Mechanical and gravitational properties of hadrons from a hadronic perspective

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Outline

- Particles+Fields
- Energy momentum tensor
- Form factors
- Meson dominance
- Pion
- Nucleon
- Lattice
- Transversity
- Conclusions

Based on recent work

- Gravitational form factors of the pion and meson dominance
  Wojciech Broniowski, Enrique Ruiz Arriola
  Phys.Lett.B 859 (2024) 139138
  e-Print: 2405.07815 [hep-ph]
- Scalar and tensor meson dominance and gravitational form factors of the pion
  Enrique Ruiz Arriola, Wojciech Broniowski
  PoS QNP2024 (2025) 068
  e-Print: 2411.10354 [hep-ph]
- Transverse densities of the energy-momentum tensor and the gravitational form factors the pion
  Wojciech Broniowski, Enrique Ruiz Arriola
  Acta Physical Polonica B (in press)
  e-Print: 2412.00848 [hep-ph]
- Gravitational form factors and mechanical properties of the nucleon in a meson dominance approach
  Wojciech Broniowski, Enrique Ruiz Arriola
  e-Print: 2503.09297 [hep-ph]

... and older papers

- Meson dominance of hadron form factors and large-Nc phenomenology
  Pere Masjuan, Enrique Ruiz Arriola, Wojciech Broniowski
  Phys.Rev.D 87 (2013) 1, 014005
  e-Print: 1210.0760 [hep-ph]
- Scalar-isoscalar states in the large-N(c) Regge approach
  Enrique Ruiz Arriola, Wojciech Broniowski
  Phys.Rev.D 81 (2010) 054009
  e-Print: 1001.1636 [hep-ph]
- Gravitational and higher-order form factors of the pion in chiral quark models
  Wojciech Broniowski, Enrique Ruiz Arriola
  Phys.Rev.D 78 (2008) 094011
  e-Print: 0809.1744 [hep-ph]
- The Energy momentum tensor of chiral quark models at low energies
  E. Megias, E. Ruiz Arriola, L.L. Salcedo
  Phys.Rev.D 72 (2005) 014001
  e-Print: hep-ph/0504271 [hep-ph]
- Low-energy chiral Lagrangian in curved space-time from the spectral quark model
  E. Megias, E. Ruiz Arriola, L.L. Salcedo, W. Broniowski
  Phys.Rev.D 70 (2004) 034031
  e-Print: hep-ph/0403139 [hep-ph]

Some reviews

- Forces inside hadrons: pressure, surface tension, mechanical radius, and all that
  Maxim V. Polyakov , Peter Schweitzer
  Int.J.Mod.Phys.A 33 (2018) 26, 1830025
  e-Print: 1805.06596
- Pressure inside hadrons: criticism, conjectures, and all that
  Cédric Lorce, , Peter Schweitzer
  Acta Phys.Polon.B 56 (2025) 3-A17
  e-Print: 2501.04622 [hep-ph]
- Colloquium: Gravitational form factors of the proton
  V.D. Burkert et al.
  Rev.Mod.Phys. 95 (2023) 4, 041002
  e-Print: 2303.08347 [hep-ph]

Particle seismology

- A space-time vacuumquake: Flat metric changes

$$\eta^{\mu\nu} \to \eta^{\mu\nu} + \Delta g^{\mu\nu}(x)$$

- Masses of particles change locally

$$M \to M + \int d^4x \Delta g^{\mu\nu}(x) \frac{\delta M}{\delta g^{\mu\nu}(x)}$$

- Gravitational densities and stress

$$T_H^{\mu\nu}(x) = \frac{\delta M}{\delta g^{\mu\nu}(x)} \equiv \langle H|\Theta^{\mu\nu}(x)|H\rangle$$

- Hadron state is a wave packet so

$$|N\rangle = \sum_s \int d^4p \Psi_s(p)\delta_+(p^2 - M^2)|p, s\rangle \qquad \begin{cases} \delta_+(p^2 - M^2) = \theta(p \cdot n)\delta(p^2 - M^2) \\ \Theta^{\mu\nu}(x) = e^{iP\cdot x}\Theta^{\mu\nu}(0)e^{-iP\cdot x} \end{cases}$$

- is the on-shell spectral condition on a given hypersurface

$$\begin{aligned} T_H^{\mu\nu}(x) &= \int d^4p d^4p' e^{ix\cdot(p-p')}\delta^+(p'^2 - M^2)\delta^+(p^2 - M^2) \\ &\times \phi_s(p')^+ \langle p', s'|\Theta^{\mu\nu}(0)|p, s\rangle \phi_s(p) \end{aligned} \qquad (1)$$

- Gravitational form factor

$$\langle p', s'|\Theta^{\mu\nu}(0)|p, s\rangle = \sum_i O_i^{\mu\nu}(p', s', p, s) G_i(q^2)$$

PARTICLES+FIELDS

Energy momentum tensor primer

- System of point particles in external potential

$$m\frac{d^2 x_i}{dt^2} = -\nabla V(x_i)$$

- Density of particles

$$n(x,t) = \sum_i \delta(x - x_i(t)) \implies \partial_t n = -\sum \frac{dx_i}{dt}\nabla\delta(x - x_i(t))$$

- Current of particles (particle flux)

$$\vec{j}(x,t) \equiv \sum_i \frac{dx_i}{dt}\nabla\delta(x - x_i(t)) \implies \partial_t n + \nabla\cdot\vec{j} = 0$$

- Momentum density $\vec{\mathcal{P}}(x,t) = m\vec{j}(x,t)$

$$\partial_t\vec{\mathcal{P}}(x,t) = \sum_i m\frac{d^2\vec{x}_i}{dt^2}\delta(x - x_i(t)) + \sum_i \frac{d\vec{x}_i}{dt}\frac{d\vec{x}_i}{dt}\cdot\nabla\delta(x - x_i(t)) = -\nabla V(x)n(x,t) + m\nabla T(x,t)$$

- Stress tensor

$$T_{ab}(x,t) = \sum_i \frac{dx_i^a}{dt}\frac{dx_i^b}{dt}\delta(x - x_i(t))$$

- Energy density

$$\mathcal{H}(x,t) = \sum_i \frac{1}{2}m\left(\frac{d\vec{x}_i}{dt}\right)^2 + V(x)n(x,t) \implies \partial_t\mathcal{H}(x,t) = V(x)\nabla n(x,t) - \nabla\cdot J_E(x,t)$$

- Energy Flux (heat current)

Phase space view

- Hamiltonian dynamics

$$H(p,x) = E(p) + V(x) \implies \begin{cases} \dot{\vec{x}} = \nabla_p H = \nabla_p E \equiv \vec{v} \\ \dot{\vec{p}} = -\nabla_x H = -\nabla V(x) \end{cases}$$

- Phase space density

$$W(x,p,t) = \sum_i \delta(x - x_i(t))\delta(p - p_i(t)) \implies \partial_t W + \partial_p H \partial_x W - \nabla_x H \partial_p W = 0$$

- Poisson bracket

$$\{A,B\} \equiv \partial_x A \partial_p B - \partial_p A \partial_x B \implies \partial_t W + \{H,W\} = 0$$

- Local quantities

$$A(x,t) = \int dp A(x,p) W(x,p,t)$$

A(x,p)	1	p	H(x,p)	$p_i p_i$	$p_i H$
O(x,t)	n(x,t)	$\mathcal{P}(x,t)$	$\mathcal{H}(x,t)$	$T_{ij}(x,t)$	$J_E(x,t)$

Energy momentum tensor primer: relativistic

- Hamiltonian dynamics

$$H(p,x) = \sqrt{p^2 + m^2} + V(x) \implies \begin{cases} \dot{\vec{x}} = \nabla_p H = \frac{\vec{p}}{\sqrt{p^2+m^2}} \equiv \vec{v} \\ \dot{\vec{p}} = -\nabla_x H = -\nabla V(x) \end{cases}$$

- Energy and momentum densities

$$\begin{cases} \mathcal{H}(x,t) = \sum_i \sqrt{p_i^2 + m^2}\delta(x - x_i) + n(x,t)V(x) \\ \mathcal{P}(x,t) = \sum_i p_i \delta(x - x_i) \end{cases} \implies \partial_t \mathcal{H}(x,t) + \nabla\cdot\vec{\mathcal{P}}(x,t) = V(x)\partial_t n(x,t) = -V\nabla$$

The heat flux= Momentum density

- Stress tensor

$$T^{ab}(x,t) = \sum_i \frac{p_i^a p_i^b}{\sqrt{p_i^2 + m^2}}\delta(x - x_i(t)) \implies \partial_t \vec{\mathcal{P}}(x,t) + \nabla T(x,t) = -\nabla V(x) n(x,t)$$

- Energy momentum tensor

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{H} & \vec{\mathcal{P}} \\ \vec{\mathcal{P}} & T^{ab} \end{pmatrix} \implies \partial_\mu T^{\mu\nu} = f^\nu$$

- Interactions between relativistic particles ?
- No interaction theorem:it is impossible to construct a Hamiltonian or Lagrangian description of a system of interacting particles that is both relativistically invariant and contains non-trivial interactions.(Leutwyler 1965)

Electrodynamics and the field energy

- Particle dynamics : Lorentz force

$$\dot{\vec{p}} = q\left[\vec{E} + \vec{v} \wedge \vec{B}\right] \qquad \begin{cases} \rho(x,t) = \sum_i q_i \delta(x - x_i(t)) \\ \vec{J}(x,t) = \sum_i q_i v_i \delta(x - x_i(t)) \end{cases} \implies \partial_t \rho(x,t) + \nabla \cdot \vec{J}(x,t)$$

- Maxwell Field equations

$$\nabla \wedge E = -\partial_t B\,, \qquad \nabla \wedge B = \mu_0 \left(J + \epsilon_0 \partial_t E\right)\,, \qquad \nabla \cdot E = \frac{\rho}{\epsilon_0}\,, \qquad \nabla \cdot B = 0$$

- From Maxwell equations IN VACUUM we have a conservation law

$$\partial_t \mathcal{H} + \nabla \mathcal{P} = 0\,, \qquad \begin{cases} \mathcal{H} = \frac{1}{2}\left[\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right] \\ \vec{\mathcal{P}} = \frac{1}{\mu_0} E \wedge B \end{cases}$$

- How do we identify $\mathcal{H}$? We place matter and obtain

$$\frac{d}{dt}\int d^3x \mathcal{H} + \sum_i v_i F_i = 0 \implies \mathcal{H}_{\text{e.m.}} \text{energy density}$$

$$\frac{d}{dt}\int d^3x \mathcal{P} + \sum_i p_i = 0 \implies \mathcal{P}_{\text{e.m.}} \text{energy density}$$

Field and matter energies are ADDITIVE (no interaction) : Energy scattering

The Schrödinger field

- Schrödinger equation in a potential: Two constants of motion

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + V\psi \implies \begin{cases} \frac{d}{dt}\int d^3x|\psi|^2 = 0 \\ \frac{d}{dt}\int d^3x\left[\frac{1}{2m}|\nabla\psi|^2 + V(x)|\psi|^2\right] = 0 \end{cases}$$

- Probability density and probability flux

$$\begin{cases} n(x,t) = |\psi(x,t)|^2 \\ \vec{J}(x,t) = \frac{1}{2mi}\left[\psi^*\nabla\psi - \nabla\psi^*\psi\right] \end{cases} \implies \partial n(x,t) + \nabla\cdot\vec{J} = 0$$

- Energy density and heat flux

$$\begin{cases} \mathcal{H}(x,t) = \frac{1}{2m}|\nabla\psi|^2 + V(x)|\psi|^2 + \frac{1}{8m}\nabla^2|\psi|^2 \\ J_E(x,t) = \frac{1}{2mi}\left[\nabla\psi^*\nabla^2\psi - \nabla^2\psi^*\nabla\psi\right] \end{cases}$$

- Adding classical particles

$$\begin{cases} i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + \sum_i V(x-x_i)\psi \\ \dot{\vec{p}}_i = -\nabla\int d^3x V(x-x_i(t))|\psi(x,t)|^2 \end{cases} \implies \begin{cases} P = \sum_i p_i + \int d^3x m\vec{J}(x,t) \\ E = \sum_i \frac{p_i^2}{2M} + \int d^3x\left[\frac{1}{2m}|\nabla\psi|^2 + \sum_i V(x-x_i)|\psi|^2\right] \end{cases}$$

- Energy non-additive : Particle Scattering vs Energy Scattering

Neutral Klein-Gordon field

- Scalar neutral particle in a external field

$$(\partial_t^2 - \nabla^2 + m^2 + U)\phi = 0 \implies \partial_t \mathcal{H} + \nabla \vec{\mathcal{J}}_E = \partial_t U \frac{1}{2}\phi^2$$

- Energy density and energy flux

$$\mathcal{H} = \frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(m^2+U)\phi^2 \,, \qquad \vec{\mathcal{J}}_E = -\partial_t\phi\nabla\phi$$

- Energy norm $U + m^2 > 0$

$$||\phi||_E^2 = \int d^3x \mathcal{H} = \frac{1}{2}\int d^3x \left[(\partial_t\phi)^2 + (\nabla\phi)^2 + (m^2+U)(\phi)^2\right] \geq 0$$

- Energy scalar product

$$\langle\phi,\varphi\rangle_E = \frac{1}{2}\int d^3x \left[\partial_t\phi\partial_t\varphi + \nabla\phi\nabla\varphi + (m^2+U)\phi\varphi\right] \implies \frac{d}{dt}\langle\phi,\varphi\rangle_E = 0$$

- Wave packet

$$\phi(x,t) = \int \frac{d\omega}{\sqrt{2\pi}}\phi_\omega(x)e^{-i\omega t} \implies (-\nabla^2 + U)\phi_\omega = (\omega^2 - m^2)\phi_\omega$$

- Scattering solutions

$$\phi_\omega(x) \to Z(\omega)\left[e^{ik\cdot x} + \frac{e^{ikr}}{r}f\right]$$

Neutral Klein-Gordon: Energy Scattering

- Energy conservation

$$\Delta E = \int_{-\infty}^{\infty} dt \frac{dE}{dt} = \int_{-\infty}^{\infty} dt d^3x \partial_t \mathcal{H} = -\int dt d^3x \nabla \vec{\mathcal{J}}_E = -\int dt d\vec{S} \vec{\mathcal{J}}_E = -\int dt d\vec{S} \partial_t \phi \nabla \phi$$

- Wave packet with scattering boundary conditions

$$\Delta E = -\int d\omega \int d\vec{S} i\omega \phi_\omega(x) \nabla \phi_\omega(x)^* |Z(\omega)|^2 \to r^2 \int d\omega i\omega \int d\Omega \phi_\omega(x) \partial_r \phi_\omega(x)^*$$

- Optical theorem

$$0 = \Delta E = \int d\omega \omega |Z|^2 \left[-\frac{4\pi}{k} \mathrm{Im} f(\hat{k}, \hat{k}) + \int d\Omega |f(\hat{k}, \hat{x})|^2 \right] = 0$$

- Cross section as energy transfer (not probability transfer)

$$\frac{d\sigma}{d\Omega} = \frac{\Delta E_{\rm out}/\Delta\Omega}{\Delta E_{\rm in}/\Delta S} \implies \sigma_T = \frac{4\pi}{k} \mathrm{Im} f(\hat{k}, \hat{k})$$

- Optical theorem for neutral scalar particle has to do with energy and NOT probability conservation
- For monocromatic wave packets $|Z(\omega)^2| = A\delta(\omega - \omega_0)$ and $\Delta E_{\rm out} \sim \omega_0 \Delta N_{\rm out}$, $\Delta E_{\rm in} \sim \omega_0 \Delta N_{\rm in}$

$$\frac{d\sigma}{d\Omega} = \frac{\Delta E_{\rm out}/\Delta\Omega}{\Delta E_{\rm in}/\Delta S} = \frac{\int d\omega \omega |Z|^2 |f(\hat{k}, \hat{x})|^2}{\int d\omega \omega |Z|^2} = \frac{\Delta N_{\rm out}/\Delta\Omega}{\Delta N_{\rm in}/\Delta S}$$

Field Theory

The energy momentum tensor $\Theta_{\mu\nu}$ is the conserved Noether current corresponding to the symmetry under space-time translations

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu \implies \phi'(x') = \phi(x) \implies \delta\phi(x) = \epsilon^\mu \partial_\mu \phi$$

The invariance of the Lagrangian gives

$$\delta\mathcal{L}(x) = \epsilon^\mu \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\delta\partial^\mu \phi = \partial^\nu \left[\frac{\partial \mathcal{L}}{\partial \partial^\nu \phi}\right]\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\delta\partial^\mu \phi \implies \epsilon^\nu \partial^\mu \Theta_{\mu\nu} = 0\,,$$

For example for scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - U(\phi) \implies \Theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu}\mathcal{L}$$

The canonical of Noether EMT is NOT always symmetric.
How to measure $\Theta^{\mu\nu}$? Natural way coupling to gravity via a curved space time.
We take the Hilbert or metric EMT

$$\Theta^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\Big|_{g^{\mu\nu}=\eta^{\mu\nu}}\,, \qquad \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \implies \Theta^{\mu\nu} = \Theta^{\nu\mu}$$

Because of derivatives the quantum operator is badly divergent The improved EMT (Coleman+Callan+)

$$\bar\Theta^{\mu\nu} = \Theta^{\mu\nu} - \frac{1}{6}\left[\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2\right]\phi^2 \implies \Theta = \Theta^\mu_\mu$$

has the property that for $U(\phi) = m^2\phi^2/2 + g\phi^4/4!$ with $m = 0$ one has scale invariance and a trace anomaly after quantization

$$\bar\Theta = 0 \implies \partial^\mu D_\mu = \Theta^\mu_\mu = \beta(g)\frac{1}{4!}\phi^4$$

Lorentz properties

$$x^\mu \to \Lambda^\mu_\alpha x^\alpha \implies \Theta^{\mu\nu} \to \Lambda^\mu_\alpha \Lambda^\nu_\beta \Theta^{\alpha\beta}$$

The (Hilbert) EMT is conserved and symmetric but not irreducible.

$$\Theta^{\mu\nu} = \Theta^{\nu\mu} . \qquad \partial_\mu \Theta^{\mu\nu} = 0 \, , \implies \text{6 independent componentes.}$$

The trace is a scalar

$$\Theta \equiv \Theta^\mu_\mu$$

A naive decomposition

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}_S + \Theta^{\mu\nu}_T \equiv \frac{1}{4} g^{\mu\nu} \Theta + \left[\Theta^{\mu\nu} - \frac{1}{4} g^{\mu\nu} \Theta \right] \implies \partial_\mu \Theta^{\mu\nu}_S = \partial^\nu \Theta \neq 0$$

A consistent decomposition where two tensor components are conserved separately.

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}_S + \Theta^{\mu\nu}_T$$

with

$$\Theta^{\mu\nu}_S = \frac{1}{6} \left[g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} \right] \Theta \implies \partial_\mu \Theta^{\mu\nu}_S = 0$$

We will analyze lattice data using the consistent decomposition.

Ward identities

- The standard canonical approach is cumbersome and is plaggued with Schwinger terms. We consider the path integral approach

$$\langle O \rangle_S = \int D\phi O e^{iS[\phi]}$$

- For instance the time ordered product

$$\langle 0| T\left[\phi(x_1)\dots\phi(x_n)\right]|0\rangle = \langle \phi(x_1)\dots\phi(x_n)\rangle_S = \int D\phi\phi(x_1)\dots\phi(x_n)e^{iS[\phi]}$$

- Invariance under a transformation $\phi \to \phi + \delta\phi$

$$\delta\langle O \rangle_S = \langle \delta O \rangle_S + \langle i\delta S O \rangle_S = 0 \implies \langle \delta O \rangle_S = -i\langle O \delta S \rangle_S$$

- Functional Feynmann-Hellmann theorem

$$\langle 0| T\left[\delta\phi(x_1)\phi(x_2)\right]|0\rangle + \langle 0| T\left[\delta\phi(x_1)\phi(x_2)\right]|0\rangle = -i\langle 0| T\left[\phi(x_1)\phi(x_2)\delta S\right]|0\rangle$$

- For a symmetry transformation with a global group generator $\delta\phi(x) = \epsilon A\phi(x)$ with A an operator yields $\delta S = 0$.
- The quantum Noether construction with a local group generator $\epsilon(x)$ yields $\delta\phi(x) = \epsilon(x)A\phi(x)$ and $\delta S = \int d^4x\epsilon(x)\partial^\mu J_\mu$ yields

$$\delta(x-x_1)\langle 0| T\left[A\phi(x_1)\phi(x_2)\right]|0\rangle + \delta(x-x_2)\langle 0| T\left[\delta\phi(x_1)A\phi(x_2)\right]|0\rangle = -i\langle 0| T\left[\phi(x_1)\phi(x_2)\partial^\mu J_\mu(x)\right]|0\rangle$$

Gravitational Ward Identity

- Scalar field under general transformation $x \to x' = x + \epsilon(x)$

$$\phi'(x') = \phi(x) \implies \delta\phi(x) = -\epsilon^\mu \partial_\mu \phi(x) \implies \delta S = \int d^4x \epsilon^\mu \partial^\nu \Theta_{\mu\nu}$$

- Ward identity

$$\delta(x-x_1)\langle 0|T\left[\partial^\mu \phi(x_1)\phi(x_2)\right]|0\rangle + \delta(x-x_2)\langle 0|T\left[\phi(x_1)\partial^\mu \phi(x_2)\right]|0\rangle = -i\langle 0|T\left[\phi(x_1)\phi(x_2)\partial_\nu \Theta^{\mu\nu}(x)\right]|0\rangle$$

- Propagator

$$i\langle 0|T\left[\phi(x_1)\phi(x_2)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x_1-x_2)}\Delta(p)$$

- Unamputated vertex function

$$\Lambda^{\mu\nu}(p',p) = \int d^4x_1 d^2x_2 e^{ip'\cdot x_1} e^{-ip\cdot x_2}\langle 0|T\left[\phi(x_1)\phi(x_2)\Theta^{\mu\nu}(0)\right]|0\rangle$$

- Amputated vertex function

$$\Theta^{\mu\nu}(p',p) = D(p')^{-1}\Lambda^{\mu\nu}(p',p)D(p)^{-1}$$

- Ward identity

$$q_\mu \Theta^{\mu\nu}(p+q,p) = p^\nu \Delta^{-1}(p+q) - (p^\nu + q^\nu)\Delta^{-1}(p)$$

Gravitational Form Factor

- Definitions

$$P^\mu = \frac{1}{2}(p^\mu + p'^\mu) \, , \qquad q^\mu = p'^\mu - p^\mu$$

- On-shell conditions

$$p^2 = p'^2 = m^2 \implies \left\{ P \cdot q = 0 \quad P^2 = 4m^2 - q^2 \right.$$

- EM Conservation on shell

$$q_\mu \Theta^{\mu\nu}(p'p) = 0$$

- Gravitational form factors for spin-0 particle

$$\Theta^{\mu\nu}(p', p) \equiv \langle p' | \Theta^{\mu\nu}(0) | p \rangle = 2P^\mu P^\nu A(q^2) + \frac{1}{2} \left(q^\mu q^\nu - g^{\mu\nu} q^2 \right) D(q^2)$$

- Normalization from Ward identity (first $q \to 0$ and then on-shell $p^2 = m^2$

$$q_\mu \Theta^{\mu\nu}(p, p)|_{q \to 0} = 2p^\nu p\dot{q} \implies \Theta^{\mu\nu}(p, p)|_{p^2 = m^2} = 2p^\mu p^\nu \implies A(0) = 1$$

- The term $D(0)$ is free and is a fundamental quantity for hadrons

QCD

SEM tensor

$$\Theta^{\mu\nu} \quad = \quad \frac{i}{4}\bar{\Psi}\left[\gamma^{\mu}\overleftrightarrow{D}^{\mu}+\gamma^{\nu}\overleftrightarrow{D}^{\mu}\right]\Psi - F^{\mu\lambda a}F^{\nu}_{\lambda a}+\frac{1}{4}g^{\mu\nu}F^{\sigma\lambda a}F_{\sigma\lambda a}+\Theta^{\mu\nu}_{\mathrm{GF-EOM}}, \tag{2}$$

Trace Anomaly

$$\partial^{\mu}D_{\mu} = \Theta^{\mu}_{\mu} \equiv \Theta = \frac{\beta(\alpha)}{2\alpha}G^{\mu\nu a}G^{a}_{\mu\nu} + \sum_{q} m_q\left[1+\gamma_m(\alpha)\right]\bar{q}q. \tag{3}$$

Here $\beta(\alpha) = \mu^2 d\alpha/d\mu^2$ denotes the beta function, $\alpha = g^2/(4\pi)$ is the running coupling constant, $\gamma_m(\alpha) = d\log m/d\log\mu^2$ is the anomalous dimension of the current quark mass m_q, and $G^a_{\mu\nu}$ is the field strength tensor of the gluon field.
Breakup of hadron mass: (Ji 1995)

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}_q + \Theta^{\mu\nu}_g$$

Scale dependent decomposition.

$$\langle p|\Theta^{\mu\nu}|p\rangle = 2p^{\mu}p^{\nu}\left[\langle x\rangle_{\mathrm{q}}+\langle x\rangle_{\mathrm{g}}\right] \implies \langle x\rangle_{\mathrm{q}}+\langle x\rangle_{\mathrm{g}} = 1$$

In Deep Inelastic Scattering we have

$$\langle x\rangle^{\pi}_{\mathrm{val}} = \langle x\rangle^{N}_{\mathrm{val}} \sim 0.6 \qquad \mu = 2\mathrm{GeV}$$

We will not analyze the separate contributions here

Gravitational Form factors

- The EMT has matrix elements between hadronic states (helicity-normality basis) $|pJ\lambda N$

$$\langle p'j'\lambda'N'|\Theta^{\mu\nu}(0)|pj\lambda N\rangle = \sum_i \chi^\dagger_{j'\lambda'} O_i^{\mu\nu}(p',p)\chi^\dagger_{j\lambda} F_i(q^2)$$

The invariant functions $F_i(q^2)$ are the corresponding gravitational form factors.

- Mechanical interpretation: M. Polyakov 2003, Polyakov, Schweitzer 2018, Ji 2021, Lorce, Metz, Pasquini, Rodini 2021, ...

$$T_H^{\mu\nu}(x) = \langle H|\Theta^{\mu\nu}(x)|H\rangle \tag{4}$$

where $|H\rangle$ is a general wave packet,

$$|H\rangle = \sum_s \int d^4p \Psi_s(p)\delta_+(p^2-M^2)|p,s\rangle\,, \qquad \delta_+(p^2-M^2) = \begin{cases}\theta(p^+)\delta(p^2-M^2)\\ \theta(p_0)\delta(p^2-M^2)\end{cases}$$

- Space-like $x=(x_0,\vec{r})=(x^+,x^-,\vec{b})$, $x^2=x_0^2-\vec{r}^2=x^+x^- - \vec{b}^2<0$ one may use two popular choices

$$\text{TOMORROW} \qquad \begin{cases} x^+ = 0\,, & x^2=-\vec{b}^2\,, & T^{++}(\vec{b}),\, T^{+-}(\vec{b}),\, T^{ij}(\vec{b})\,, & \text{transverse} \\ x_0 = 0\,, & x^2 = -r^2\,, & T^{00}(\vec{r}),\, T^{0i}(\vec{r}),\, T^{ij}(\vec{r})\,, & \text{3D}\end{cases}$$

- D - Druck term= Intrinsic hadronic property (Polyakov+Weiss, 1999)

$$O_D^{\mu\nu}(p',p) = q^\mu q^\nu - g^{\mu\nu}q^2 \implies D(q^2)\,, \qquad D(0)$$

Pion GFF

- The spin-0 particle like the pion is the simplest case

$$\boxed{\langle \pi^a(p')|\Theta^{\mu\nu}(0)|\pi^b(p)\rangle = \delta_{ab}\left[2P^\mu P^\nu A(t) + \frac{1}{2}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)D(t)\right]}$$

a, b - isospin, $P = \frac{1}{2}(p' + p)$, $q = p' + p$, $t = q^2 = -Q^2$

- Trace form factor

$$\Theta^\mu_\mu \equiv \Theta(q^2) = 2\left(m_\pi^2 - \frac{q^2}{4}\right)A(q^2) - \frac{3}{2}q^2 D(q^2). \qquad (5)$$

- Raman decomposition (Raman:1971jg) conserved irreducible tensors corresponding to well-defined total angular momentum, $J^{PC} = 0^{++}$ (scalar) and 2^{++} (tensor)

$$\Theta^{\mu\nu} = \Theta_S^{\mu\nu} + \Theta_T^{\mu\nu}, \qquad \begin{cases} \Theta_S^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Theta \\ \Theta_T^{\mu\nu} = 2\left[P^\mu P^\nu - \frac{P^2}{3}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\right]A \end{cases} \underbrace{\Longrightarrow}_{q^2\to 0} 2A(0)m_\pi^2 = \Theta(0)$$

- Since Θ and A carry the information on good J^{PC} channels, they should be regarded as the primary objects, whereas the D-term form factor mixes the quantum numbers, with the explicit formula

$$D = -\frac{2}{3t}\left[\Theta - \left(2m_\pi^2 - \frac{1}{2}t\right)A\right], \qquad D_\pi(0) = -1 + \mathcal{O}(m_\pi^2), \qquad \text{(chiral theorem)}$$

Nucleon GFF

- Matrix elements

$$\langle p', s' | \Theta_{\mu\nu}(0) | p, s \rangle = \bar{u}(p', s') \Gamma_{\mu\nu} u(p, s)$$

- Gordon identity

$$2m\bar{u}' \gamma^\alpha u = \bar{u}' (2P^\alpha + i\sigma^{\alpha\rho} q_\rho) u,$$

- Three representations

$$\begin{aligned}
\Gamma_{\mu\nu} &= A(t)\, \gamma_{\{\mu} P_{\nu\}} + B(t)\, \frac{i\, P_{\{\mu}\sigma_{\nu\}\rho} q^\rho}{2m_N} + D(t)\, \frac{q_\mu q_\nu - g_{\mu\nu} q^2}{4m_N} \\
&= A(t) P_\mu P_\nu + J(t)\, i\, P_{\{\mu}\sigma_{\nu\}\rho} q^\rho + D(t)\, \frac{q_\mu q_\nu - g_{\mu\nu} q^2}{4} \\
&= 2J(t)\, \gamma_{\{\mu} P_{\nu\}} - B(t) \frac{P_\mu P_\nu}{m_N} + D(t)\, \frac{q_\mu q_\nu - g_{\mu\nu} q^2}{4m_N}
\end{aligned}$$

- Relations and normalizations

$$J(t) = \frac{1}{2}(A(t) + B(t)), \qquad A(0) = 1, \qquad B(0) = 1, \qquad J(0) = \frac{1}{2}, \qquad D(0) = ?$$

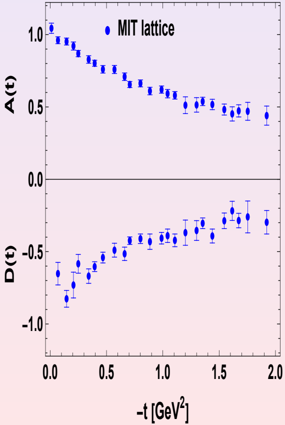
- Raman decomposition: Trace

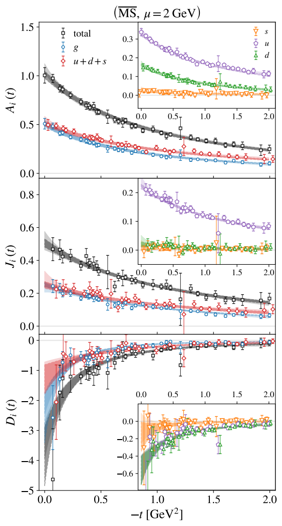
$$\Theta(t) = \frac{1}{m_N}\left[(m_N^2 - \frac{t}{4}) A(t) - \frac{3}{4} t D(t) + \frac{1}{2} t J(t) \right], \quad \Theta(0) = m_N, \qquad D(0) = \frac{4m_N}{3} \left[m_N A'(0) - \Theta'(0) \right]$$

$$m_N \Gamma_T^{\mu\nu} = \left[P^\mu P^\nu - \frac{P^2}{3} Q^{\mu\nu} \right] A(t) + \left[i\, P^{\{\mu} \sigma^{\nu\}\rho} q_\rho - \frac{t}{6} Q^{\mu\nu} \right] J(t),$$

MIT data

[Phys.Rev.D 108 (2023) 11, 114504 & D. Pefkou, PhD Thesis]
Unprecedented accuracy, both quarks and gluons, $m_\pi = 170$ MeV (SPACE-LIKE RESULTS)
(below the total q+g used, as it corresponds to the conserved current $\to$ renorm invariant)





PION VECTOR FORM FACTOR

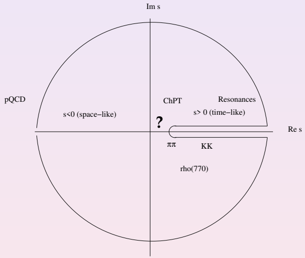
Dispersion relations and sum rules

Example: pion vector form factor

$$e^-\pi^+ \to e^-\pi^+\,, \qquad \langle \pi^+(p')|J_3^\mu(0)|\pi^+(p)\rangle = F_\pi(q^2)(p'^\mu + p^\mu)\,, \qquad q^2 < 0 \qquad \text{space-like}$$

$$e^+e^- \to \pi^+\pi^-\,, \qquad \langle \pi^+(-p')\pi^-(p)|J_3^\mu(0)|0\rangle = F_\pi(q^2)(p'^\mu + p^\mu)\,, \qquad q^2 > 4m_\pi^2 \qquad \text{time-like}$$

Analyticity: the two processes correspond to the same function in different domains



$$F(q^2) = F(q^2)^*\,, \quad q^2 < 0 \implies F(z^*) = F(z)^* \implies \mathrm{Disc}F(q^2) = 2i\mathrm{Im}F(q^2 + i\epsilon)\,, \qquad q^2 > 4m_\pi^2$$

Unitarity cuts: line $q^2 > 4m_\pi^2 \implies$ Two Riemann sheets $F_{\rm I}(s)$ and $F_{\rm II}(s)$
Resonances:

$$F_{\rm II}(s) = S_{\rm II}(s)F_{\rm I}(s) \implies F_{\rm II}(s) \to \frac{Z_R}{s - m_R^2 + im_R\Gamma_R} + \ldots$$

Large momentum behaviour (pQCD)

$$F(-Q^2) = \frac{16\pi F_\pi^2 \alpha_s(Q^2)}{Q^2} \sim \frac{1}{Q^2 \log Q^2} \underbrace{\to}_{Q^2 \to e^{-i\pi} s} \frac{1}{s(\log s - i\pi)} \implies \mathrm{Im} F(s) = -\frac{\pi}{s(\log s^2 + \pi^2)} < 0 \quad (7)$$

Unsubtracted Dispersion relations

$$F(-Q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im} F(s)}{s + Q^2}$$

Normalization

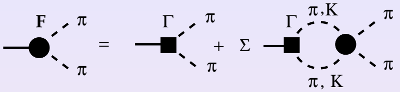
$$\boxed{F(0) = 1 = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im} F(s)}{s}}$$

Superconvergent sum rule (Donoghue:1996bt)

$$Q^2 F(-Q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds \mathrm{Im} F(s) + \mathcal{O}\left(\frac{1}{\log Q^2}\right) \implies \boxed{\int_{4m_\pi^2}^{\infty} ds \mathrm{Im} F(s) = 0} \implies \mathrm{Im} F(s) \quad \text{changes sign}$$

t-channel unitarity

- Bethe-Salpeter (coupled channel)



- Watson's theorem: $\pi\pi$ scattering in $J = I = 1$ channel

$$F(s) = |F(s)| e^{i\delta_{11}(s)} \implies \boxed{\mathrm{Im} F(s) = |F(s)| \sin\delta_{11}(s) > 0}\,, \qquad 4m_\pi^2 < s < 4m_K^2$$

- Threshold behaviour

$$\delta_{11}(s) \sim a_{11}(s/4 - m_\pi^2)^{\frac{3}{2}} \implies \mathrm{Im} F(s) \sim |F(4m_\pi^2)| a_{11}(s/4 - m_\pi^2)^{\frac{3}{2}}$$

- Omnes-Mushkelisvili solution in the spacelike region

$$F(-Q^2) = \exp\left[-\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{Q^2}{s}\,\frac{\delta_{11}(s)}{s+Q^2}\right] \sim \underbrace{\rightarrow}_{\Gamma_\rho \to 0} = \frac{m_\rho^2}{m_\rho^2 + Q^2}$$

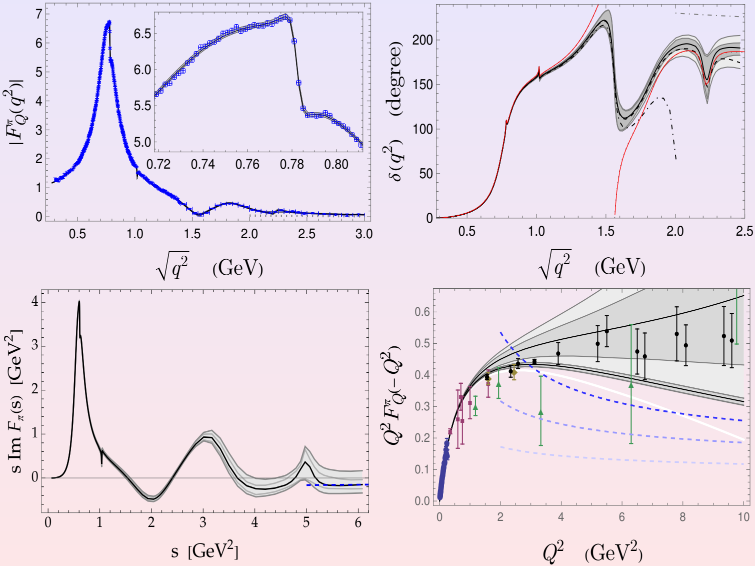
- Question of modeling/using the spectral density

$$\rho(s) = \frac{1}{\pi}\mathrm{Im} F(s) = \begin{cases} \rho_{\mathrm{ChPT}}(s) & 4m_\pi^2 \le s \le 16 m_\pi^2\,, & \text{threshold region} \\ \rho_{\mathrm{R}}(s) & 16 m_\pi^2 \le s \le \Lambda_{\mathrm{pQCD}}^2\,, & \text{resonance region} \\ \rho_{\mathrm{pQCD}}(s) & \Lambda_{\mathrm{pQCD}}^2 \le s \le \infty\,, & \text{pQCD region} \end{cases}$$

- What would be a reasonable Λ_{pQCD} ?

Analysis of Babar

ERA, Pablo Sanchez-Puertas (RuizArriola:2024gwb) for $\sim 3m_\pi \leq \sqrt{s} \leq 3\mathrm{GeV}$, DR: $|F(s)| \to \arg F(s)$



The incompleteness problem

ERA, Pablo Sanchez-Puertas, Christian Weiss (2025 Transverse dist)

- The maximum Babar $s_{\max} = 9\mathrm{GeV}^2$

$$\frac{1}{\pi}\int_{s_0}^{s_{\max}} ds \frac{\operatorname{Im} F(s)}{s}\Big|_{\rm Data} = 1.01(1)_{\rm st}(^{+2}_{-1})_{\rm syst},$$

$$\frac{1}{\pi}\int_{s_0}^{s_{\max}} ds \operatorname{Im} F(s)\Big|_{\rm Data} = 0.63(2)_{\rm st}(^{+7}_{-4})_{\rm syst}\ \mathrm{GeV}^2. \qquad m_\rho^2 = 0.6\mathrm{GeV}^2$$

- The pQCD part extrapolated

$$\frac{1}{\pi}\int_{s_{\max}}^{\infty} ds \frac{\operatorname{Im} F(s)}{s}\Big|_{\rm pQCD} = -\underbrace{0.0025}_{\rm LO} - \underbrace{0.0011}_{\rm NLO} - \underbrace{0.0006}_{\rm NNLO},$$

$$\frac{1}{\pi}\int_{s_{\max}}^{\infty} ds \operatorname{Im} F(s)\Big|_{\rm pQCD} = -\underbrace{0.114}_{\rm LO} - \underbrace{0.030}_{\rm NLO} - \underbrace{0.013}_{\rm NNLO}\ \mathrm{GeV}^2.$$

- Superconvergence is a theorem but pQCD is far away
- Solution: subtractions (but need constants independently)

$$F(-Q^2) = 1 - Q^2 F'(0) + \frac{1}{\pi}\left[\int_{4m_\pi^2}^{s_{\max}} + \int_{s_{\max}}^{\infty}\right] ds \frac{Q^4}{s^2}\frac{\mathrm{Im}F(s)}{s+Q^2}, \qquad \text{Last term} \quad \mathcal{O}(Q^4/s_{\max}^2)$$

- Space-like looks very much as Vector-Meson Dominance

$$F(-Q^2) = \frac{m_\rho^2}{m_\rho^2 + Q^2} \Longrightarrow J_3^\mu = f_\rho m_\rho^2 \rho_3^\mu, \qquad \text{current-field identity (Sakurai)}$$

- Space-like physics is INDEPENDENT of time-like details.

Extended meson dominance

- Generalized Current field identity

$$J_V^\mu = \sum_{\rho,\rho',\ldots} f_V M_V^2 V^\mu \implies F_V(t) = \sum_V c_V \frac{M_V^2}{M_V^2 - t}, \qquad c_V = f_V g_{V\pi\pi}$$

- Short distance constraints

$$F_V(t) \sim \frac{\sum_T c_T m_V^2}{Q^2} + \ldots$$

- Normalization

$$F_V(0) = 1 = \sum_T c_T$$

- Minimal hadronic ansatz

$$F_V(t) = \frac{m_\rho^2}{m_\rho^2 - t}$$

- Improved hadronic ansatz

$$F_V(t) = (1 + at)\frac{m_\rho^2}{m_\rho^2 - t}\frac{m_{\rho'}^2}{m_{\rho'}^2 - t}$$

Current-field identities for conserved SEM tensor

- Saturation with O^{++} and 2^{++} isoscalar states (Krolikowski:1967ryy,Raman:1970wq,Raman:1971ur) (Raman decomposition manifest)

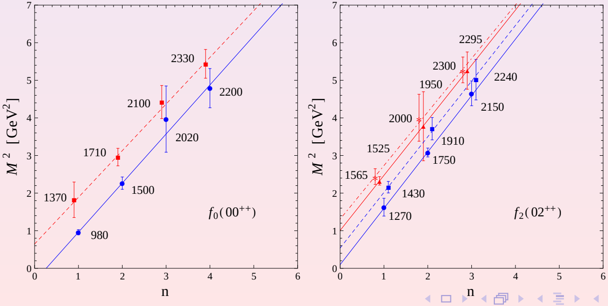
$$\Theta^{\mu\nu} = \sum_S \frac{1}{3} f_S \left(\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2 \right) S + \sum_T f_T m_T^2 T^{\mu\nu} ,$$

- Matrix elements

$$\langle A | \Theta^{\mu\nu} | B \rangle = \sum_S \frac{f_S}{3} \frac{g^{\mu\nu} q^2 - q^\mu q^\nu}{m_S^2 - q^2 - i\epsilon} \langle A | J_S | B \rangle + \sum_T f_T \frac{m_T^2}{m_T^2 - q^2 - i\epsilon} \langle A | \sum_\lambda \epsilon_\lambda^{\mu\nu} \epsilon_{\alpha\beta}^\lambda J_T^{\alpha\beta} | B \rangle$$

- PDG resonances follow radial regge trajectories (Masjuan:2012gc)

$$M_{nJ}^2 = a(n + J) + b$$



PION GRAVITATIONAL FORM FACTOR

Spectral Properties

- pQCD

$$A(t) = -3D(t)\,(1+\mathcal{O}(\alpha)) = -\frac{48\pi\alpha(t) f_\pi^2}{t}\,(1+\mathcal{O}(\alpha))\,,$$

- Watson's theorem implies $4m_\pi^2 < s < 4m_K^2$

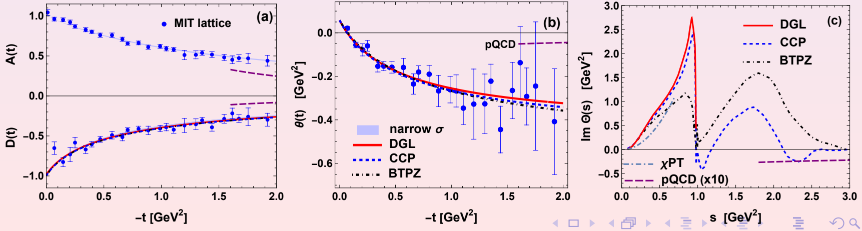
$$\mathrm{Im}\Theta(s) = |\Theta(s)| \sin\delta_{00}(s)\,, \qquad \mathrm{Im}A(s) = |A(s)| \sin\delta_{02}(s)$$

Question of modeling/using the spectral density

$$\rho(s) = \begin{cases} \rho_{\rm ChPT}(s) & 4m_\pi^2 \le s \le 16m_\pi^2 \\ \rho_{\rm R}(s) & 16m_\pi^2 \le s \le \Lambda_{pQCD}^2 \\ \rho_{\rm pQCD}(s) & \Lambda_{pQCD}^2 \le s \end{cases}$$

- Meson dominance ($m_\pi = 170\mathrm{MeV}$)

$$A^*(-Q^2) = \frac{m_{f_2}^{*2}}{m_{f_2}^{*2} + Q^2}\,, \qquad \Theta^*(-Q^2) = 2m_\pi^{*2} - \frac{m_\sigma^{*2} Q^2}{m_\sigma^{*2} + Q^2}\,.$$



Finite widths at space-like momenta

Energy dependent Breit-Wigner parametrization

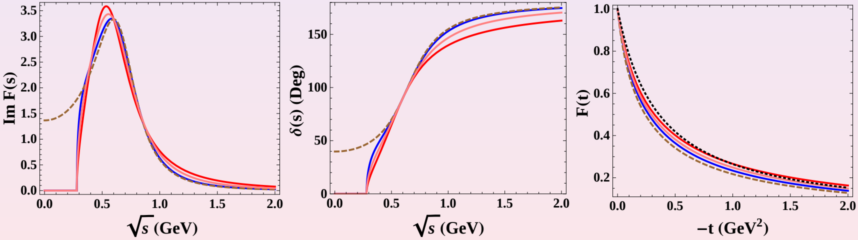
$$N(s) = M^2 - s + i\Gamma M \Gamma(s) \qquad S(s) = e^{2i\delta(s)} = \frac{N(s)}{N(s)^*} \implies \delta(M^2) = \frac{\pi}{2}$$

Resonance = Pole in the second Riemann sheet

$$1/S_{II}(s_R) = S_I(s_R) = 0$$

Omnes representation complies with Watson's theorem

$$F(t) = \exp\left[\frac{t}{\pi}\int_{4m_\pi^2} \frac{ds}{s}\frac{\delta(s)}{s-t}\right] \qquad F(0) = 1 \implies \frac{F(t+i0)}{F(t-i0)} = e^{2i\delta(s)}$$



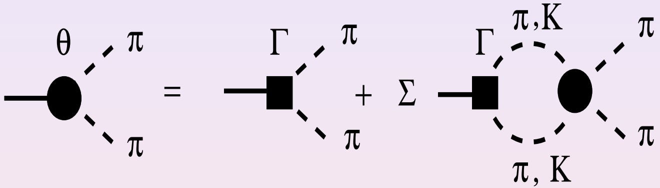
Even for a broad S-wave resonance the Form Factor resembles a monopole for space-like momenta

$$F(t) \sim \frac{M^2}{M^2 - t}$$

Hadronic representacion

For two coupled channels

$$\begin{pmatrix} \Theta_\pi(s) \\ \Theta_K(s) \end{pmatrix} = \begin{pmatrix} \Gamma_\pi(s) \\ \Gamma_K(s) \end{pmatrix} + \begin{pmatrix} T_{\pi\pi\to\pi\pi}(s) & T_{\pi\pi\to KK}(s) \\ T_{KK\to\pi\pi}(s) & T_{KK\to KK}(s) \end{pmatrix} \begin{pmatrix} \Delta_{\pi\pi}(s) & 0 \\ 0 & \Delta_{KK}(s) \end{pmatrix} \begin{pmatrix} \Gamma_\pi(s) \\ \Gamma_K(s) \end{pmatrix}$$



Watson's final state theorem

$$F = \Gamma + VG_0 F = \Gamma + TG_0\Gamma \implies \mathrm{Im}F(s) = \mathrm{Im}\left[T(s)G_0(s)\right]\Gamma(s) \implies F(t) = F(0) + \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{t}{s}\frac{\mathrm{Im}F(s)}{s-t}$$

The poles of the FF in the second Riemann sheet coincide with the resonances of the S-matrix.

$$\Theta_{\mathrm{II}}(s) = S_{\mathrm{II}}(s)\Theta_{\mathrm{I}}(s)$$

Chiral extrapolation from $m_\pi = 170$MeV to physical

- In order to relate *different* pion masses a mass independent renormalization scheme is needed, such as $\overline{\rm MS}$ in chiral perturbation theory

$$\theta^{(0)}_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L}^{(0)}, \quad (8)$$

$$\theta^{(2)}_{\mu\nu} = \frac{f^2}{4}\langle D_\mu U^\dagger D_\nu U\rangle - \eta_{\mu\nu}\mathcal{L}^{(2)}, \quad (9)$$

$$\begin{aligned}\theta^{(4)}_{\mu\nu} &= -\eta_{\mu\nu}\mathcal{L}^{(4)} + 2L_4\langle D_\mu U^\dagger D_\nu U\rangle\langle\chi^\dagger U + U^\dagger\chi\rangle + L_5\langle D_\mu U^\dagger D_\nu U + D_\nu U^\dagger D_\mu U\rangle\langle\chi^\dagger U + U^\dagger\chi\rangle \\ &- 2L_{11}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle D_\alpha U^\dagger D^\alpha U\rangle - 2L_{13}\left(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\langle\chi^\dagger U + U^\dagger\chi\rangle \\ &- L_{12}\left(\eta_{\mu\alpha}\eta_{\nu\beta}\partial^2 + \eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\mu\alpha}\partial_\nu\partial_\beta - \eta_{\nu\alpha}\partial_\mu\partial_\beta\right)\langle D^\alpha U^\dagger D^\beta U\rangle\,, \end{aligned} \quad (10)$$

- We compute Θ and A in ChPT and obtain from MIT lattice

$$10^3\cdot L_{11}(m_\rho^2) = 1.06(15)\,,\qquad 10^3\cdot L_{12}(m_\rho^2) = -2.2(1)\,,\qquad 10^3\cdot L_{13}(m_\rho^2) = -0.7(1.1).$$

- This implies for $m_\pi = 140$ MeV yields

$$m^*_\sigma = 0.65(3)\to m_\sigma = 0.63(6)\,,\qquad m^*_{f_2} = 1.24(3)\to m_{f_2} = 1.27(4)$$

- Druck term at $m_\pi = 140$MeV.

$$D(0) = -0.95(3)$$

NUCLEON GRAVITATIONAL FORM FACTOR

Spectral Properties

- Large Q^2

$$A(t) \sim +\frac{\alpha(t)^2}{(-t)^2}\,, \qquad J(t) \sim +\frac{\alpha(t)^2}{(-t)^2}\,, \qquad B(t) \sim -\frac{\alpha(t)^2}{(-t)^3}\,, \qquad , D(t) \sim -\frac{\alpha(t)^2}{(-t)^3}.$$

- Large s

$$\mathrm{Im}\,A(s) \sim +\frac{1}{s^2 L^3}\,, \qquad \mathrm{Im}\,J(s) \sim +\frac{1}{s^2 L^3}\,, \qquad \mathrm{Im}\,B(s) \sim +\frac{1}{s^3 L^3},\,, \qquad \mathrm{Im}\,D(s) \sim +\frac{1}{s^3 L^3},$$

- Watson's theorem $4m_\pi^2 < s < 4m_K^2$ (Raman decomposition: helicity-flip $\pi\pi \to N\bar{N}$) where $\sigma_\pi = \sqrt{1 - 4m_\pi^2/t}$.

$$\mathrm{Im}\,\Theta(t) = \frac{3\sigma_\pi |f_{0,+}(t)||\Theta_\pi(t)|}{2(4m_N^2 - t)} > 0,$$

$$\mathrm{Im}\,J(t) = \frac{3t^2\sigma_\pi^5}{64\sqrt{6}}|f_{2,-}(t)||A_\pi(t)| > 0,$$

$$\mathrm{Im}\,A(t) + \frac{2t\mathrm{Im}J(t)}{4m_N^2 - t} = \frac{3t^2 m_N \sigma_\pi^5}{32\sqrt{6}}|f_{2,+}(t)||A_\pi(t)| > 0,$$

- Unsubtracted disperion relations
- Superconvergence sum rules

Meson dominance I

- Normalization

$$A(0) = 1\,, \qquad B(0) = 0\,, \qquad \Theta(0) = m_N$$

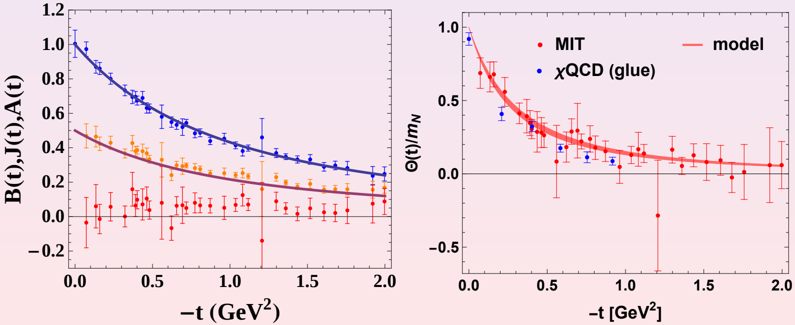
- High energy behaviour

$$A(t) \sim \frac{\alpha^2}{t^2}\; B(t) \sim\sim \frac{\alpha^2}{t^3}\,, \qquad , \qquad \Theta_N(t) \sim \frac{\alpha^2}{t^2}$$

- Minimal hadronic ansatz

$$A(t) = 2J(t) = \frac{1}{(1 - t/m_{f_2}^2)(1 - t/m_{f_2'}^2)}\,, \qquad B(t) = 0\,, \qquad \Theta(t) = \frac{m_N}{(1 - t/m_\sigma^2)(1 - t/m_{f_0}^2)}$$

We use PDG for masses (NO FIT) and and $m_\sigma = 650(50)\mathrm{MeV}$ (Consistent with pion)



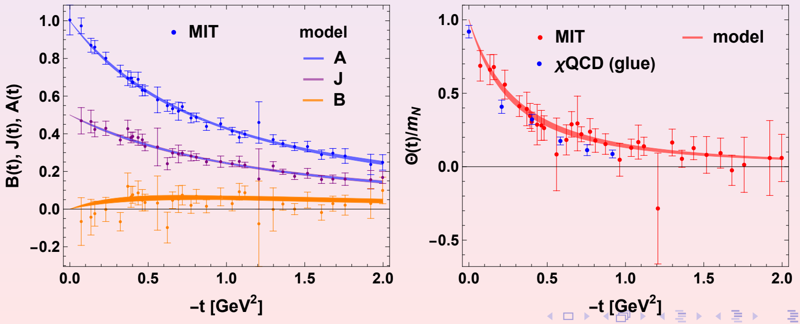
Meson dominance II

$$\Theta(t) = \frac{m_N}{(1 - t/m_\sigma^2)(1 - t/m_{f_0}^2)},$$

$$A(t) = \frac{1 - c_A t + c_2 t^2}{(1 - t/m_{f_2}^2)(1 - t/m_{f_2'}^2)(1 - t/m_{f_2''}^2)(1 - t/m_{f_2'''}^2)},$$

$$J(t) = \frac{1 - c_J t + c_2 t^2}{2(1 - t/m_{f_2}^2)(1 - t/m_{f_2'}^2)(1 - t/m_{f_2''}^2)(1 - t/m_{f_2'''}^2)}. \tag{11}$$

We use PDG and fit c_J, c_A, c_2 and $m_\sigma = 650(50)\mathrm{MeV}$ (Consistent with pion)



Summary part I

1. Lattice results for gravitational ff of the pion and nucleon fully compatible with meson dominance at "intermediate" values of Q^2
2. Important to look at the data in good spin channels - all expected features satisfied
3. Matter radius larger due to small $\sigma-$mass $m_\sigma = 0.64(4)$GeV.

$$\langle r^2 \rangle_{\theta,\pi} = \frac{6}{m_\sigma^2} = \qquad \langle r^2 \rangle_{\theta,N} = \frac{6}{m_\sigma^2} + \frac{6}{m_{f_0}^2} = [0.90(4)\mathrm{fm}]^2$$

4. $D(t)$ (the Druck term) is a combination of good spin form factors

$$D_\pi(0) = -0.95(3) \qquad D_N(0) = -3.0(4)$$

5. Higher Q^2 desired approach pQCD ... Modeling involves the broad σ meson!
6. This was already expected [Masjuan, ERA, WB, 2013]

One sees mesons all over the lattice!

TRANSVERSITY

Interpretation of Form factors

- Charge form factor of relativistic particle (spin 0)

$$\langle p'|J^\mu(0)|p\rangle = (p'^\mu + p^\mu)F(q^2) \implies q^\mu \langle p'|J^\mu(0)|p\rangle = 0 \,, \qquad F(0) = 1$$

- This is NOT an expectation value of ANYTHING. For a normalized state —H⟩ we have the expectation value of the current

$$J_H^\mu(x) = \langle H|J^\mu(x)|H\rangle$$

- A wave packet

$$|H\rangle = \int d^4p\Psi(p)\delta_+(p^2 - M^2)|p,s\rangle \,, \qquad \delta_+(p^2 - M^2) = \begin{cases} \theta(p^+)\delta(p^2 - M^2) \\ \theta(p_0)\delta(p^2 - M^2) \end{cases}$$

- Using translational invariance $J^\mu(x) = e^{iP\cdot x}J^\mu(0)e^{-iP\cdot x}$

$$\begin{aligned} J_H^\mu(x) &= \int d^4pd^4p' e^{ix\cdot(p-p')}\delta^+(p'^2 - M^2)\delta^+(p^2 - M^2)\phi(p')^+\langle p'|J^\mu(0)|p\rangle\phi(p) \\ &= \int d^4pd^4p' e^{ix\cdot(p-p')}\delta^+(p'^2 - M^2)\delta^+(p^2 - M^2)\phi(p')^+(p'^\mu + p^\mu)F(q^2) \qquad (12) \end{aligned}$$

Interpretation of Form factors: Equal time

- Assuming covariant normalization

$$\langle p'|p\rangle = (2\pi)^3 2E\delta(\vec{p}' - \vec{p}) , \qquad p^0 = E = \sqrt{p^2 + m^2}$$

- Equal time wave packet :

$$|H\rangle = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E}} \phi(p)|\vec{p}\rangle \implies \langle H|H\rangle = \int d^3p|\phi(p)|^2 = \int d^3r|\phi(r)|^2$$

- Current density

$$J_H^\mu(\vec{r}, t) = \int \frac{d^3p d^3p'}{(2\pi)^3} e^{-i\vec{r}\cdot(\vec{p}-\vec{p}') + i(E-E')t} \phi(\vec{p}')^* \phi(\vec{p}) \frac{(p'^\mu + p^\mu)}{\sqrt{2E2E'}} F(q^2) \tag{13}$$

- Static charge density

$$\rho(\vec{x}, 0) = J_H^0(\vec{r}, 0) = \int \frac{d^3p d^3p'}{(2\pi)^3} e^{-i\vec{r}\cdot(\vec{p}-\vec{p}')} \phi(\vec{p}')^* \phi(\vec{p}) \frac{(E' + E)}{\sqrt{2E2E'}} F((E - E')^2 - \vec{q}^2) \tag{14}$$

- Heavy particles $E - E' = \sqrt{\vec{p}^2 + m^2} - \sqrt{\vec{p}'^2 + m^2} \to 0$

$$\begin{aligned} \rho_H(\vec{x}, 0) &= \int \frac{d^3p d^3p'}{(2\pi)^3} e^{-i\vec{r}\cdot(\vec{p}-\vec{p}')} \phi(\vec{p}')^* \phi(\vec{p})^* \phi(\vec{p}') F(-\vec{q}^2) \\ &= \int d^3r|\phi(\vec{r} - \vec{r}')|^2 \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}'} F(-\vec{q}^2) \underbrace{\to}_{\text{loc.}} \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{x}} F(-\vec{q}^2) \end{aligned}$$

- The form factor is the Fourier transform of the density ONLY for heavy particles

Interpretation of Form factors: Light cone coordinates

- We take the conventions $p^{\pm} = (p^0 \pm p^3)/\sqrt{2} = p_{\mp}$, such that $x \cdot p = p^+ x^- + p^- x^+ - p_\perp \cdot x_\perp$ and $d^4 p = dp^+ dp^- d^2 p_\perp$. Also, $g^{++} = g^{--} = 0$ and $g^{+-} = g^{-+} = 1$.
- Wave packet

$$|\phi\rangle = \int \frac{d^2 p_\perp dp^+}{(2\pi)^3 2p^+} \tilde{\phi}(p_\perp, p^+) |p_\perp, p^+\rangle. \tag{15}$$

- From here we have the scalar product

$$\begin{aligned} \langle \phi | \psi \rangle &= \int \frac{d^2 p_\perp dp^+}{(2\pi)^3 2p^+} \tilde{\phi}(p_\perp, p^+)^* \tilde{\psi}(p_\perp, p^+) \\ &= \int d^2 x_\perp dx^- \phi(x_\perp, x^-)^* \psi(x_\perp, x^-). \end{aligned} \tag{16}$$

- The coordinate and momentum representations are related via the Fourier transform,

$$\psi(x_\perp, x^-) = \int \frac{d^2 p_\perp dp^+}{\sqrt{(2\pi)^3 2p^+}} \tilde{\psi}(p_\perp, p^+) e^{i(x_\perp \cdot p_\perp - p^+ x^-)}. \tag{17}$$

- Integrate over the x^- coordinate in the local operator, and define the transverse wave packet distribution in the transverse coordinate $b = x_\perp$,

$$n_\psi(b) = \int dx^- |\psi(b, x^-)|^2 = \int_0^\infty \frac{dp^+}{4\pi p^+} \left| \int \frac{d^2 p_\perp}{(2\pi)^2} e^{ib \cdot p_\perp} \tilde{\psi}(p_\perp, p^+) \right|^2. \tag{18}$$

Interpretation of Form factors: Light cone coordinates

- We consider the $x^+ = 0$ quantization surface. Using translational invariance, $O(x) = e^{iP\cdot x}O(0)e^{-iP\cdot x}$, and after some straightforward manipulations, one obtains the intuitive formula for the expectation value of the electromagnetic current J^μ,

$$\langle\psi|\int dx^- J^+(b,x^-)|\psi\rangle = \int d^2b' n_\psi(b-b')F(b'), \quad (19)$$

- $F(b)$ is the Fourier transform of the charge form factor in the space-like momentum space,

$$F(b) = \int \frac{d^2q_\perp}{(2\pi)^2} F(-q_\perp^2)e^{-iq_\perp\cdot b}. \quad (20)$$

- For a localized wave packet $n_\psi(b) \to \delta^{(2)}(b)$ and $n^+_\psi(b) \to p^+\delta^{(2)}(b)$, hence one has

$$\langle\psi|\int dx^- J^+(b,x^-)|\psi\rangle \to F(b) \quad (21)$$

- Transverse charge density is invariant

Transverse densities

- We consider
  Assuming covariant normalization

$$\langle p'_T p^{+\prime} | p_T p^+ \rangle = (2\pi)^2 2p^+ \delta(\vec{p}'_T - \vec{p}_T)\delta(p'^+ - p^+)\,, \qquad p^- = \frac{\sqrt{p_T^2 + m^2}}{2p^+}$$

- Equal time wave packet :

$$|H\rangle = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E}} \phi(p)|\vec{p}\rangle \implies \langle H|H\rangle = \int d^3p |\phi(p)|^2 = \int d^3r |\phi(r)|^2$$

- Current density

$$J_H^\mu(\vec{r}, t) = \int \frac{d^3p d^3p'}{(2\pi)^3} e^{-i\vec{r}\cdot(\vec{p}-\vec{p}')+i(E-E')t} \phi(\vec{p}')^* \phi(\vec{p}) \frac{(p'^\mu + p^\mu)}{\sqrt{2E2E'}} F(q^2) \qquad (22)$$

- Static charge density

$$\rho(\vec{x}, 0) = J_H^0(\vec{r}, 0) = \int \frac{d^3p d^3p'}{(2\pi)^3} e^{-i\vec{r}\cdot(\vec{p}-\vec{p}')} \phi(\vec{p}')^* \phi(\vec{p}) \frac{(E' + E)}{\sqrt{2E2E'}} F((E - E')^2 - \vec{q}^2) \qquad (23)$$

- Heavy particles $E - E' = \sqrt{\vec{p}^2 + m^2} - \sqrt{\vec{p}'^2 + m^2} \to 0$

$$\begin{aligned}\rho_H(\vec{x}, 0) &= \int \frac{d^3p d^3p'}{(2\pi)^3} e^{-i\vec{r}\cdot(\vec{p}-\vec{p}')} \phi(\vec{p}')^* \phi(\vec{p})^* \phi(\vec{p}') F(-\vec{q}^2) \\ &= \int d^3r |\phi(\vec{r} - \vec{r}')|^2 \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}'} F(-\vec{q}^2) \underbrace{\to}_{\text{loc.}} \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{x}} F(-\vec{q}^2)\end{aligned}$$

- The form factor is the Fourier transform of the density ONLY for heavy particles

Positivity

- LC spinors

$$\Psi_\pm = \mathcal{P}_\pm \Psi \qquad \mathcal{P}_\pm = \gamma^0\gamma^\pm = (1 \pm \gamma^0\gamma^3)/\sqrt{2} \implies \mathcal{P}_+ + \mathcal{P}_- = 1 \qquad \mathcal{P}_\pm^2 = \mathcal{P}_\pm = \mathcal{P}_\pm^\dagger$$

  and $\mathcal{P}_\pm \mathcal{P}_\mp = 0$
- In QCD, the EM current and SEM in LC coordinates and with the gauge $A^+ = 0$ (which is ghost free), one has

$$\begin{aligned} J^+ &= \Psi_+^\dagger Q \Psi_+ , \\ \Theta_q^{++} &= \frac{i}{2}\left(\Psi_+^\dagger \partial^+ \Psi_+ - \partial^+ \Psi_+^\dagger \Psi_+\right), \quad \Theta_g^{++} = (\partial^+ A_\perp^a)^2 , \\ \Theta^{++} &= \Theta_q^{++} + \Theta_g^{++} . \end{aligned} \tag{24}$$

- The field expansion for the quark field in the transverse coordinate space [**?**] at $x^+ = 0$ is

$$q_+(b, x^-) = \int_0^\infty \frac{dp^+}{4\pi p^+} \sum_\lambda \qquad [b_\lambda(b, p^+) u_{\lambda,+}(p^+) e^{-ip^+x^-} + d_\lambda^\dagger(b, p^+) v_{\lambda,+}(p^+) e^{ip^+x^-}], \tag{25}$$

  with $b_\lambda^\dagger(b, p^+)$ and $d_\lambda^\dagger(b, p^+)$ denoting the particle and antiparticle creation operators with LC helicity λ

Positivity

$$\int dx^- q_+^\dagger q_+ = \sum_\lambda \int \frac{dp^+}{4\pi p^+} \left[n(b,p^+) - \bar{n}_\lambda(b,p^+) \right] ,$$

$$\int dx^- q_+^\dagger i\partial^+ q_+ = \sum_\lambda \int \frac{dp^+}{4\pi p^+} \left[p^+ n_\lambda(b,p^+) - p^+ \bar{n}_\lambda(b,p^+) \right] , \quad (26)$$

with $n_\lambda(b,p^+) = b_\lambda^\dagger(b,p^+) b_\lambda(b,p^+)$ and $\bar{n}_\lambda(b,p^+) = d_\lambda^\dagger(b,p^+) d_\lambda(b,p^+)$ denoting the particle and antiparticle number operators, respectively. Thus, for $\pi^+ = u\bar{d}$,

$$\int dx^- J^+(b,x^-) \underbrace{\to}_{\pi^+} \sum_\lambda \int \frac{dp^+}{4\pi p^+} \left[\frac{2}{3} n_{u,\lambda}(b,p^+) + \frac{1}{3} n_{\bar{d},\lambda}(b,p^+) \right] , \quad (27)$$

since generally $q_+^\dagger q_+$ is positive for quarks and negative for antiquarks. Thus (27), and consequently $F(b)$, are positive definite. For Θ_q^{++} one also finds positivity,

$$\int dx^- \frac{i}{2} \left(\Psi_+^\dagger \partial^+ \Psi_+ - \partial^+ \Psi_+^\dagger \Psi_+ \right) = i \int dx^- \Psi_+^\dagger \partial^+ \Psi_+$$

$$\underbrace{\to}_{\pi^+} \sum_\lambda \int \frac{dp^+}{4\pi p^+} \left[p^+ n_{u,\lambda}(b,p^+) + p^+ n_{\bar{d},\lambda}(b,p^+) \right] \quad (28)$$

Dispersion relations

- Space like form factors

$$F(-Q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}F(s)}{s+Q^2}, \tag{29}$$

- Sum rules

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}f(s)}{s} = 1, \tag{30}$$

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds\, \mathrm{Im}f(s) = 0. \tag{31}$$

- Transverse density

$$\begin{aligned} \rho(b) &= \int \frac{d^2 q_T}{(2\pi)^2} e^{iq_T\cdot b} F(-q_T^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds \mathrm{Im}F(s) \int \frac{d^2 q_T}{(2\pi)^2} \frac{e^{iq_T\cdot b}}{s+Q^2} \\ &= \frac{1}{2\pi^2}\int_{4m_\pi^2}^{\infty} ds K_0(b\sqrt{s})\, \mathrm{Im}\, F(s), \end{aligned} \tag{32}$$

- Pion Vector form factor: Transverse charge distribution

$$F_\pi(t) = \frac{m_\rho^2}{m_\rho^2 - t} \implies \mathrm{Im}F_\pi(s) = \pi(s - m_\rho^2) \implies \rho_\pi(b) = \frac{1}{2\pi^2} K_0(bm_\rho)$$

Transverse distributions

- $A(b)$ as the relative distribution of P^+ in the transverse coordinate space

$$\Theta^{++}(b) = \int \frac{d^2 q_\perp}{2P^+(2\pi)^2} e^{-iq_\perp \cdot b} 2P^{+2} A(q_\perp^2) = P^+ A(b) \qquad \int d^2 b\, \Theta^{++}(b) = P^+ \tag{33}$$

- Transverse energy density

$$\Theta^{+-}(b) = \int \frac{d^2 q_\perp}{2P^+(2\pi)^2} e^{-iq_\perp \cdot b} \left[2P^+ P^- A(q_\perp^2) + \frac{1}{2} q_\perp^2 D(q_\perp^2) \right], \tag{34}$$

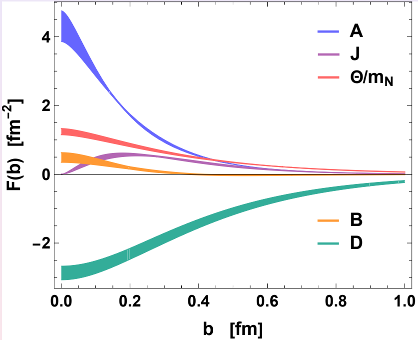
- $p(b)$ is the transverse pressure and $s(b)$ denotes the transverse shear forces

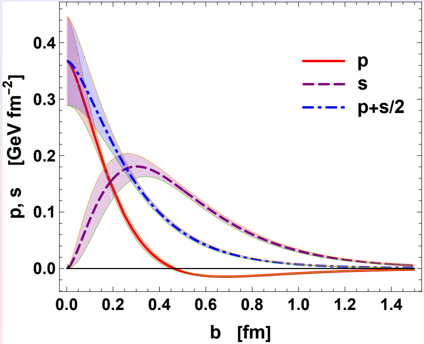
$$\Theta^{ij}(b) = \frac{1}{2P^+} \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-iq_\perp \cdot b} \frac{1}{2} \left[q_\perp^i q_\perp^j - \delta^{ij} q_\perp^2 \right] D(q_\perp^2) = \delta^{ij} p(b) + \left[\frac{b^i b^j}{b^2} - \frac{1}{2}\delta^{ij} \right] s(b)$$

- The trace GFF is

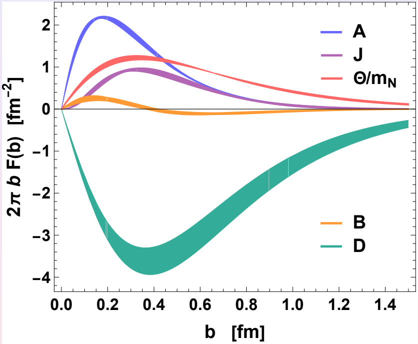
$$\begin{aligned} \Theta^\mu_\mu(b) &= 2\Theta^{+-}(b) - \Theta^{11}(b) - \Theta^{22}(b) = \epsilon(b) - 2p(b) \\ & \frac{1}{2P^+} \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-iq_\perp \cdot b} \left[2(m_\pi^2 + \tfrac{1}{4} q_\perp^2) A(q_\perp^2) + \frac{3}{2} q_\perp^2 D(q_\perp^2) \right] \\ &= \frac{1}{2P^+} \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-iq_\perp \cdot b} \Theta(q_\perp^2) = \frac{1}{2P^+} \Theta(b). \end{aligned} \tag{35}$$

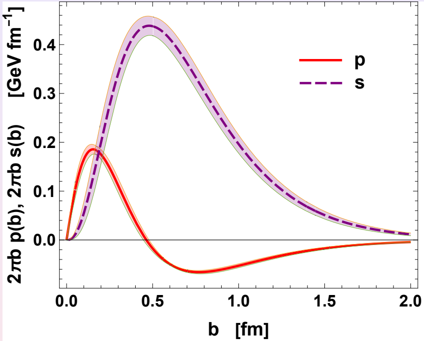
Transverse densities and mechanical in meson dominance





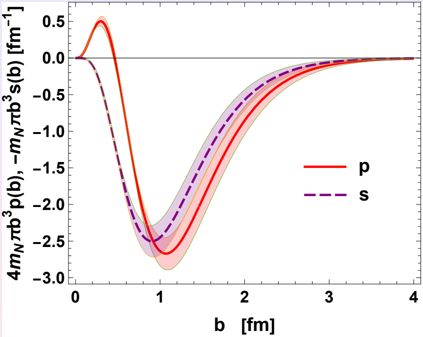
$2\pi b \times$ densities/mechanical





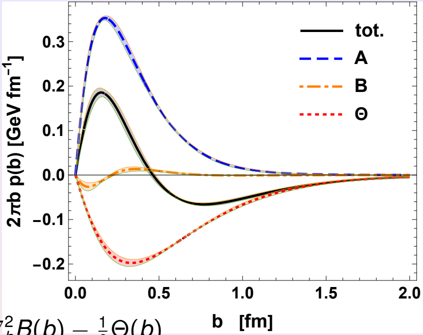
$\int_0^\infty 2\pi b \, db \, p(b) = 0$

$2\pi b^3 \times$ mechanical



quantities integrate to $D(0) = -3.0(4)$

Anatomy of pressure



$p(b) = \frac{m_N}{6} A(b) + \frac{1}{24 m_N} \nabla_b^2 B(b) - \frac{1}{6} \Theta(b)$

2^{++} repulsion in the core, 0^{++} repulsion in the tail [cf. Ji, Yang 2025, Fujii, Kawaguchi, Tanaka 2025]
In meson dominance it simply reflects the hierarchy of masses

Transverse radii

$$\langle b^2\rangle_F = \frac{\int_0^\infty 2\pi b\, b^2 F(b)}{\int_0^\infty 2\pi b\, F(b)} = \frac{4}{F(0)} \left.\frac{dF(t)}{dt}\right|_{t=0}$$

In our model

$$\langle b^2\rangle_A = 4\left(-c_A + \frac{1}{m_{f_2}^2} + \frac{1}{m_{f_2'}^2} + \frac{1}{m_{f_2''}^2} + \frac{1}{m_{f_2'''}^2}\right) = [0.34(1)\ \mathrm{fm}]^2$$

c_A approximately cancels the contribution $1/m_{f_2''}^2 + 1/m_{f_2'''}^2$

$$\langle b^2\rangle_\Theta = 4\left(\frac{1}{m_\sigma^2} + \frac{1}{m_{f_0}^2}\right) = [0.60(3)\ \mathrm{fm}]^2$$

$$\langle b^2\rangle_{\mathrm{mech}} = \frac{\int_0^\infty 2\pi b\, b^2[p(b) + \frac{1}{2}s(b)]}{\int_0^\infty 2\pi b[p(b) + \frac{1}{2}s(b)]} = \frac{4D(0)}{\int_0^\infty d(-t)D(t)} = [0.48(3)\ \mathrm{fm}]^2$$

Hierarchy reflects the meson mass pattern

$$\langle b^2\rangle_A < \langle b^2\rangle_{\mathrm{mech}} < \langle b^2\rangle_\Theta$$

$2D \leftrightarrow 3D$

For N – Abel transform [Panteleeva, Polyakov 2021, Freese, Miller 2021], for π 3D makes no sense

Radii hierarchy

$$\langle r^2\rangle_A^{1/2} < \langle r^2\rangle_J^{1/2} < \langle r^2\rangle_E^{1/2} < \langle r^2\rangle_{\rm mech}^{1/2} < \langle r^2\rangle_\Theta^{1/2}$$
$$0.51(1) < 0.57(3) < 0.67(2) < 0.72(5) < 0.90(4)[\rm fm]$$

$b = 0$ sum rules

$$F(r=0) = -\frac{1}{4\pi^2}\int_{4m_\pi^2}^\infty ds\sqrt{s}\,{\rm Im}\,F(s)$$

$$J^{\rm mon}(r) = -\frac{2}{3}J^{\rm quad}(r) = \frac{r^2}{36\pi^2}\int_{4m_\pi^2}^\infty ds\, s^{3/2}{\rm Im}\,J(s) + \mathcal{O}(r^3)\ \text{[Lorce et al. 2017]}$$

$$p(r=0) = -\frac{1}{24\pi^2 m_N}\int_{4m_\pi^2}^\infty ds\, s^{3/2}\,{\rm Im}\,D(s)$$

$$\left.\frac{ds(r)}{dr^2}\right|_{r=0} = \frac{1}{240\pi^2 m_N}\int_{4m_\pi^2}^\infty ds\, s^{5/2}\,{\rm Im}\,D(s)$$

...

Summary tables

Pion

quantity	low limit		intermediate range	high limit	
$\mathrm{Im}\, A(s)$	$+$	2π	changes sign	$-$	pQCD
$\mathrm{Im}\, D(s)$	$-$		changes sign	$+$	
$\mathrm{Im}\, \Theta(s)$	$+$		changes sign	$-$	
$A(-Q^2)$	1	sym.		$+$	pQCD
$D(-Q^2)$	$-1 + \mathcal{O}(m_\pi^2)$			$-$	
$\Theta(-Q^2)$	$2m_\pi^2$		changes sign	$-$	
$A(b)$	$+\infty$	pQCD	positive definite	$+$	2π
$\Theta(b)$	$-\infty$		changes sign	$+$	
$p(b)$	$+\infty$		changes sign	$-$	

Nucleon

quantity	low limit		intermediate range	high limit	
$\mathrm{Im}\, A(s)$	$+$	2π	changes sign	$+$	pQCD
$\mathrm{Im}\, J(s)$	$+$		changes sign	$+$	
$\mathrm{Im}\, B(s)$	$+$		changes sign	$+$	
$\mathrm{Im}\, D(s)$	$-$		changes sign	$+$	
$\mathrm{Im}\, \Theta(s)$	$+$		changes sign	$-$	
$A(-Q^2)$	1	sym.		$+$	pQCD
$J(-Q^2)$	$\frac{1}{2}$			$+$	
$B(-Q^2)$	0			$-$	
$D(-Q^2)$				$-$	
$\Theta(-Q^2)$	m_N		changes sign	$-$	
$A(b)$	+		positive definite	$+$	2π
$\Theta(b)$				$+$	
$p(b)$			changes sign	$-$	

Conclusions

1. Gravitational form factors provide insight on the mass and forces inside hadrons.
2. They are related to GPSs as low moment
3. MIT lattice benchmark data have prodived high accuracy GFFs for nucleon and pion directly.
4. Lattice results for gravitational ff of the pion and nucleon fully compatible with meson dominance at “intermediate” values of Q^2
5. Important to look at the data in good spin channels - all expected features satisfied
6. Matter radius larger due to small $\sigma-$mass $m_\sigma = 0.64(4)$GeV.

$$\langle r^2\rangle_{\theta,\pi} = \frac{6}{m_\sigma^2} = \qquad \langle r^2\rangle_{\theta,N} = \frac{6}{m_\sigma^2} + \frac{6}{m_{f_0}^2} = [0.90(4)\mathrm{fm}]^2$$

7. $D(t)$ (the Druck term) is a combination of good spin form factors

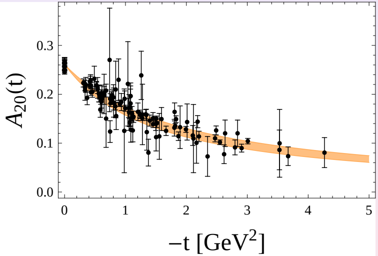
$$D_\pi(0) = -0.95(3) \qquad D_N(0) = -3.0(4)$$

8. Transverse distributions are intrinsice properties of hadrons and meson dominance provides a clear description for transverse distances larger than $b > 0.1$fm
9. This was already expected [Masjuan, ERA, WB, 2013] and extents to all known form factors

BACK-UP SLIDES

Early estimates

Chiral quark models: $\langle r_2 \rangle_A = \frac{1}{2} \langle r_2 \rangle_{EM}$ - mass distribution more compact than charge [WB, ERA, 2008]

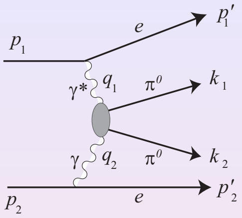


Lattice [Brommel 2007] vs meson dominance [Masjuan, ERA, WB, Phys.Rev.D 87 (2013) 1, 014005]

$(A_{20}(t) \equiv \frac{1}{2} A_q(t)$ - quark part)

At that time $D_q(t)$ very noisy, no gluons

Determination from the Belle data



[Belle, 2015]

[Kumano, Song, Teryaev, 2015] (GDAs, quark parts only) $\rightarrow$

$$\langle r^2 \rangle_A = (0.32 - 0.39 \text{ fm})^2$$
$$\langle r^2 \rangle_D = (0.82 - 0.88 \text{ fm})^2$$

recall $\langle r^2 \rangle_{EM} = (0.656 \pm 0.005 \text{ fm})^2$ (PDG 2021)

(case of A in line with our earlier quark model estimate)

Resonance Saturation with narrow states

$$\langle \pi\pi | \Theta^{\mu\nu} | 0 \rangle = \sum_R \langle \pi\pi | R \rangle \frac{1}{m_R^2 - q^2} \langle R | \Theta^{\mu\nu} | 0 \rangle, \tag{36}$$

Problems with subtractions for higher spin $J \geq 2$ particles.
We take the absortive part and use the disersion relation with pertinent subtractions

$$\frac{1}{\pi} \mathrm{Im} \langle \pi\pi | \Theta^{\mu\nu} | 0 \rangle = \sum_R \langle \pi\pi | R \rangle \langle R | \Theta^{\mu\nu} | 0 \rangle \delta(m_R^2 - s), \tag{37}$$

$$\begin{aligned} \langle S | \Theta^{\mu\nu} | 0 \rangle &= f_S (g^{\mu\nu} q^2 - q^\mu q^\nu)/3, \\ \langle T | \Theta^{\mu\nu} | 0 \rangle &= f_T m_T^2 \epsilon_\lambda^{\mu\nu}, \end{aligned} \tag{38}$$

where $\epsilon_\lambda^{\mu\nu}$ is the spin-2 polarization tensor, which is symmetric $\epsilon_\lambda^{\mu\nu} = \epsilon_\lambda^{\nu\mu}$, traceless $g_{\mu\nu}\epsilon_\lambda^{\mu\nu} = 0$, and transverse $q_\mu \epsilon_\lambda^{\mu\nu} = 0$. The extra factor 3 in the definition is conventional such that $\langle S | \Theta | 0 \rangle = f_S m_S^2$. The *on-shell* couplings of the resonances to the $\pi\pi$ continuum are taken as

$$\begin{aligned} \langle S | \pi\pi \rangle &= g_{S\pi\pi}, \\ \langle T | \pi\pi \rangle &= g_{T\pi\pi} \epsilon_\lambda^{\alpha\beta} P^\alpha P^\beta = g_{T\pi\pi} \epsilon_\lambda^{\alpha\beta} p'^\alpha p^\beta . \end{aligned} \tag{39}$$

Thus, we get

$$\frac{1}{\pi} \mathrm{Im} \langle \pi\pi | \Theta^{\mu\nu} | 0 \rangle = \sum_S \frac{g_{S\pi\pi} f_S}{3} \delta(m_S^2 - q^2)(g^{\mu\nu} q^2 - q^\mu q^\nu) + \sum_{T,\lambda} \epsilon_\lambda^{\alpha\beta} P^\alpha P^\beta \epsilon_\lambda^{\mu\nu} g_{T\pi\pi} f_T \delta(m_T^2 - q^2), \tag{40}$$

which naturally complies with separate conservation for each contribution when contracting with q^μ.

The sum over tensor polarizations is given by

$$\sum_\lambda \epsilon_\lambda^{\alpha\beta}\epsilon_\lambda^{\mu\nu} = \frac{1}{2}\left(X^{\mu\alpha}X^{\nu\beta} + X^{\nu\alpha}X^{\mu\beta}\right) - \frac{1}{3}X^{\mu\nu}X^{\alpha\beta}, \quad (41)$$

with $X^{\mu\nu} = g^{\mu\nu} - q^\mu q^\nu/q^2$, hence the on-shell condition $P\cdot q = 0$ implies $P_\alpha X^{\alpha,\beta} = P^\beta$ and we get

$$\sum_\lambda \epsilon_\lambda^{\alpha\beta}P_\alpha P_\beta \epsilon_\lambda^{\mu\nu} = P^\mu P^\nu - \frac{1}{3}(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2})P^2 \quad (42)$$

(cf. the tensor structure in Eq. (**??**)). Therefore, in the narrow resonance, large-N_c motivated approach

$$\begin{aligned} \frac{1}{\pi}\mathrm{Im}A(s) &= \frac{1}{2}\sum_T g_{T\pi\pi}f_T\delta(m_T^2 - q^2), \\ \frac{1}{\pi}\mathrm{Im}\Theta(s) &= \sum_S g_{S\pi\pi}f_S m_S^2\delta(m_S^2 - q^2), \end{aligned} \quad (43)$$

where, as expected, A and Θ get contributions exclusively from the 2^{++} and 0^{++} states, respectively.