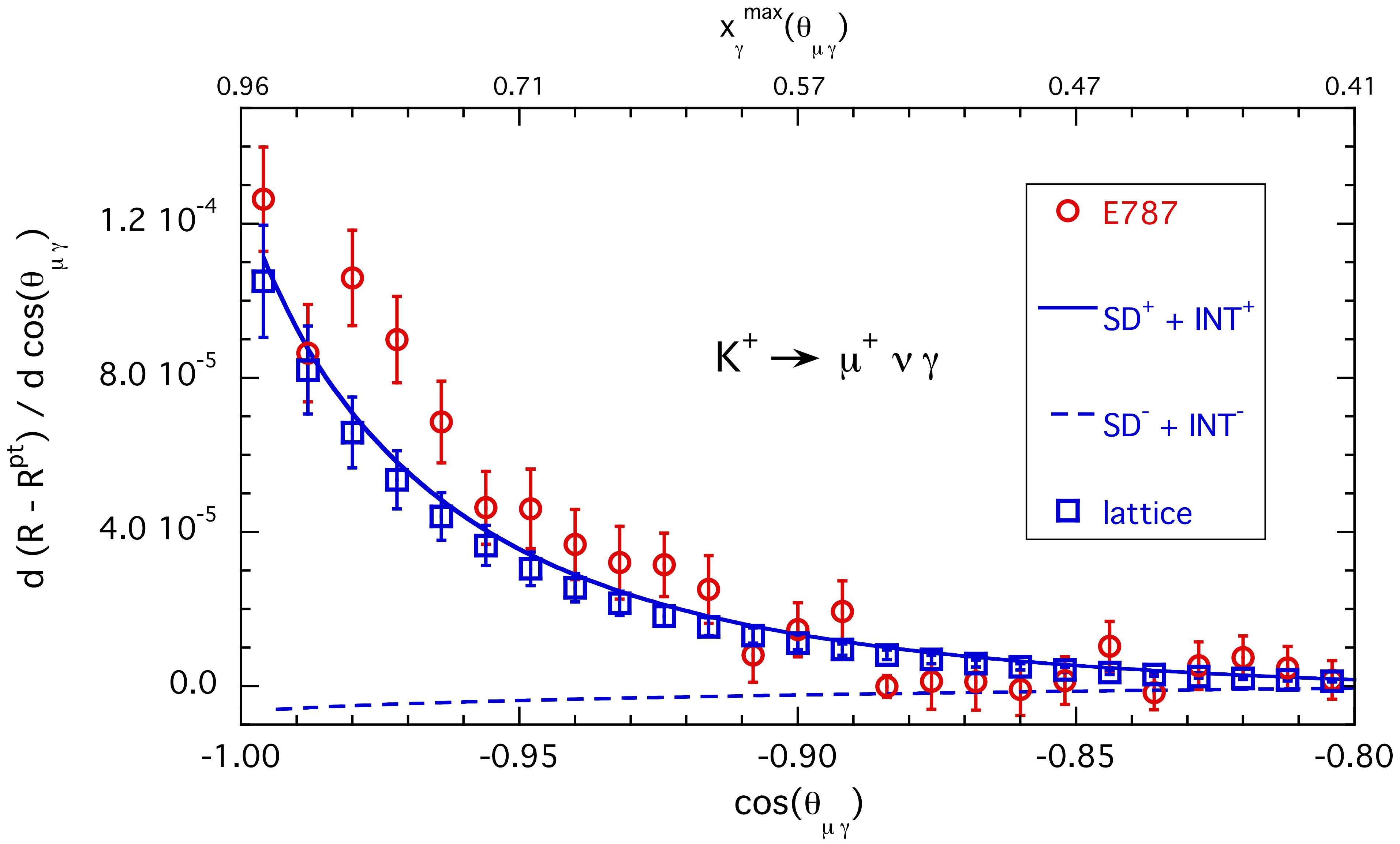
- $\pi \to e\nu_e\gamma$ PIBETA@πE1 beam line PSI, arXiv:0804.1815

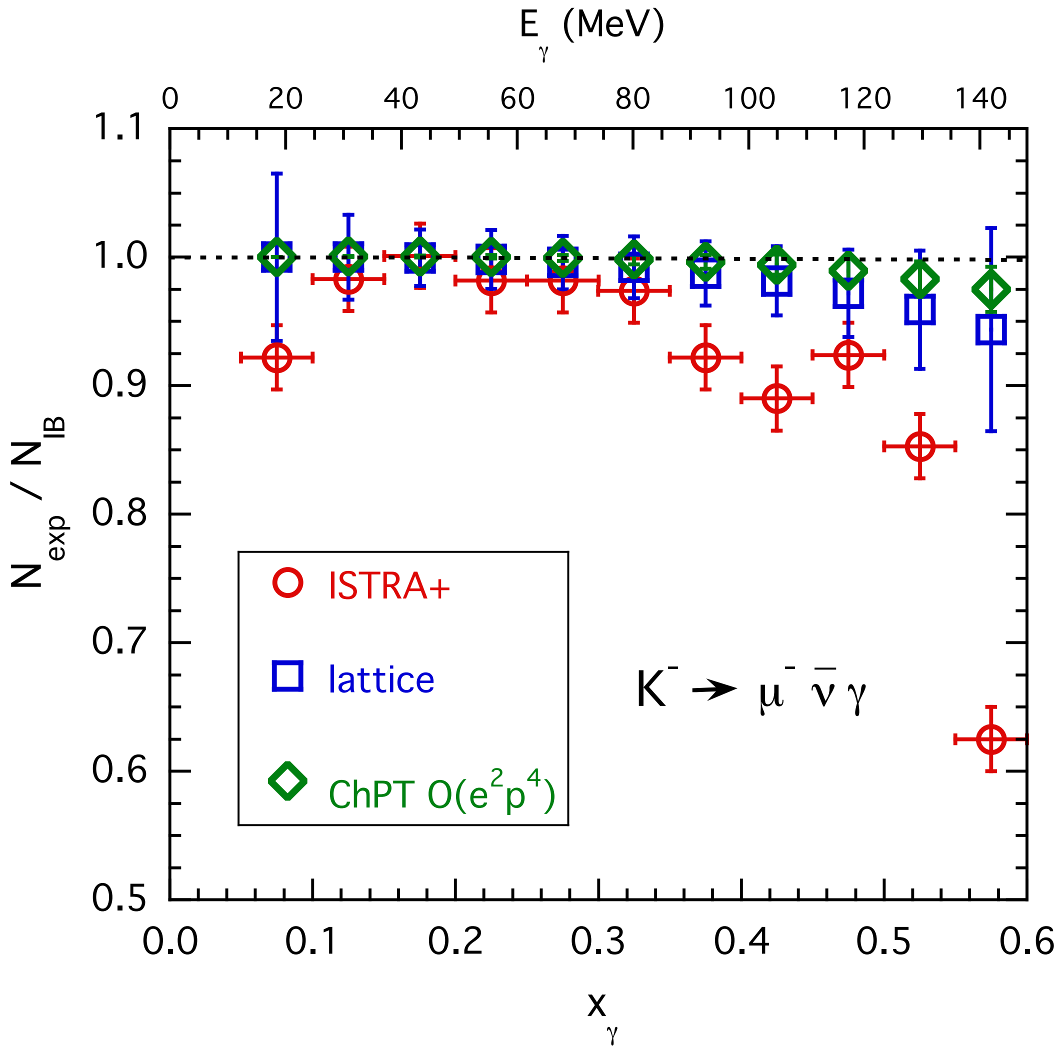
- The different experiments introduce different cuts on E_γ , E_ℓ and $\cos\theta_{\ell\gamma}$, resulting in sensitivities to different form factors.

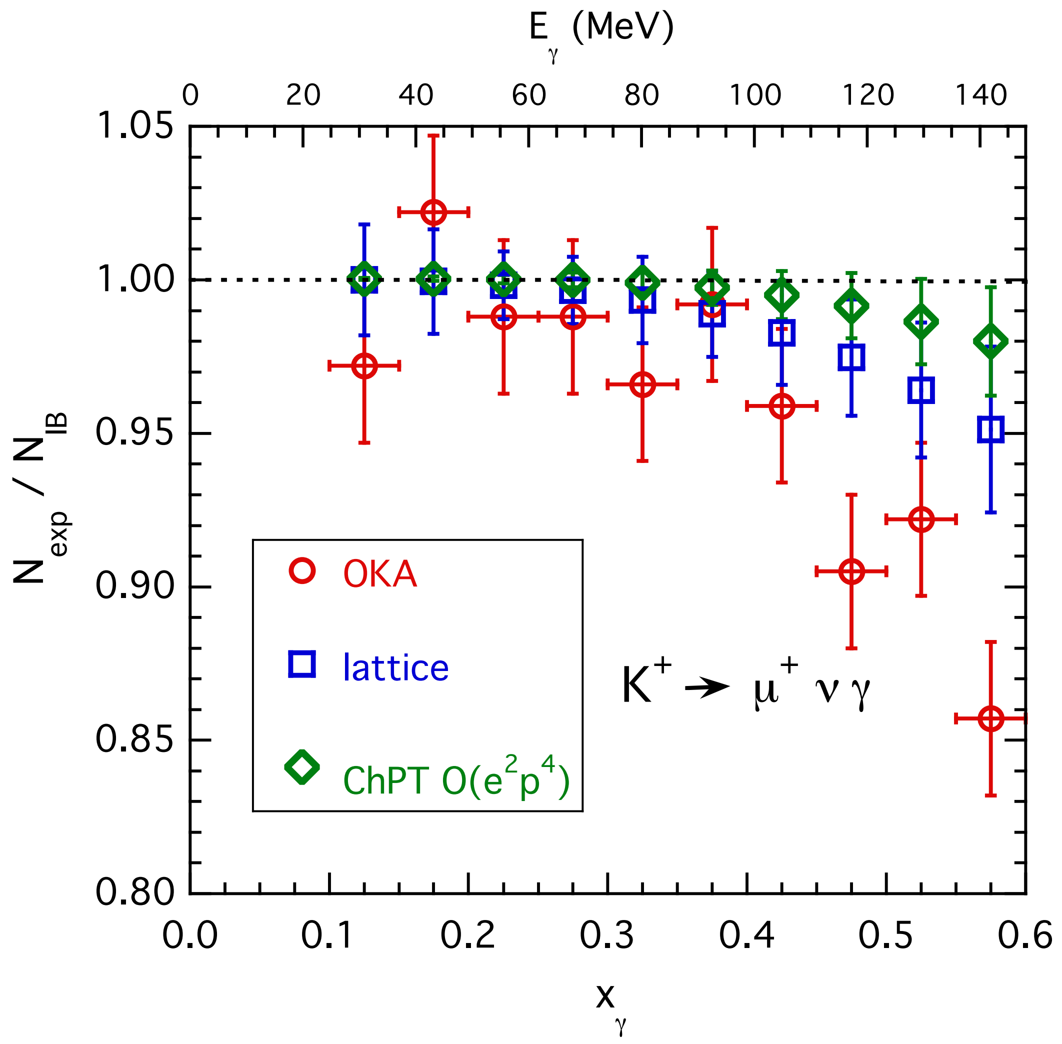
Comparison with Experimental Data — Kaon Decays





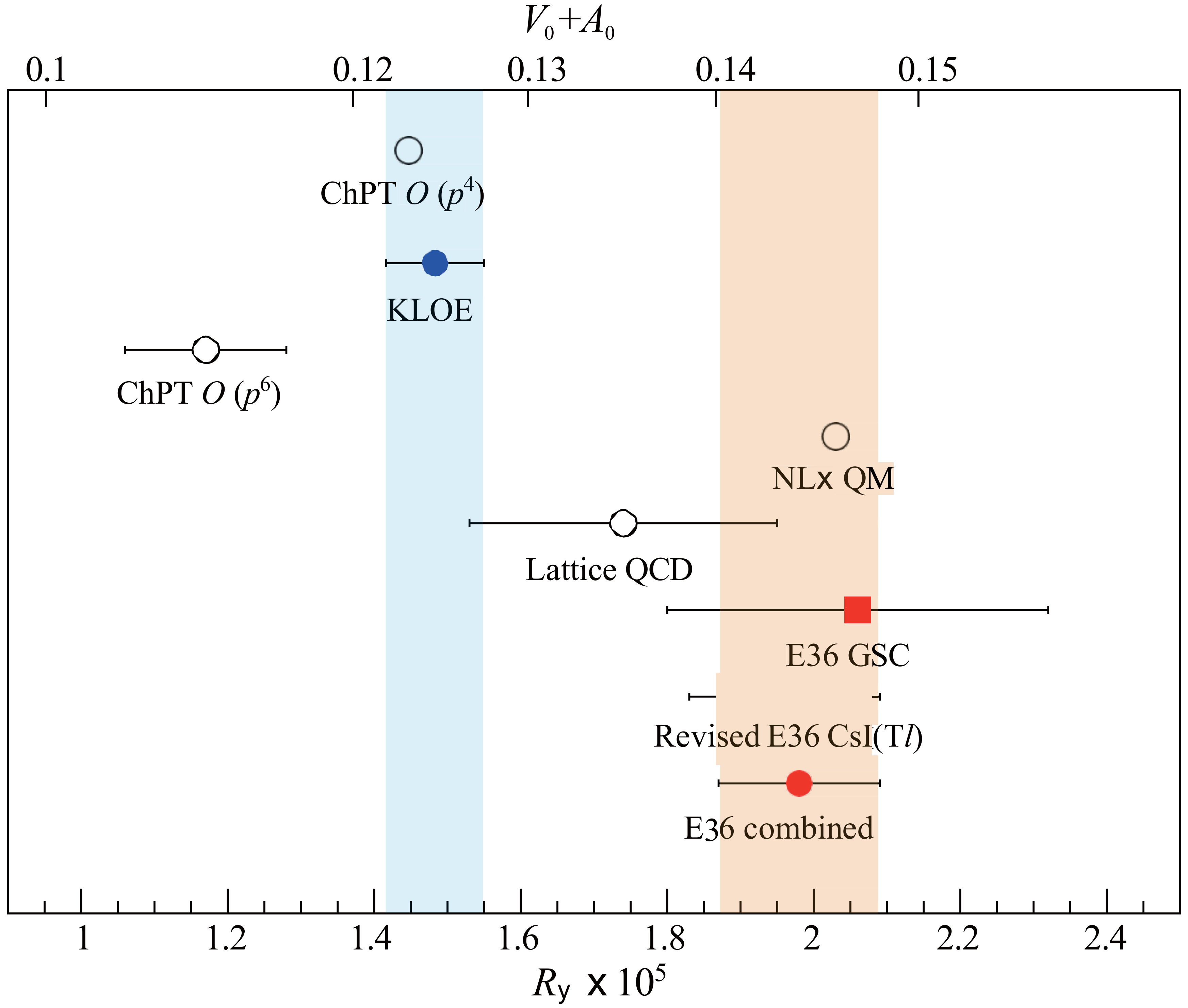






- Good Agreement with KLOE
- Significant tensions with $K \to \mu\nu_\mu\gamma$ experiments
- Unable to find a set of phenomenological form factors to account for all the data.
- NA62 will soon have the most precise results for $K \to e\nu_e\gamma$ decay rates.
- Is it conceivable that we have LFU-violation here?

Comparing JPARC and KLOE's Results



J-PARC E36 Collaboration, A.Kobayashi et al., arXiv:2212.10702

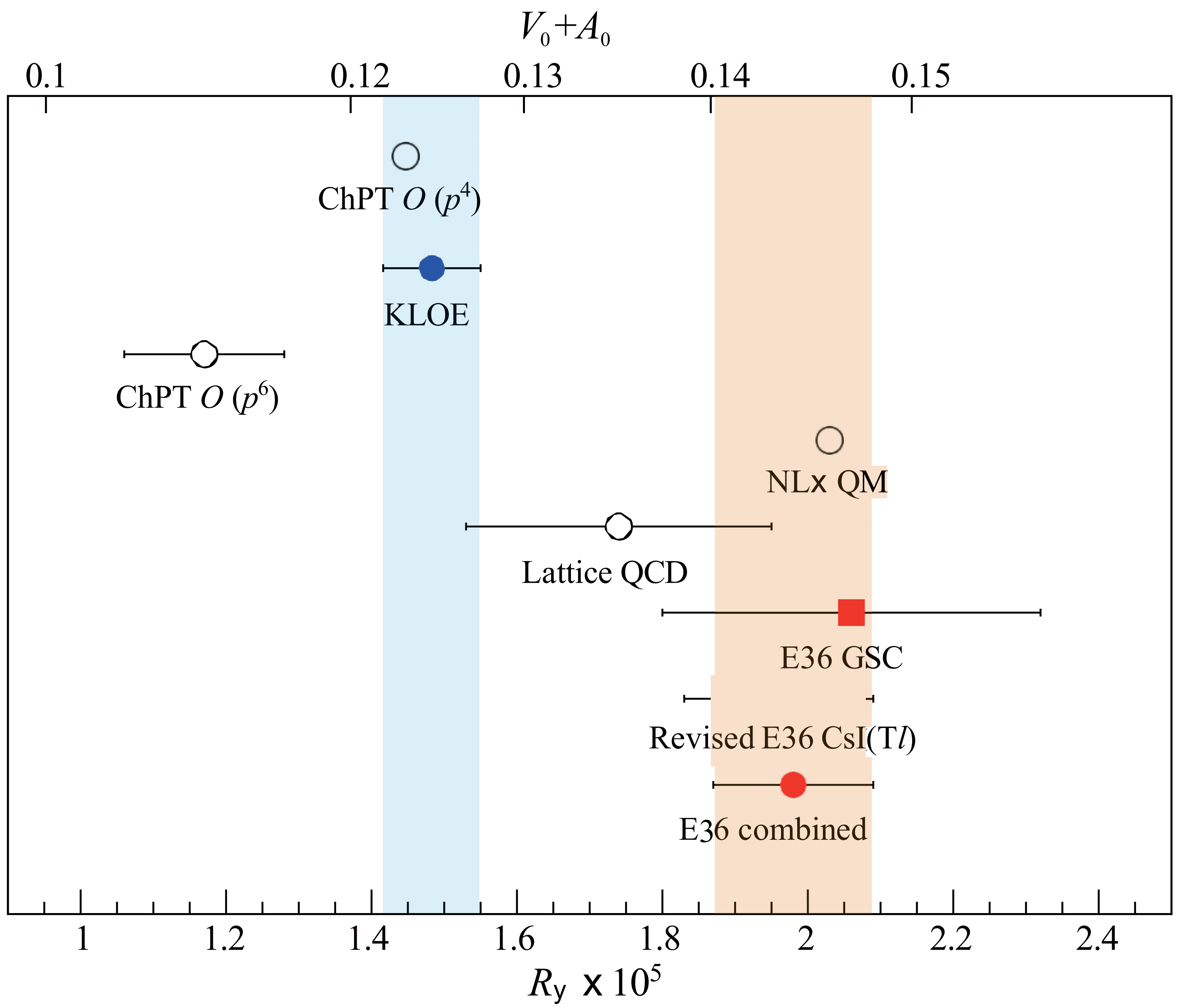
E_γ (MeV)	p_e (MeV)	KLOE [10]	J-PARC E36 [11]	lattice [9]	ChPT
10 - 250	> 200	1.483 ± 0.066 ± 0.013	1.85 ± 0.11 ± 0.07	1.743 ± 0.212	1.279 ± 0.324

(Units of 10^{-5})

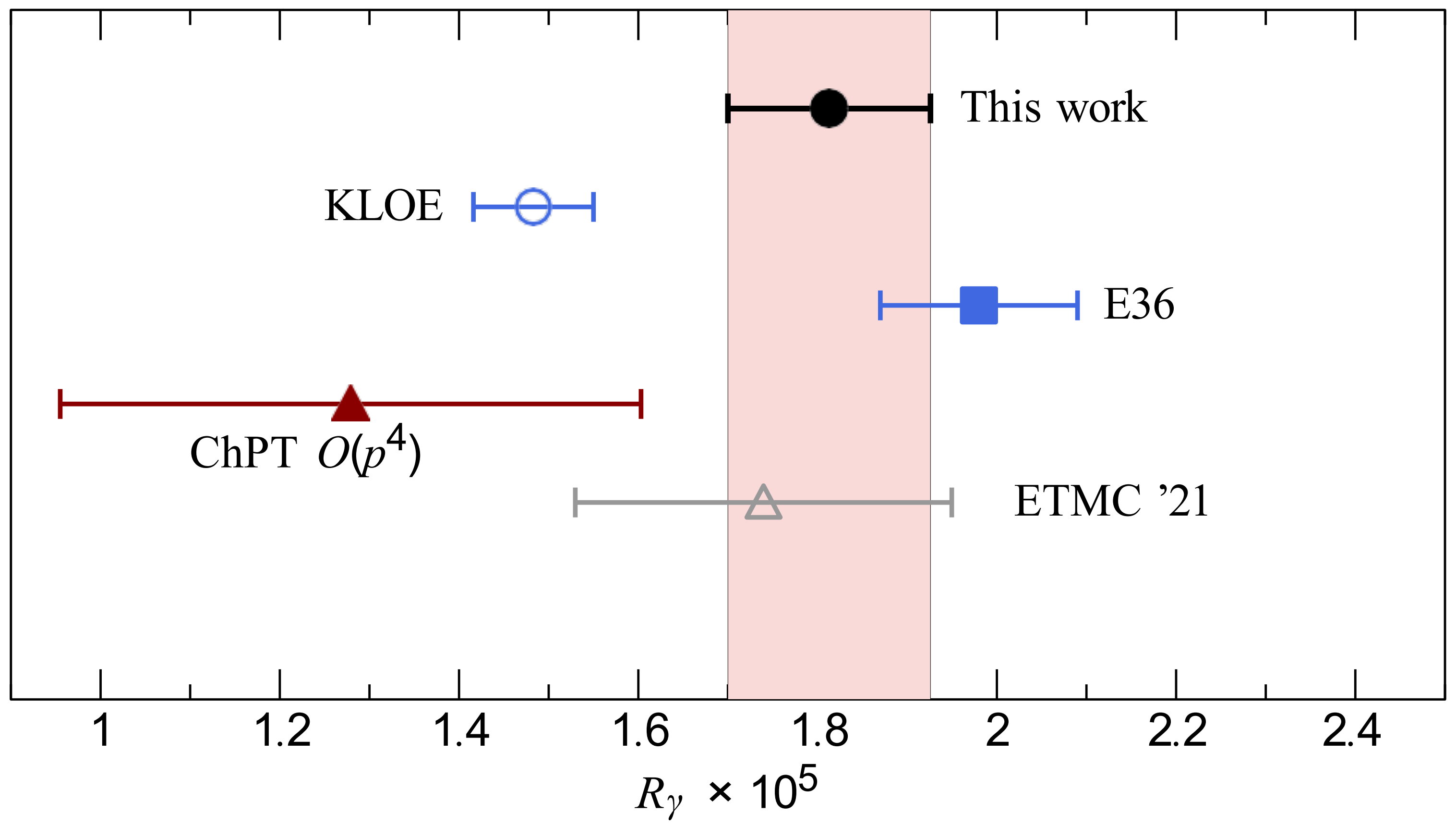
S.Simula et al., PoS Lattice 2021 (2022) 631

- E36 Result was subsequently updated to $(1.98 \pm 0.11) \times 10^{-5}$ (as in the figure above).

Comparing JPARC and KLOE's Results



J-PARC E36 Collaboration, A.Kobayashi et al., arXiv:2212.10702



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S.Simula et al., PoS Lattice 2021 (2022) 631

- E36 Result was subsequently updated to $(1.98 \pm 0.11) \times 10^{-5}$ (as in the figures above).
- Our result has been updated this year to $(1.84 \pm 0.12) \times 10^{-5}$ (as in the right-hand figure).

R. Di Palma et al., arXiv:2504.08680

D_s Decays

- In the paper discussed above, we have also computed the form factors for the D_s meson but only for $E_\gamma < 0.4\,\text{GeV}$.

- In a subsequent paper we have computed them over the full kinematic range.

R.Frezzotti, G.Gagliardi, V.Lubicz, G.Martinelli, F.Mazzetti, CTS, F.Sanfilippo, S.Simula, and N.Tantalo, arXiv:2306.05904

- The calculations were performed using four ETMC ensembles with $a \in [0.058, 0.09]\,\text{fm}$, three of which have approximately physical pion masses and the coarsest has $m_\pi = 174.5\,\text{MeV}$.
  - Sea Quarks - Wilson Clover TM Fermions and maximal twist
  - Valence Quarks - Osterwalder-Seiler Fermions
  - Physical m_s and m_c.

$\mathrm{D_s} \to \ell \nu_\ell \gamma$ - Results for the Form Factors

x_γ	F_A	ΔF_A	F_V	ΔF_V
0.1	0.0813	0.0054	-0.1048	0.0097
0.2	0.0715	0.0041	-0.0819	0.0028
0.3	0.0641	0.0033	-0.0643	0.0013
0.4	0.0582	0.0028	-0.0519	0.0008
0.5	0.0534	0.0021	-0.0431	0.0008
0.6	0.0495	0.0024	-0.0363	0.0008
0.7	0.0463	0.0031	-0.0316	0.0007
0.8	0.0432	0.0032	-0.0291	0.0010
0.9	0.0433	0.0083	-0.0297	0.0056
1.0	0.0489	0.0229	-0.0315	0.0152

- Appendix A for an explanation of why the errors grow at large x_γ.
- Discussion of method to reduce such errors studied in

D.Giusti et al., arXiv:2302.01298, arXiv:2505.11757

- Our Results for the form factors are well represented by the following VMD-inspired ansatz:

$$F_W(x_\gamma) = \frac{C_W}{\sqrt{R_W^2 + x_\gamma^2/4}\left(\sqrt{R_W^2 + x_\gamma^2/4} + x_\gamma/2 - 1\right)} + B_W$$

where $W = A, V$ and R_W, B_W and C_W are fit parameters.

- For single pole dominance $R_W = m_{\mathrm{res}}/m_{D_s}$ and $B_W = 0$.

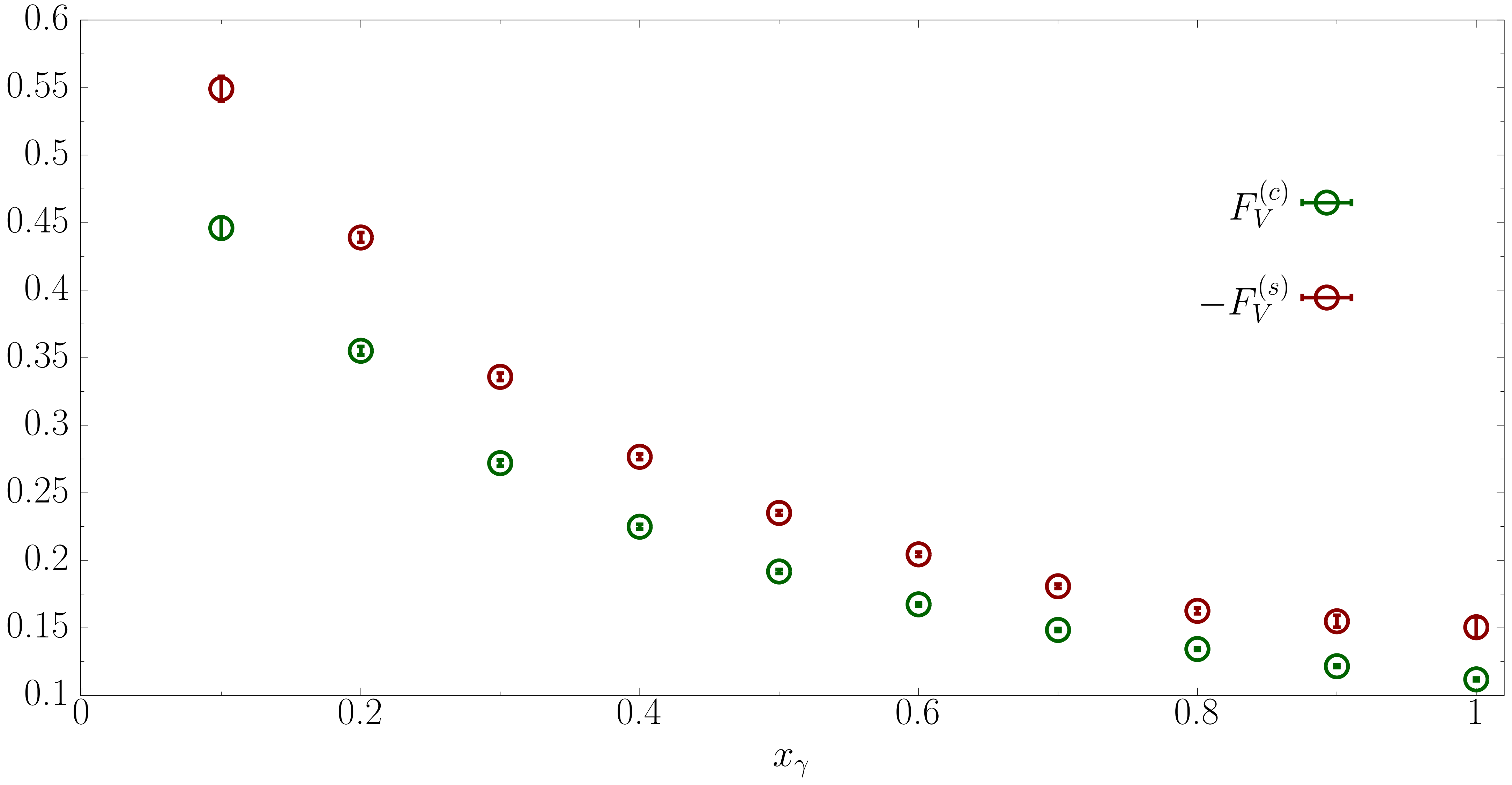
- For F_V we obtain stable results for C_V, and hence deduce the coupling $g_{D_s^* D_s \gamma}$ using

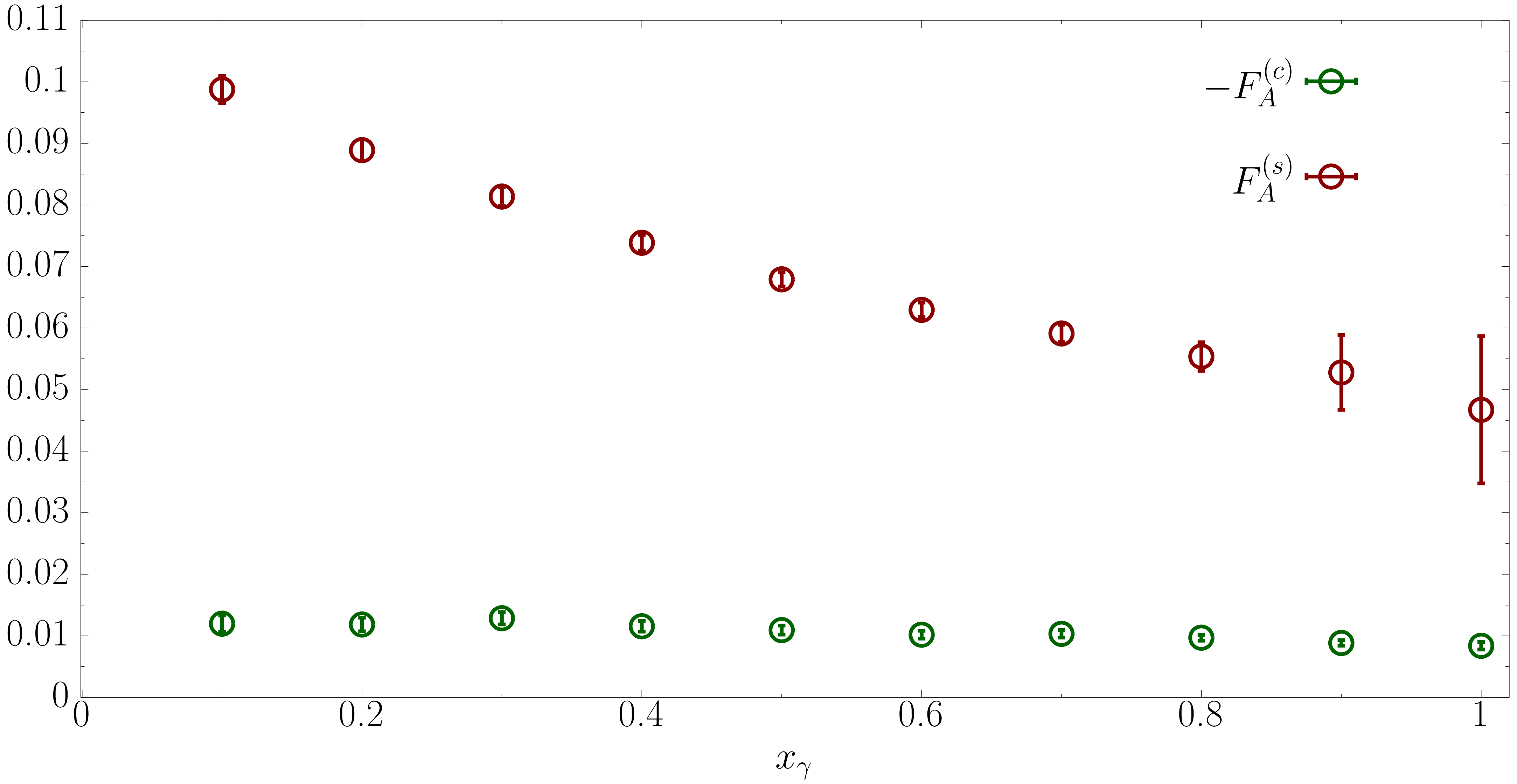
$$C_V = -\frac{m_{D_s^*} f_{D_s^*} g_{D_s^* D_s \gamma}}{2 m_{D_s}}$$

and $f_{D_s^*} = 268.6(6.6)$ MeV.

ETM Collaboration , V.Lubicz et al., arXiv 1707.04529

Cancellation in $F_V = F_V^{(c)} + F_V^{(s)}$





- There is a significant partial cancellation in F_V between the contributions from the emission of the photon from the strange and charm quarks.

- This had been observed previously by the HPQCD collaboration in their computation of the $D_s^* \to D_s\gamma$ decay amplitude.

HPQCD Collaboration , arXiv:1312.5264

	LCSR	HPQCD	This work
$g_{D_s^* D_s \gamma}$	0.60(19)	0.10(2)	0.118(13)
$g^{(s)}_{D_s^* D_s \gamma}$	1.0	0.50(3)	0.532(15)
$g^{(c)}_{D_s^* D_s \gamma}$	-0.4	-0.40(2)	-0.415(16)
$g^{(s)}_{D_s^* D_s \gamma}/g^{(c)}_{D_s^* D_s \gamma}$	-2.5	-1.25(10)	-1.282(61)

- $g_{D_s^* D_s\gamma}$ in GeV^{-1}

LCSR = B.Pullin and R.Zwicky, arXiv:2106.13617

$D_s \to \ell \nu_\ell \gamma$ — Conclusions

- We find $B(D_s \to e\nu_e\gamma) = 4.4(3) \times 10^{-6}$ for $E_\gamma > 10\,\text{MeV}$ in the rest frame of the D_s meson. This is consistent with the corresponding bound $B(D_s \to e\nu_e\gamma) < 1.3 \times 10^{-4}$ at 90% confidence level from BESIII (quoted in PDG).

- Even for photon energies as low as $10\,\text{MeV}$, we find that the Structure Dependent contribution dominates the branching fraction because of the strong helicity suppression of the point-like term by a factor of $(m_e/m_{D_s})^2$.
  - Such radiative decays therefore provide excellent test of the SM and Beyond.

- We use our results to test the validity and applicability of model dependent calculations.
  - LCSR calculations at NLO fail to reproduce our results for the form factors.

    B.Pullin and R.Zwicky, arXiv:2106.13617, J.Lyon and R.Zwicky, arXiv:1210.6546
  - Pure VMD parametrisation does not always reproduce the momentum dependence of the form factors.
  - There are also quark model predictions for the branching ratio in the range $10^{-3} - 10^{-5}$.