Loop calculations in light cone perturbation theory

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Outline

Lecture 1: basics

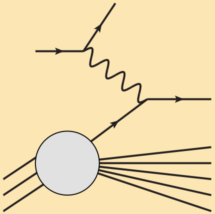
- Dipole picture of DIS
- Light cone quantization
- Color Glass Condensate
- BK equation

Lecture 2: NLO

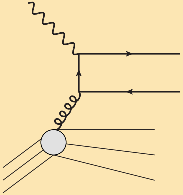
- Dipole picture DIS at NLO
- Diffractive structure function at NLO
- Mass renormalization LCPT and dipole picture DIS with quark masses

Dipole picture of DIS

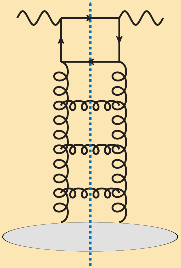
Inclusive DIS in collinear factorization



- Proton consists of partons
- Struck by γ^*: interaction time short
  $\Longrightarrow$ partonic interaction factorizes
- LO $\gamma^* + q \to q$ counts quarks
  $\Longrightarrow$ PDF's
- NLO: also gluon-initiated



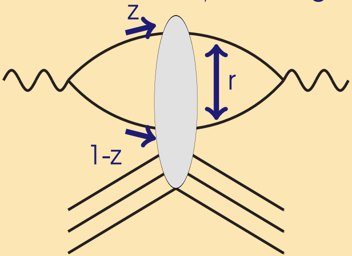
- High Q^2 logs resummed into DGLAP
- $Q^2 \gg \Lambda^2_{\mathrm{QCD}}$ and $W^2 \gg \Lambda^2_{\mathrm{QCD}}$



Nonperturbative physics parametrized by PDF's

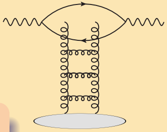
Inclusive DIS in dipole picture

Different frame: γ^* moving fast; not proton



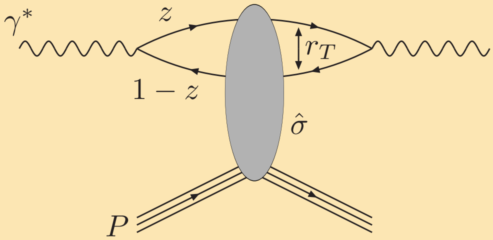
- γ^* consists of partons
- Partons struck by target gluon field: interaction time short $\Longrightarrow$ structure of γ^* frozen
- LO: γ^* is only $q\bar{q}$ dipole
- NLO corrections: also include $\gamma^* \to q\bar{q}g$
- $q\bar{q}$ dipole of size $r \sim 1/Q$

- High W^2 logs resummed into BK/BFKL
- $W^2 \gg Q^2$



Nonperturbative physics: q/g+p scattering amplitudes

Photon wavefunction and dipole amplitude

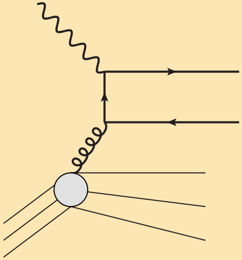


High energy: we assume (lifetime/timescale) factorization between

- $\left|\psi^{\gamma^*\to q\bar{q}}(\mathbf{r}_\perp, z)_{T,L}\right|^2$: probability for photon to fluctuate into $\bar{q}q$
- $2\mathcal{N}$ imaginary part of the forward elastic scattering amplitude,
  i.e. the total $q\bar{q}$ cross section; optical theorem

$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{r}_\perp \, \mathrm{d}z \left|\psi^{\gamma^*\to q\bar{q}}(\mathbf{r}_\perp, z)_{T,L}\right|^2 2\mathrm{Im}\mathcal{A}$$

Remarks: same process in IMF



Dipole picture diagram in collinear picture

- Looks like formally higher order (NLO DIS)
- Does not describe valence quarks

However: leading contribution at small-x

- The valence quark distribution is small
- DGLAP sea quarks come from gluons: $xq(x, Q^2) \sim \alpha_s \ln Q^2 xg(x, Q^2)$ $\alpha_s \ln Q^2 \sim \alpha_s$
- Same diagram contains both
  - $\gamma^* + g \to q + \bar{q}$: a NLO DIS process $\sim \alpha_s$
  - 1 DGLAP split $g \to q + \bar{q}$, $\sim \alpha_s$ + LO DIS process $\gamma^* + q \to q$
- Split between these two is scheme dependent
- Matching dipole and collinear pictures not very clear beyond leading log
  (where only $xg(x, Q^2)$ matters)

Light Cone Perturbation Theory and γ^* light cone wave function

Quantizing the photon: LCPT

- Recall: want to understand the **partonic content of the photon**
- Unlike proton, γ^* is a perturbative object: calculable
- The theoretical tool of choice is Light Cone Perturbation Theory

What is Light Cone Perturbation Theory (LCPT)?

- Heisenberg picture: time-dependent operators $\hat{A}(t)$, **equal-time** commutation relations $[\hat{A}(t), \hat{B}(t)]$ known.
- Then solve equation of motion $\partial_t \hat{A}(t) = -i[\hat{A}(t), \hat{H}(t)]$
- LCPT: choose the "time" variable to be light-like: $x^+ = \frac{1}{\sqrt{2}}(t+z)$

Advantages and disadvantages

+ Only physical degrees of freedom $\Longrightarrow$ partonic interpretation
+ Maximal number of commuting Lorentz generators
+ Longitudinal boosts are easy $\Longrightarrow$ high energy
- 3d rotational invariance is hard
- Connection to lattice, hadron rest frame difficult

Idea of LCPT calculation

- Know free particle Fock states: $|\gamma^*\rangle_0, \quad |q\bar{q}\rangle_0, \quad |q\bar{q}g\rangle_0$ etc.
- **Interacting** states are superpositions of these:

$$|\gamma^*\rangle = (1+\dots)|\gamma^*\rangle_0 + \psi^{\gamma^*\to q\bar{q}} \otimes |q\bar{q}\rangle_0 + \psi^{\gamma^*\to q\bar{q}g} \otimes |q\bar{q}g\rangle_0 + \dots$$

- Calculate in QM perturbation theory, e.g. ground state $|0\rangle$ wavefunction:

$$\psi^{0\to n} = \sum_n \frac{\langle n|\,\hat{V}\,|0\rangle}{E_n - E_0} + \dots$$

  - Here $1/\Delta E$ is $\sim$ the lifetime of the quantum fluctuation from 0 to n
- "Energy" E is conjugate to "time", LC time is $x^+ \implies$ LC energy k^-
- Note: energy not "conserved," only 3-momentum $\vec{k} = (k^+, \mathbf{k}_\perp)$ is

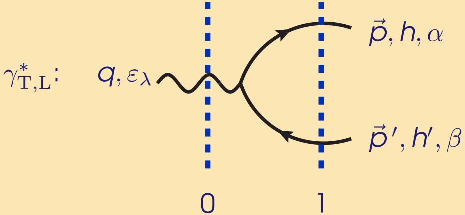
Connection to Feynman perturbation theory

- Matrix elements $\langle n|\,\hat{V}\,|m\rangle$ are vertices in Feynman rules
- LC energy denominators from propagators, integrating over k^- with pole

Let's calculate $\psi^{\gamma^* \to q\bar{q}}$

Steps for calculating light cone wave function:

$$\psi^{\gamma^*_{\mathrm{T/L}} \to q\bar{q}} = \frac{-ee_f \delta_{\alpha\beta}}{q^- - p^- - p'^-} \left[\bar{u}_h(p) \not{\varepsilon}_{\lambda,\mathrm{T/L}}(q) v_{h'}(p') \right]$$



- Color factor $\delta_{\alpha\beta}$ and electric charge ee_f
- Energy denominator $k_0^- - k_1^-$
- Matrix element $\bar{u} \not{\varepsilon} v_{h'}$
- Fourier transform to transverse coordinate space
  (Why coordinate space? Eikonal scattering, will come back to this)

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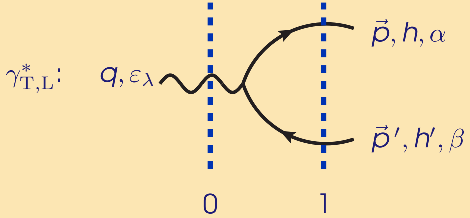

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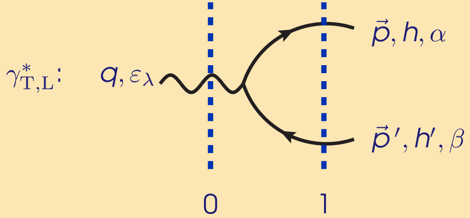

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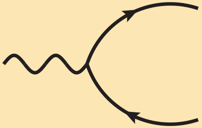
Matrix element

$\bar{u}_h(p)\not\varepsilon_{\lambda,\mathrm{T/L}}(q)v_{h'}(p')$ with



$$h, h' = \pm\frac{1}{2}; \quad \lambda = 0 = L, \quad \lambda = \pm 1 = T$$

- Transv. polarization vector (LC gauge $\varepsilon^+ = 0$!!)
  $\varepsilon^\mu_{\lambda=\pm}(q) = (0, \frac{\mathbf{q}_\perp \cdot \boldsymbol{\varepsilon}_{\perp\lambda}}{q^+}, \boldsymbol{\varepsilon}_{\perp\lambda}) \xrightarrow{\mathbf{q}_\perp=\mathbf{0}} (0, 0, \boldsymbol{\varepsilon}_{\perp\lambda}),$
- Circularly polarized 2d polarization vectors $\boldsymbol{\varepsilon}_{\perp\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp 1 \\ -i \end{pmatrix}$
- Longit. polarization effectively $\varepsilon^\mu_{\lambda=0}(q) = (0, \frac{\sqrt{Q^2 - \mathbf{q}_\perp{}^2}}{q^+}, \frac{\mathbf{q}_\perp}{\sqrt{Q^2 - \mathbf{q}_\perp{}^2}}) \xrightarrow{\mathbf{q}_\perp=\mathbf{0}} (0, \frac{Q}{q^+}, \mathbf{0}),$
- Using tables for matrix elements T polarization is

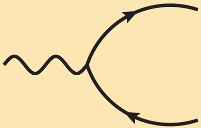
$$\bar{u}\not\varepsilon v = \frac{2\delta_{h,-h'}}{\sqrt{z(1-z)}}\left(z\delta_{\lambda,2h} - (1-z)\delta_{\lambda,-2h}\right)\boldsymbol{\varepsilon}_{\perp\lambda} \cdot \mathbf{k}_\perp + \delta_{h,h'}\delta_{\lambda,2s}\frac{\sqrt{2}m}{\sqrt{z(1-z)}}.$$

Remarks:

- Only relative center-of-mass momentum $\mathbf{k}_\perp = \mathbf{p}_\perp - z\mathbf{q}_\perp$
- Quark helicity conserving $\sim \mathbf{k}_\perp$ + helicity-flip $\sim m$

Energy denominator

$\mathbf{q}_\perp, q^+$



$\mathbf{p}_\perp,\ p^+ = zq^+$

$\mathbf{k}_\perp \equiv \mathbf{p}_\perp - z\mathbf{q}_\perp$

$\mathbf{p}'_\perp = \mathbf{q}_\perp - \mathbf{p}_\perp,\ p'^+ = (1-z)q^+$

Energy denominator $(q^- - k^- - k'^-)^{-1}$ (On-shell momenta!)

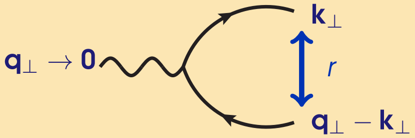
$$q^- - p^- - p'^- = -\left(\overbrace{\frac{Q^2}{2q^+}}^{-q^-} + \overbrace{\frac{\mathbf{p}_\perp{}^2 + m^2}{2zq^+}}^{p^-} + \overbrace{\frac{(\mathbf{p}_\perp - \mathbf{q}_\perp)^2 + m^2}{2(1-z)q^+}}^{p'^-} \right)$$

$$\frac{1}{q^- - p^- - p'^-} = \frac{-2q^+z(1-z)}{\underbrace{Q^2z(1-z) + m^2}_{\equiv \bar{Q}^2} + \mathbf{k}_\perp{}^2}.$$

- Also only relative center-of-mass momentum $\mathbf{k}_\perp = \mathbf{p}_\perp - z\mathbf{q}_\perp$

Fourier transform

Scattering at high energy is **eikonal**: transverse **position** of parton does not change
$\Longrightarrow$ Fourier-transform $\mathbf{k}_\perp \to \mathbf{r}_\perp$



$$\text{L:} \quad \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \frac{1}{\mathbf{k}_\perp{}^2 + \bar{Q}^2} \sim K_0(r\bar{Q})$$

$$\text{T:} \quad \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} \frac{k^i}{\mathbf{k}_\perp{}^2 + \bar{Q}^2} \sim \frac{r^i}{r} K_1(r\bar{Q})$$

- Recall $\bar{Q}^2 \equiv z(1-z)Q^2 + m^2$.
- Note asymptotics $K_{0,1}(x) \sim e^{-x} \Longrightarrow$ enforces $r \sim 1/Q$.

DIS dipole frame: summary

- High energy DIS: γ^* fluctuates into $q\bar{q}$, which scatters off target

$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{r}_\perp \, \mathrm{d}z \left| \psi^{\gamma^* \to q\bar{q}}(r,z)_{T,L} \right|^2 2\mathcal{N}$$

- Typical dipole size: $r \sim 1/Q$
- Used optical theorem: $2\mathcal{N}$ is total cross section
  - can also take $|\mathcal{N}|^2$: elastic scattering (diffractive DIS)
- Assuming that fixed-size dipoles are basis that diagonalizes $\mathrm{Im}T$
  - In general: high energy/eikonal approximation: $\mathbf{x}_\perp$ is fixed;
    (does not imply zero momentum transfer!)

The Color Glass Condensate

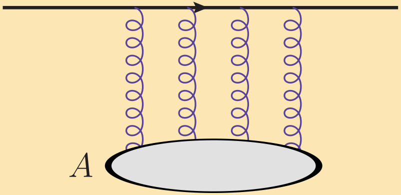
What is the target made of?

- So what is the dipole amplitude?
- At high energy: dominantly gluons
  - Experimentally: small x gluon distribution is larger than the quark one.
  - High $\sqrt{s}$: QCD radiation builds up the target by adding gluons to it.

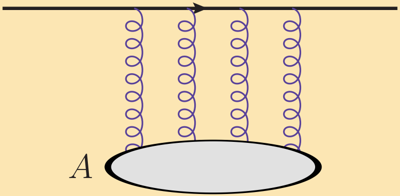
Color Glass Condensate (CGC)

Many gluons in the target $\Longrightarrow$ classical gluon field A_μ.

Sum all diagrams with n gluon lines



Scattering off CGC target



Quark in classical color field: Dirac equation!

$$(i\partial\!\!\!/ - gA\!\!\!/)\psi(x) = 0$$

(Note: $A\!\!\!/ = A_a^\mu \gamma_\mu t^a$ is $N_c \times N_c \otimes 4 \times 4$-matrix)

High energy: **eikonal** approximation

- Gluon is spin 1: it couples to a vector: $\sim p^\mu A_\mu$
- For high energy particle the only vector available is p^μ
- p^μ has one large component: $p^+ \implies p^\mu A_\mu \sim p^+ A^- \implies$ only need A^-

Ansatz for DE: $\psi(x) = V(x) e^{-ip\cdot x} u(p)$, plug in eq.

$N_c \times N_c$-matrix!

$$\implies \quad \partial_+ V(x^+, x^-, \mathbf{x}_\perp) = -igA^-(x^+, x^-, \mathbf{x}_\perp) V(x^+, x^-, \mathbf{x}_\perp)$$

This is solved by path-ordered exponential

$$V(x^+, x^-, \mathbf{x}_\perp) = \mathbb{P} \exp\left\{ -ig \int^{x^+} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x}_\perp) \right\}$$

Eikonal propagation

- Now we know the scattering S-matrix for quark states

Incoming free quark $|q_i(\mathbf{x}_\perp)\rangle$ at $x^+ \to -\infty$ is, at $x^+ \to \infty$

$$\hat{S}(-\infty,\infty)\,|q_i(p^+,\mathbf{x}_\perp)\rangle = \left[\mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}y^+ A^-(y^+,x^-,\mathbf{x}_\perp)\right\}\right]_{ji}|q_j(p^+,\mathbf{x}_\perp)\rangle$$

outgoing state linear superposition of color rotated quarks.

- In the high energy limit quark wavefunction oscillates like $e^{ip^+x^-}$ with large p^+
  $\Longrightarrow$ x^--dependence negligible compared to this $\Longrightarrow$ approximate $x^- = 0$
  glass in CGC

Scattering described by 2-d field of SU(N_c)-matrices

$$V(\mathbf{x}_\perp) \equiv \mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}x^+ A^-(x^+,x^-=0,\mathbf{x}_\perp)\right\}$$

— The **Wilson line**

Dipole amplitude and Wilson lines

Incoming dipole becomes (color neutral, normalized! ; $V(\mathbf{y}_\perp)^\dagger_{jk} = V(\mathbf{y}_\perp)^*_{kj}$ for antiquark)

$$|\text{in}\rangle = \frac{\delta_{ii'}}{\sqrt{N_c}} \left| q_i(\mathbf{x}_\perp)\bar{q}_{i'}(\mathbf{y}_\perp)\right\rangle \qquad \hat{S}\,|\text{in}\rangle = \frac{\delta_{ii'}}{\sqrt{N_c}} V_{ji}(\mathbf{x}_\perp)V^\dagger_{i'j'}(\mathbf{y}_\perp)\left| q(\mathbf{x}_\perp)_j \bar{q}(\mathbf{y}_\perp)_{j'}\right\rangle$$

Count outgoing dipoles in this state

$$S = \frac{1}{\sqrt{N_c}} \left\langle q_k(\mathbf{x}_\perp)\bar{q}_k(\mathbf{y}_\perp)\right|\text{in}\rangle = \frac{\delta_{ii'}}{N_c}\delta_{kj}\delta_{kj'} V_{ji}(\mathbf{x}_\perp)V^\dagger_{i'j'}(\mathbf{y}_\perp) = \frac{1}{N_c}\,\mathrm{tr}\,V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)$$

Dipole amplitude in the CGC

Relate DIS amplitude $\mathcal{N}$ to a **microscopical description of the target**:

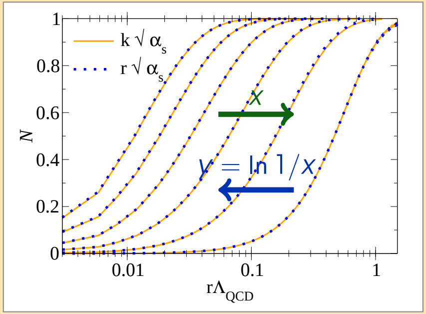
$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c}\,\mathrm{tr}\,V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)$$

(Imaginary unit convention $S_{fi} = \langle f|\,\hat{S}\,|f\rangle = 1 + iT_{fi}$ $\quad \sigma_{\text{tot}} = 2\,\mathrm{Im}T_{ii}$ $\quad \mathcal{N} \equiv \mathrm{Im}T_{ii}$ $\quad S_{ii} = \delta_{ii} - \mathcal{N} + \text{imag}$)

Saturation of the dipole cross section

Basic features following from

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c} \operatorname{tr} V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp)$$



- Small dipoles:
  - At $\mathbf{x}_\perp = \mathbf{y}_\perp$: $\quad V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) = \mathbb{1}$.
  - At $r = 0$ color neutral system, should not scatter by the strong interaction!
  - $\mathcal{N}_{q\bar{q}}(r) \sim r^2 \implies$ perturbative limit
- Large dipoles
  - Fully uncorrelated Wilson lines: $\langle \operatorname{tr} V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\rangle = 0$
  - Large dipoles $r \gtrsim 1/Q_s$ scatter with $\mathcal{N}_{q\bar{q}} \lesssim 1$
- Know that $\mathcal{N}_{q\bar{q}}$ grows at $x \to 0$

  $\implies$ turnover from $\mathcal{N} \ll 1$ to $\mathcal{N} \sim 1$ happens at smaller $r \sim R_s$ when $\sqrt{s} \to 0$

Nonperturbative weak coupling unitarization = Saturation

Saturation scale $Q_s = 1/R_s$ = inverse distance for turnover

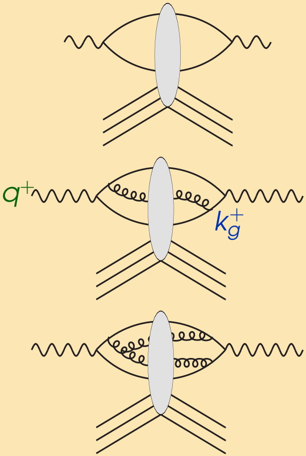
The Balitsky-Kovchegov equation

Power counting

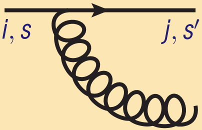
- Leading order (LO)
- Add one **soft** gluon $\Longrightarrow$ Leading Log (LL), resum by BK evolution
- Add one gluon, but **not** necessarily soft: Next-to-Leading Order (NLO) (need to subtract the soft gluon!)
- Add two gluons, one of them soft: NLL $\Longrightarrow$ resum by NLO BK equation

$$\sigma \sim \underbrace{\mathcal{O}(1)}_{\text{LO}} + \underbrace{\mathcal{O}(\alpha_s \ln 1/x)}_{\text{LL}} + \underbrace{\mathcal{O}(\alpha_s)}_{\text{NLO}} + \underbrace{\mathcal{O}(\alpha_s^2 \ln 1/x)}_{\text{NLL}}$$

- Current research: NLO & NLL



What happens if one radiates a gluon?



$$\psi^{q\to qg}(z, \mathbf{k}_\perp) = \frac{1}{\frac{\mathbf{p}_\perp{}^2}{2p^+} - \frac{\mathbf{k}_\perp{}^2}{2k^+} - \frac{\mathbf{p}'_\perp{}^2}{2p'^+}} \bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p)$$

Full matrix element (take quark mass $m = 0$)

$$\bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p) = \frac{-2gt^a_{ji}}{z\sqrt{1-z}}(\delta_{\lambda,2s} + (1-z)\delta_{\lambda,-2s})\mathbf{q}_\perp \cdot \varepsilon_\perp{}^*_\lambda, \qquad \mathbf{q}_\perp = \mathbf{k}_\perp - z\mathbf{p}_\perp$$

Focus on the **soft** (or slow) gluon limit $z \to 0$:

$$\psi^{q\to qg}(k^+, \mathbf{k}_\perp) \approx \frac{-2zp^+}{\mathbf{k}_\perp{}^2}\frac{-2gt^a_{ji}\delta_{ss'}}{z}\mathbf{k}_\perp \cdot \varepsilon_\perp{}^*_\lambda = \frac{4gt^a_{ji}p^+}{\mathbf{k}_\perp{}^2}\mathbf{k}_\perp \cdot \varepsilon_\perp{}^*_\lambda$$

IR divergences in gluon emission

Squaring the wave function we get a "probability for gluon emission"

$$\mathrm{d}P_{q\to qg} = \left|\psi^{q\to qg}(k^+,\mathbf{k}_\perp)\right|^2 \frac{\mathrm{d}k^+\,\mathrm{d}^2\mathbf{k}_\perp}{2k^+(2\pi)^3} \sim \frac{\mathrm{d}z}{z}\frac{\mathrm{d}^2\mathbf{k}_\perp}{\mathbf{k}_\perp{}^2} \qquad \left(\sum_{\lambda=\pm 1}\varepsilon_i\varepsilon_j^* = \delta_{ij}\right)$$

This has 2 types of divergences:

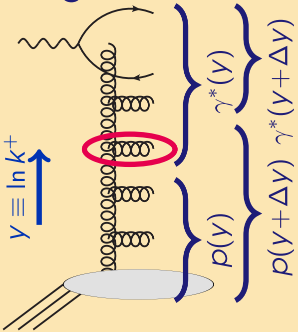
collinear $\int_0 \frac{\mathrm{d}^2\mathbf{k}_\perp}{\mathbf{k}_\perp{}^2}$ Cancels in emission from color neutral dipole.

soft $\int_{\sim 0}\frac{\mathrm{d}z}{z}$ Not really divergence, but needs to be resummed

- The limit $z\to 0$: large $q\bar{q}g$ invariant mass: $M_{q\bar{q}g}\to\infty$
- We are working in the "$s=\infty$" eikonal approximation
- Physically, one must however have $M^2_{q\bar{q}g} < s \implies z \gtrsim [\perp]/s$
  (where "$\perp$" is some relevant transverse momentum scale.)
- Thus "divergence" is a sign of a large log $\sim \alpha_s \ln s \sim \alpha_s \ln 1/x$

Resum this large log using the Balitsky-Kovchegov equation

Soft gluons and large logs, idea of RGE



- Emitted gluons have z between 1 and x: each gluon contributes $\sim \alpha_s \ln 1/x$
- For x small $\alpha_s \ln 1/x \sim 1 \implies$ all n gluon emissions contribute same $\implies$ resum
- Resummation by renormalization

Is the gluon at y a part of γ^* or of p?

Physical cross section is the same.

$$\sigma^{\gamma^* p} = \overbrace{\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_y \otimes 2\mathcal{N}_y^{q\bar{q}+p} + \left|\psi^{\gamma^*\to q\bar{q}g}\right|^2_y \otimes 2\mathcal{N}_y^{q\bar{q}g+p} + \dots}^{\text{gluons up to } y \text{ are part of proton}}$$

$$= \underbrace{\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_{y+\Delta y} \otimes 2\mathcal{N}_{y+\Delta y}^{q\bar{q}+p} + \left|\psi^{\gamma^*\to q\bar{q}g}\right|^2_{y+\Delta y} \otimes 2\mathcal{N}_{y+\Delta y}^{q\bar{q}g+p} + \dots}_{\text{gluons up to } y+\Delta y \text{ are part of proton}}$$

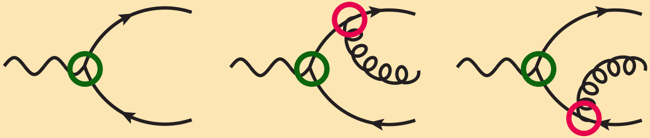
Can calculate $\left|\psi^{\gamma^*\to q\bar{q}}\right|^2_y$'s $\implies$ get differential equation for unknown $\mathcal{N}$

Quick derivation of the BK equation

Let's put this idea into practice. We will

- Calculate $\psi^{\gamma^* \to q\bar{q}g}(z)$
- Take soft gluon limit $z \to 0$
- Reabsorb the gluon to become a part of the target
- Get evolution equation for $q\bar{q}$ cross section

We need:



In the soft gluon limit $z \to 0$ this calculation simpfies, because

$$\psi^{\gamma^* \to q\bar{q}g} \approx \psi^{\gamma^* \to q\bar{q}} \left(\psi^{q \to qg} + \psi^{\bar{q} \to \bar{q}g} \right)$$

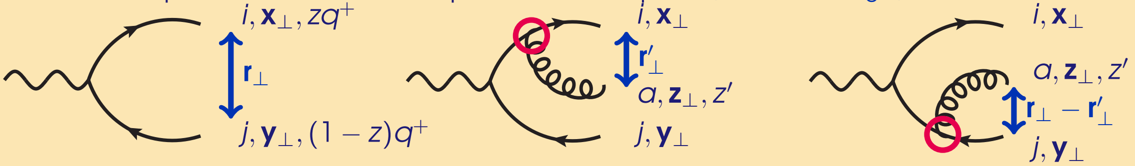
This is true only for limit $z \to 0$ where $k^-_{q\bar{q}g} - k^-_{\gamma^*} \approx k^-_g \sim 1/z$
(In the full kinematics the gluon emission knows about the γ^*, not just the emitting parent $q/\bar{q}$ $\Longrightarrow$ this makes full NLO cross section computation much complex)

Gluon emission from coordinate space dipole

First step: Fourier-transform the gluon emission wavefunction to coordinate space

$$\psi^{q\to qg}(k^+,\mathbf{r}_\perp) = \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\psi^{q\to qg}(k^+,\mathbf{k}_\perp) = -i2p^+\frac{2gt^a_{ji}}{2\pi}\frac{\varepsilon_\perp\cdot\mathbf{r}_\perp}{\mathbf{r}_\perp{}^2}\delta_{s,s'}$$

Second step: sum emission from quark and antiquark (note relative sign!)



$$\begin{aligned}|\gamma^*\rangle_{\text{int}} = |\gamma^*\rangle &+ \int_{z,\mathbf{r}_\perp} C(\mathbf{r}_\perp)\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\,|q_i(\mathbf{x}_\perp,z)\bar{q}_i(\mathbf{y}_\perp,1-z)\rangle \\ &+ \int_{z,\mathbf{r}_\perp,\mathbf{r}'_\perp}\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\int\frac{dz'}{4\pi z'}\frac{-i2g}{2\pi}t^a_{ji}\left[\frac{(\mathbf{x}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_\perp}{(\mathbf{x}_\perp-\mathbf{z}_\perp)^2}-\frac{(\mathbf{y}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_\perp}{(\mathbf{y}_\perp-\mathbf{z}_\perp)^2}\right] \\ &\qquad\times|q_i(\mathbf{x}_\perp,z)\bar{q}_j(\mathbf{y}_\perp,1-z)g_a(\mathbf{z}_\perp,z')\rangle\end{aligned}$$

Virtual term

(Take a "unitarity" shortcut instead of calculating loop diagram)

$$|\gamma^*\rangle_{\text{int}} = |\gamma^*\rangle + \int_{z,\mathbf{r}_\perp} C(\mathbf{r}_\perp)\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\,|q_i(\mathbf{x}_\perp,z)\bar{q}_i(\mathbf{y}_\perp,1-z)\rangle + \dots$$

Gluon emission $\Longrightarrow$ correct the normalization of $|q\bar{q}\rangle$ by $C(\mathbf{r}_\perp) = 1 + \mathcal{O}(\alpha_s)$

$$N_c\,|C(\mathbf{r}_\perp)|^2 =$$

$$N_c - \frac{(2g)^2}{(2\pi)^2}\frac{1}{4\pi}t^a_{ij}t^a_{ji}\int\frac{dz'}{z'}\int d^2\mathbf{r}'_\perp \sum_{\lambda=\pm 1}\left|\frac{(\mathbf{x}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_{\perp\lambda}}{(\mathbf{x}_\perp-\mathbf{z}_\perp)^2} - \frac{(\mathbf{y}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_{\perp\lambda}}{(\mathbf{y}_\perp-\mathbf{z}_\perp)^2}\right|^2$$

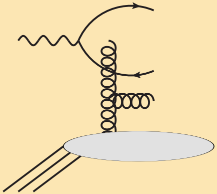
$$= N_c - \frac{\alpha_s}{\pi^2}\frac{{N_c}^2-1}{2}\Delta y\int d^2\mathbf{r}'_\perp\frac{{\mathbf{r}_\perp}^2}{{\mathbf{r}'_\perp}^2(\mathbf{r}_\perp-\mathbf{r}'_\perp)^2}$$

$$\sum_{\lambda=\pm 1}\varepsilon_i^{(\lambda)}\varepsilon_j^{(\lambda)*} = \delta_{ij}$$

Crucial step: move the gluon to the target

Absorb corrections from gluon to a redefinition of the $q\bar{q}$ amplitude

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2}\frac{N_c{}^2-1}{2N_c}\int_y^{y+\Delta y} \mathrm{d}\ln 1/z' \int \mathrm{d}^2\mathbf{r}'_\perp \frac{\mathbf{r}_\perp{}^2}{\mathbf{r}'_\perp{}^2(\mathbf{r}_\perp-\mathbf{r}'_\perp)^2}\left[\mathcal{N}_{q\bar{q}g}^{\ln 1/z'} - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}\right]$$



Dipole scattering on new target $\mathcal{N}_{q\bar{q}}^{y+\Delta y}$ is

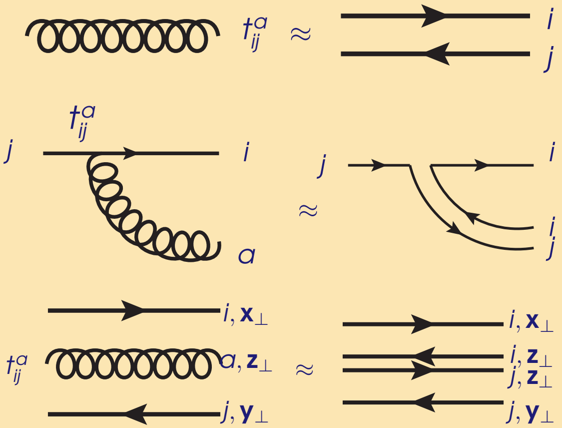
- Dipole scattering off original target $\mathcal{N}_{q\bar{q}}^{y}$
- Dipole emits a gluon into rapidity interval $[y, y+\Delta y]$, which scatters off target
- Normalization of original dipole is corrected
  (There are now less dipoles in γ^*)

Almost there

Want equation for $\mathcal{N}_{q\bar{q}}$: but enocuntered new quantity $\mathcal{N}_{q\bar{q}g}$.
$\Longrightarrow$ Relate to $\mathcal{N}_{q\bar{q}}$ in large N_c approximation

Gluon at large N_c

- At large $N_c \implies$ gluon = $q\bar{q}$ pair
  (not dipole!)
- $N_c^2 - 1$ gluon colors $\approx N_c^2$ quark-antiquark pair colors.
- Had $|q(\mathbf{x}_\perp)\bar{q}(\mathbf{y}_\perp)g(\mathbf{z}_\perp)\rangle$
- Approximate by
  $|q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)\rangle$



Now, instead of $\mathcal{N}_{q\bar{q}g}$, we need $\mathcal{N}_{q\bar{q}q\bar{q}}$:
amplitude for simultaneous scattering of two dipoles.

(Note: the gluon is not becoming a new dipole, but the common end of two new dipoles.)

Two dipole scattering amplitude

- $\mathcal{N}$ is really **scattering probability**;
- $S = 1 - \mathcal{N}$ is probability **not to scatter**

For two dipoles:

- No scattering: neither dipole scatters $\Longrightarrow S_{q\bar{q}q\bar{q}} = S_{q\bar{q}} S_{q\bar{q}}$
- Scattering probability $\mathcal{N}_{q\bar{q}q\bar{q}} = 1 - S_{q\bar{q}q\bar{q}} = 1 - (1 - \mathcal{N}_{q\bar{q}})(1 - \mathcal{N}_{q\bar{q}})$

Thus we end up with the approximation:

$$\mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{y}_\perp)g(\mathbf{z}_\perp)} \approx \mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)} + \mathcal{N}_{q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)} - \mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)} \mathcal{N}_{q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)}$$

and our equation is

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2} \frac{{N_c}^2 - 1}{2N_c} \int_y^{y+\Delta y} \mathrm{d} \ln 1/z' \int \mathrm{d}^2\mathbf{z}_\perp \frac{(\mathbf{x}_\perp - \mathbf{y}_\perp)^2}{(\mathbf{x}_\perp - \mathbf{z}_\perp)^2 (\mathbf{z}_\perp - \mathbf{y}_\perp)^2}$$
$$\times \Big[\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{z}_\perp) + \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z}_\perp, \mathbf{y}_\perp) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{z}_\perp) \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z}_\perp, \mathbf{y}_\perp)$$
$$- \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{y}_\perp)) \Big]$$

Which is easy to write differentially in y

Summary

Balitsky-Kovchegov equation (∼1995)

$$\partial_y \mathcal{N}(\mathbf{r}_\perp) = \frac{\alpha_s N_c}{2\pi^2} \int d^2\mathbf{r}'_\perp \frac{\mathbf{r}_\perp{}^2}{\mathbf{r}'_\perp{}^2 (\mathbf{r}'_\perp - \mathbf{r}_\perp)^2} \left[\mathcal{N}(\mathbf{r}'_\perp) + \mathcal{N}(\mathbf{r}_\perp - \mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r}'_\perp)\mathcal{N}(\mathbf{r}_\perp - \mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r}_\perp)\right]$$

This is the basic tool of saturation physics.

- Given initial condition $\mathcal{N}(\mathbf{r}_\perp)$ at $y = y_0$ the equation predicts the scattering amplitude at larger y = smaller x = higher $\sqrt{s}$.
- Drop nonlinear term: BFKL equation
- Divergences at $\mathbf{r}'_\perp \to 0$ and $\mathbf{r}'_\perp \to \mathbf{r}_\perp$ regulated because $\mathcal{N}(0) = 0$ due to color neutrality.
- Enforces black disk limit $\mathcal{N} < 1$
- Coupling α_s should depend on distance (some combination of $\mathbf{r}_\perp, \mathbf{r}'_\perp, \mathbf{r}_\perp - \mathbf{r}'_\perp$)

Conclusions, lecture 1

- Dipole picture of DIS: $\gamma^* \to q\bar{q}$
- Light cone quantization: partons in γ^*
- CGC: target is dense gluon field
- BK equation: add one soft gluon, absorb into redefinition of target

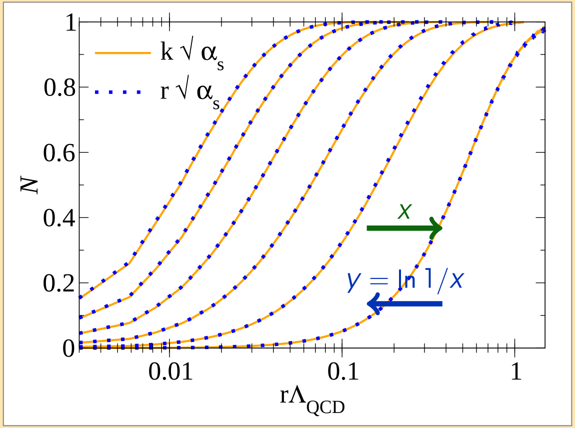
Lecture 2: NLO

- Dipole picture DIS at NLO
- Diffractive structure function at NLO
- Mass renormalization LCPT and dipole picture DIS with quark masses

What the solution of BK looks like

BK can be solved numerically

- Small dipoles $r \lesssim 1/Q_s$ scatter very little
- Large dipoles $r \gtrsim 1/Q_s$ scatter with probability almost one, but not more
- **Saturation** scale in between grows with $\ln 1/x$.



(Actually cheating, this plot is a solution of JIMWLK, which generalizes BK)

Remember, for F_2, F_L convolute this with the $\psi^{\gamma^* \to q\bar{q}}$

$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{b}_\perp \, \mathrm{d}^2\mathbf{r}_\perp \, \mathrm{d}z \left| \psi^{\gamma^* \to q\bar{q}}(r,z)_{T,L} \right|^2 2\mathcal{N}(\mathbf{r}_\perp, \mathbf{b}_\perp, x)$$