From para-positronium to para-charmonium

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1 Introduction

2 Composite model

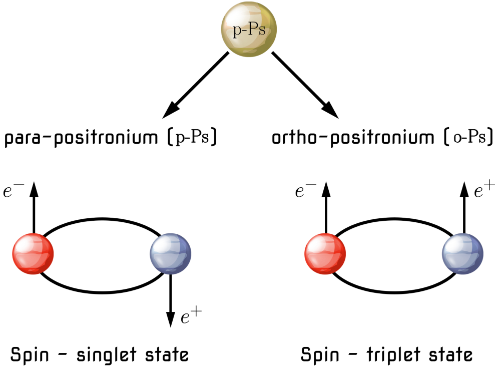
3 Summary

Motivation

- QFT techniques for positronium.
- QFT corrections (on top of non-rel results) should be small: is it true?
- Can we learn something about the QFT treatment out of the comparison with the known positronium results?
- Can QFT tells us something on its own interesting about positronium?
- Positronium shares some similarities with the pion.
- Para-positronium and para-charmonium: similarities in structure, differences in dynamics – can both be captured in one approach?

Introduction:positronium

Positronium (Ps): non-relativistic electron-positron bound state



Introduction: para-positronium

PARA-POSITRONIUM (p-Ps)

Mass of positronium

$$2m_e - \left(\frac{\alpha^2 m_e}{4}\right)$$

m_e- mass of the electron $\alpha-$ fine structure constant

Quantum numbers

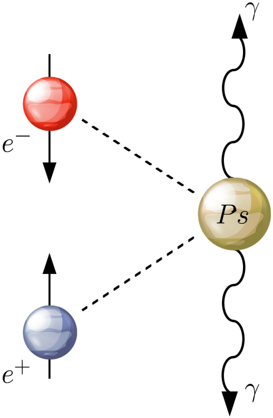
Non-relativistic notation	relativistic notation
$n\ ^{2S+1}L_J = 1\ ^1S_0$	$J^{PC} = 0^{-+}$

Wave function

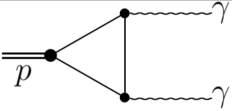
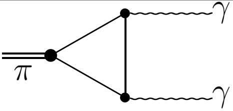
$$\psi(\vec{x}) = \frac{1}{(\pi a^3)^{1/2}} e^{-r/a}$$

$a-$ twice the Bohr radius of atomic hydrogen

Decays of $p - Ps$



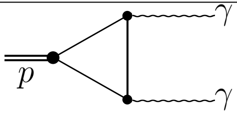
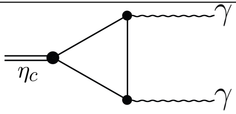
Decays into any even number of photons $(2, 4, 6, ...)$ are also possible

PARA-POSITRONIUM (p-Ps)	**PION (π^0)**
Ground state for electron-positron system $(n=1)$	Ground state for quark-antiquark system $(n=1)$
Non-relativistic state	Relativistic state (Goldstone boson)
$J^{PC}=0^{-+}$	$J^{PC}=0^{-+}$
Decays into $\gamma\gamma$	Decays into $\gamma\gamma$
 electrons are going around	 quarks are going around

e^- propagator $\neq$ quark propagator

Yet, in first approximation quark is taken

as a free propagator

PARA-POSITRONIUM (p-Ps)	PARA-CHARMONIUM (η_c)
Ground state for electron-positron system ($n = 1$)	Ground state for charm-anticharm system ($n = 1$)
Non-relativistic state	Non-relativistic state (with sizable relativistic corrections)
$J^{PC} = 0^{-+}$	$J^{PC} = 0^{-+}$
Decays into $\gamma\gamma$	Decays into $\gamma\gamma$
 electrons are going around	 quarks are going around

e^- propagator $\neq$ quark propagator

Yet, in first approximation quark is taken

as a free propagator

Lowest order decay width

$$\Gamma(\text{p-Ps} \to 2\gamma) = \frac{\alpha^5 m_e}{2}$$

Theory*	Experiment**
$8032.5028(1)\ \mu s^{-1}$	$7990.9(1.7)\ \mu s^{-1}$

mean lifetime of ~ 0.12 ns

* J.A. Wheeler, Ann. N.Y. Acad. Sci. 48, 219 (1946).
J. Pirenne, Arch. Sci. Phys. Nat. 29, 265 (1947)
** Al-Ramadhan, A. H., and D. Gidley (1994), Phys. Rev. Lett. 72, 1632.

Corrections to the decay width

- One loop level

$$\Gamma(\text{p-Ps} \to \gamma\gamma) = \Gamma_0 \left\{ 1 + \frac{\alpha}{\pi} \left(\frac{\pi^2}{4} - 5 \right) \right\}$$

$$= 7985.249 \mu s^{-1}$$

I. Harris and L.M. Brown, Phys. Rev. 105, 1656 (1957)

- Two loop level

$$\Gamma_{\text{p-Ps}} = \Gamma_0 \left\{ -2\alpha^2 ln\alpha + B_{2\gamma} \left(\frac{\alpha}{\pi} \right)^2 - \frac{3\alpha^3}{2\pi} ln^2\alpha + C\frac{\alpha^3}{\pi} ln\alpha + D \left(\frac{\alpha}{\pi} \right)^3 \right\}$$

$$= 7989.6178(2) \mu s^{-1}$$

G. S. Adkins, N. M. McGovern, R. N. Fell and J. Sapirstein, Phys. Rev. A **68** (2003), 032512
A. Czarnecki and S. G. Karshenboim, [arXiv:hep-ph/9911410 [hep-ph]].
Y. Tomozawa, "Radiative Corrections to Parapositronium Decay ," Annals of Physics **128** (1980),463-490
G. Adkins, "Radiative Corrections to Positronium Decay ," Annals of Physics **146** (1983), 78-128

Decay width-general formula

$$\Gamma(Ps \to n\gamma) = \frac{1}{2J+1} |\psi(0)|^2 \lim_{v \to 0} \left[4v\sigma(e^+e^- \to n\gamma)\right]$$

where:

$|\psi(0)|^2$- a probability that e^- and e^+ meet each other in the positronium
v- electron-positron relative velocity
σ- electron-positron annihilation cross-section
J- total spin of the positronium

Still, wave function at the origin only!

A. Sen and Z. K. Silagadze, Can. J. Phys. **97** (2019) no.7, 693-700

The lowest order

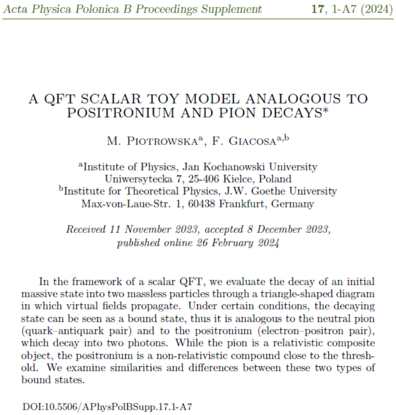
At the lowest order it becomes:

$$\Gamma\left({}^1S_0 \to 2\gamma\right) = \frac{1}{2}\frac{e^4|\psi(\vec{x}=0)|^2}{\pi m^4}\int_0^\infty |\vec{k}_1|^2\delta(2m-2|\vec{k}_1|)d|\vec{k}_1| =$$

$$= \frac{e^4|\psi(\vec{x}=0)|^2}{4^2} = \frac{4\pi\alpha^2}{m^2}|\psi(\vec{x}=0)|^2$$

$|\psi(\vec{x}=0)|^2 \sim \alpha^3$
$|\psi(\vec{x}=0)| \sim \alpha^{3/2}$

Scalar model


Acta Physica Polonica B Proceedings Supplement **17**, 1-A7 (2024)

A QFT SCALAR TOY MODEL ANALOGOUS TO POSITRONIUM AND PION DECAYS*

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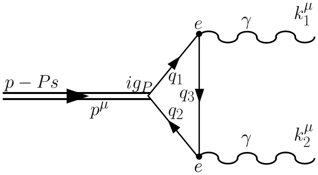
In the framework of a scalar QFT, we evaluate the decay of an initial massive state into two massless particles through a triangle-shaped diagram in which virtual fields propagate. Under certain conditions, the decaying state can be seen as a bound state, thus it is analogous to the neutral pion (quark–antiquark pair) and to the positronium (electron–positron pair), which decay into two photons. While the pion is a relativistic composite object, the positronium is a non-relativistic compound close to the threshold. We examine similarities and differences between these two types of bound states.

DOI:10.5506/APhysPolBSupp.17.1-A7

The Lagrangian

TRIANGLE DIAGRAM



EXTERNAL MOMENTA

- $p^\mu = (M_P, \vec{0})$
- $k_1^\mu = (\omega, 0, 0, \omega)$
- $k_2^\mu = (\omega, 0, 0, -\omega)$

  $(\omega = \frac{M_P}{2})$

INTERNAL MOMENTA

- $q_1 = \frac{p}{2} + q$
- $q_2 = \frac{p}{2} - q$
- $q_3 = \frac{p}{2} + q - k_1$

THE LAGRANGIAN

$$\mathcal{L}_{int} = g_P P(x)\bar{\psi}(x) i\gamma^5 \psi(x) - eA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

- $P(x)$ is the pseudoscalar positronium field
- $\psi(x)$ is the electron field
- $A_\mu(x)$ is the photon field
- e is the electric charge of the proton
- g_P is the positronium-constituent coupling constant

TRIANGLE AMPLITUDE I

$$I = \int \frac{d^4q}{(2\pi)^4} \frac{\mathcal{F}(q,p)}{\left(q_1^2 - m_e^2 + i\varepsilon\right)\left(q_2^2 - m_e^2 + i\varepsilon\right)\left(q_3^2 - m_e^2 + i\varepsilon\right)}$$

Solved by using two independent methods:

- WICK ROTATION METHOD
- RESIDUE THEOREM

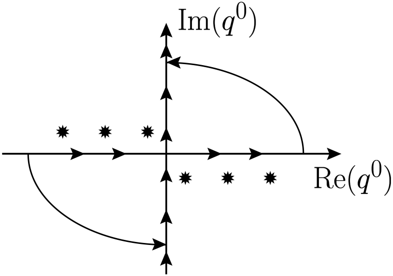
Wick rotation method

Initial variables:
q^0, q_x, q_y, q_z

Replacement:
$q^0 = iw$
$\rho^2 = q_x^2 + q_y^2$

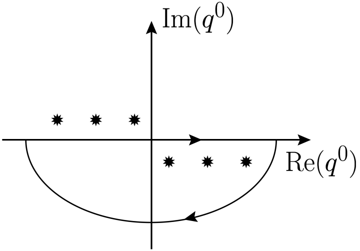
Final variables:
ρ, w, q_z

$$I = \int \frac{d^4q}{(2\pi)^4} \frac{\mathcal{F}(q,p)}{\text{den}} \overset{\text{replacement}}{\longrightarrow} i \int_0^\infty \frac{\rho d\rho}{(2\pi)^3} \int_{-\infty}^{\infty} dq_z \int_{-\infty}^{\infty} \frac{\mathcal{F}}{D_1 D_2 D_3} dw,$$



Residue theorem

$$\int dq^4 = \int d^3q \int dq^0 \overset{\text{Residue theorem}}{=} \int d^3q = \int \rho d\rho dq_z$$



TRIANGLE AMPLITUDE I: $I = \int \frac{d^3q}{(2\pi)^3} \left[\int \frac{dq^0}{2\pi} \frac{\mathcal{F}}{D_1 D_2 D_3} \right]$

Triangle amplitude

TRIANGLE AMPLITUDE

$$I = i \int \frac{d^4 q}{(2\pi)^4} \frac{\mathcal{F}(q,p)}{D_1 D_2 D_3}$$

where:

$$\begin{aligned} D_{1,2} &= (p/2 \pm q)^2 - m_e^2 + i\varepsilon = \left(M_P/2 \pm q^0\right)^2 - \vec{q}^2 - m_e^2 + i\varepsilon, \\ D_3 &= \left(M_P/2 + q^0 - k_1^0\right)^2 - (\vec{q} - \vec{k_1})^2 - m_e^2 + i\varepsilon \end{aligned}$$

Analytical formulas

By setting $D_{1,2,3}$=0 one gets:

Poles of D_1	Poles of D_2
$L_1 = -\frac{M_P}{2} - \sqrt{\rho^2 + q_z^2 + m_e^2} + i\delta$ $R_1 = -\frac{M_P}{2} + \sqrt{\rho^2 + q_z^2 + m_e^2} - i\delta$	$L_2 = \frac{M_P}{2} - \sqrt{\rho^2 + q_z^2 + m_e^2} + i\delta$ $R_2 = \frac{M_P}{2} + \sqrt{\rho^2 + q_z^2 + m_e^2} - i\delta$
Poles of D_3	
$L_3 = -\sqrt{\rho^2 + (q_z - k_z)^2 + m_e^2} + i\delta$ $R_3 = \sqrt{\rho^2 + (q_z - k_z)^2 + m_e^2} - i\delta$	

Resulting decay width into $\gamma\gamma$:

$$\Gamma_{P-ps\to\gamma\gamma} = \frac{1}{2}\frac{|\vec{k}_1|}{8\pi M_P^2} 2 \left| 8 m_e 4\pi\alpha g_P I \frac{M_P^2}{4} \right|^2$$

with $|\vec{k}_1| = \frac{M_P}{2}$.

Composite model

COMPOSITE MODEL

Positronium (Ps) is a bound state.

How to describe it?

WEINBERG COMPOSITENESS CONDITIONS
(The positronium is not an elementary object, just as the deuteron)

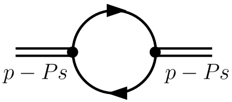
PARA-POSITRONIUM

- form factor $\sim$ wave function
- coupling constant (g) is fixed:

$$g_P = \sqrt{\frac{1}{\Sigma'(s=M_p^2)}}$$

$\Sigma(s = M_p^2)$- the loop function

- Loop diagram



Weinberg compositeness conditions

PHYSICAL REVIEW VOLUME 137, NUMBER 3B 8 FEBRUARY 1965

Evidence That the Deuteron Is Not an Elementary Particle*

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Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 30 September 1964)

If the deuteron were an elementary particle then the triplet n-p effective range would be approximately $-ZR/(I-Z)$, where $R=4.31$F is the usual deuteron radius and Z is the probability of finding the deuteron in a bare elementary-particle state. This formula is model-independent, but has an error of the order of the range $m_\pi^{-1}=1.41$F of the n-p force, so it becomes exact only in the limit of small deuteron binding energy, i.e., $R\gg m_\pi^{-1}$. The experimental value of the effective range is not of order R and negative, but rather of order m_π^{-1} and positive, so Z is small or zero and the deuteron is mostly or wholly composite.

I. INTRODUCTION

MANY physicists believe that low-energy experiments can never decide whether a given particle is composite or elementary. I will try to show here that low-energy n-p scattering data already provide very strong model-independent evidence that the deuteron is in fact composite, or more precisely, that the probability Z of finding the deuteron in a bare elementary-particle state is very small.

This conclusion is based on a theorem proven in Secs. II and III, which give formulas[1] for the triplet n-p scattering length and effective range in the limit of small deuteron binding energy:

$$a_s=[2(1-Z)/(2-Z)]R+O(m_\pi^{-1}), \quad (1)$$

$$r_e=[-Z/(1-Z)]R+O(m_\pi^{-1}), \quad (2)$$

where Z is the famous deuteron "field renormalization" constant, and R is the usual deuteron radius

$$R\equiv(2\mu B)^{-1/2}=4.31\text{ F} \quad (3)$$

with B the deuteron binding energy and μ the n-p reduced mass. The first terms in (1) and (2) are model-independent and become very large for small B, while the second terms called $O(m_\pi^{-1})$ cannot be calculated without specific information on the n-p interaction but are expected to be of the order of magnitude of the range $m_\pi^{-1}=1.41$ F, and will in any case become negligible for $B\to 0$. In actuality R is three times larger than m_π^{-1}, so the separation between terms in (1) and (2) is reasonably clear cut.

If the deuteron is purely composite then $Z=0$,[2] and (1) and (2) give in this case

$$a_s\approx R;\quad r_e=O(m_\pi^{-1}). \quad (4)$$

This is in agreement with the conclusions of simple potential theory, and, as is well known, it also agrees with the experimental values:

$$a_s=+5.41\text{ F};\quad r_e=+1.75\text{ F}. \quad (5)$$

In contrast, if the deuteron had an appreciable probability Z of being found in an elementary bare-particle state then a_s would be less than R, and more striking, *r_e would be large and negative.* This is clearly contradicted by the experimental values (5), so we may conclude that Z is small (say <0.2), and therefore the deuteron is at least mostly composite.[3]

The large values for both a_s and r_e when Z is not zero may suggest to the reader that the effective-range approximation,

$$k\cot\delta\cong-1/a_s+r_ek^2/2, \quad (6)$$

may itself break down when the deuteron is elementary. In fact, we will see that this does not happen; it is only the first two terms in the expansion of $k\cot\delta$ in powers of k^2 that become of order R^{-1} for $Z\neq 0$ and $k\cong 1/R$, the third and higher terms being smaller by powers of $(Rm_\pi)^{-1}$. One well-known consequence of (6) is the relation between a_s, r_e, and R

$$1/R=1/a_s+r_e/2R^2 \quad (7)$$

which is satisfied by (1) and (2) for all Z. It should be stressed that (7) itself tells us nothing about the elementarity of the deuteron, since (7) follows directly from the requirement that (6) give $\cot\delta=+i$ (i.e., $e^{2i\delta}=\infty$) when k is extrapolated to the deuteron pole at $k=i/R$. The true token that the deuteron is com-

* Research supported in part by the U. S. Air Force Office of Scientific Research, Grant No. AF-AFOSR-232-63 and in part by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Foundation Fellow.

[1] After deriving these formulas I became aware that they could also be obtained in the nonrelativistic limit of the Zachariasen model, as treated by J. S. Dowker, Nuovo Cimento **25**, 224 (1962), by using his Eq. (9) in his Eq. (13), and then passing to the limit $\mu R\gg 1$. However, Dowker's derivation does not show that for small binding energy this result is actually model-independent and hence applicable to the deuteron, and he does not make this application. (There seems to be a factor of 4 lost from Dowker's equation for the effective range, but his equation for $k\cot\delta$ is correct.)

[2] The use of $Z=0$ to distinguish composite from elementary particles has been discussed by many authors, including J. C. Howard and B. Jouvet, Nuovo Cimento **18**, 466 (1960); M. T. Vaughan, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961); R. Acharya, Nuovo Cimento **24**, 870 (1962); S. Weinberg, *Proceedings of the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 683; A. Salam, Nuovo Cimento **25**, 224 (1962); J. S. Dowker, *ibid.* **25**, 1135 (1962); S. Weinberg, Phys. Rev. **130**, 776 (1963).

[3] The point that the experimental values (5) of r_e and a_s are consistent with $Z=0$ has been made by H. Ezawa, T. Muta, and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **29**, 877 (1963). However, these authors do not compute r_e and a_s for $Z\neq 0$, and hence miss the point that an elementary deuteron would entail a large negative n-p effective range.

Comparison with quark model

$$f_{\pi,\eta_c} \sim \int d^3q \frac{A(\vec{q})}{\sqrt{\vec{q} + m^2}}$$

Matching

$$\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = A(\vec{q}^2)(\vec{q}^2 + \gamma^2)$$

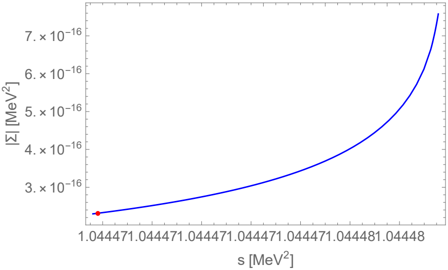
S. Godfrey and N. Isgur, "Mesons in a Relativized Quark Model with Chromodynamics ," Phys. Rev. D **32** (1985),189

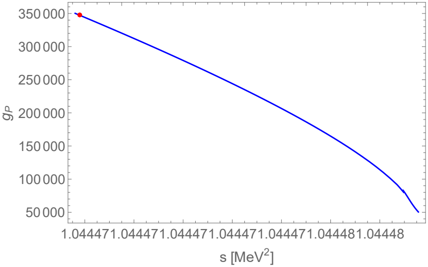
J. Pestieau, C. Smith and S. Trine, Int. J. Mod. Phys. A **17** (2002), 1355-1398

Vertex function

- Vertex function $\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = \frac{1}{\left(1+\frac{\vec{q}^2}{\gamma^2}\right)^2}\left(\vec{q}^2+\gamma^2\right)$

with $\gamma^2 = m^2 - \frac{M_P^2}{4}$





* J. Pestieau, C. Smith and S. Trine, "Positronium decay: Gauge invariance and analyticity," Int. J. Mod. Phys. A **17** (2002), 1355-1398 doi:10.1142/S0217751X02009606

Vertex function

- Vertex function $\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = \frac{1}{\left(1+\frac{\vec{q}^2}{\gamma^2}\right)^2} \left(\vec{q}^2 + \gamma^2\right)$

with $\gamma^2 = m^2 - \frac{M_P^2}{4}$

RESULTS

PARA-POSITRONIUM	$\Gamma_{P-ps\rightarrow\gamma\gamma}$ $[\mu s^{-1}]$
Experimental result*	$7990.9(1.7)$
pole 1	7968.1
pole 1 + pole 2	7995.1
pole 1 + pole 2 + pole 3	7917.9

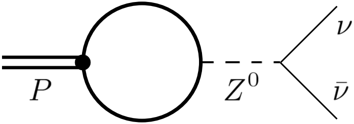
* Al-Ramadhan, A. H., and D. Gidley (1994), Phys. Rev. Lett. 72, 1632.

Discussion

- idea of poles

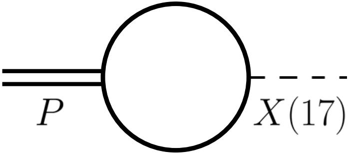
  -The contribution to the total decay rate from the first pole is by far the dominant one.
  -The first and second pole contributions to the decay width is positive
  -Interestingly, the third pole gives a negative contribution to the decay width. This contribution goes in good direction but it is even too strong.
- $\alpha-$corrections-only some are included.

f_P weak decay constant of positronium



$$\Gamma \sim \frac{m_\nu^2}{m_{Z^0}^4}$$

Proof of principle



η_c

- Vertex function $\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = \frac{1}{\left(1+\frac{\vec{q}^2}{\gamma^2}\right)^2} \left(\vec{q}^2 + \gamma^2\right)$

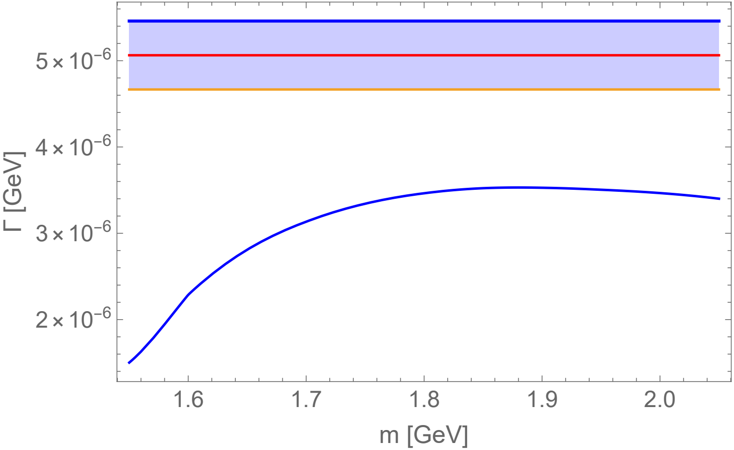
  The same as for p-Ps
- $m_c = 1.7$ GeV

RESULTS

η_c	$\Gamma_{\eta_c \to \gamma\gamma}$ [GeV]
Experimental result*	$5.063 \cdot 10^{-6}$
pole 1	$9.976 \cdot 10^{-6}$
pole 1 + pole 2	$1.553 \cdot 10^{-5}$
pole 1 + pole 2 + pole 3	$3.139 \cdot 10^{-6}$
f_{η_c}	0.465

* P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020)

$\Gamma(m)$



η_c

- we changed $A(\vec{q})$

$$A(\vec{q}) = e^{\frac{-\vec{q}^2}{2\Lambda^2}}$$

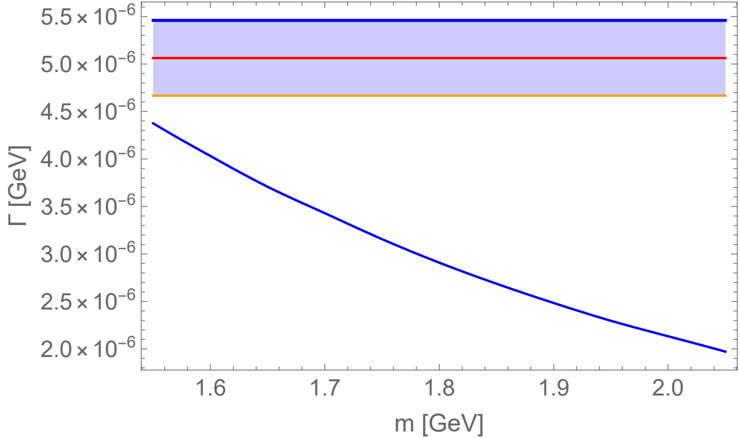
- $m_c = 1.5$ GeV

RESULTS

η_c	$\Gamma_{\eta_c \to \gamma\gamma}$ [GeV]
Experimental result*	$5.063 \cdot 10^{-6}$
pole 1	$8.83 \cdot 10^{-6}$
pole 1 + pole 2	$1.13 \cdot 10^{-5}$
pole 1 + pole 2 + pole 3	$4.76 \cdot 10^{-6}$
f_{η_c}	0.427

* P.A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020)

$\Gamma(m)$


5.5 × 10⁻⁶
5.0 × 10⁻⁶
4.5 × 10⁻⁶
4.0 × 10⁻⁶
3.5 × 10⁻⁶
3.0 × 10⁻⁶
2.5 × 10⁻⁶
2.0 × 10⁻⁶
Γ [GeV]
1.6
1.7
1.8
1.9
2.0
m [GeV]

π^0

- Vertex function $\mathcal{F}(q,p) = \mathcal{F}(\vec{q}^2) = \frac{1}{\left(1+\frac{\vec{q}^2}{\gamma^2}\right)^2}(\vec{q}^2+\gamma^2)$

  The same as for p-Ps
- $m_c = 0.25$ GeV

RESULTS

π^0	$\Gamma_{\pi^0\to\gamma\gamma}$ [GeV]
Experimental result	$7.80 \cdot 10^{-9}$
Our model	$8.1 \cdot 10^{-9}$
f_π exp.	0.130
f_π our model	0.122

Summary

- QFT composite model for the positronium studied
- The same framework was extended to η_c state
- Weak and exotic decay channels (e.g. to Z^0 or $X(17)$) can be explored within the same formalism.
- Future applications to excited states $(n = 2, ...p - P_s)$

THANK YOU FOR YOUR ATTENTION

Covariance

Question: is it covariant?

$$\vec{q}^2 = \frac{-(pq)^2 + p^2 q^2}{p^2}$$

$$\mathcal{F}(p,q) = \mathcal{F}\left(\frac{-(pq)^2 + p^2 q^2}{p^2}\right) = \mathcal{F}_{\mathrm{RF}}(\vec{q}^2)$$

Answer: Yes, it can be seen as covariant.

M. Soltysiak and F. Giacosa, "A covariant nonlocal Lagrangian for the description of the scalar kaonic sector," Acta Phys. Polon. Supp. **9** (2016), 467-472