Finite size effects on the conserved charge fluctuations

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What are the typical sizes?

- Typical size of the fireball in **heavy ion collisions** is a few fm.
- Neutron stars and **compact stars** built up from strongly interacting matter (with extra structure) with a size $\sim$ 10 km.
- Several **models with finite** (different) size.
- In field theoretical calculations (LSM, NJL, DS, etc): **infinite size**.

Why does it matter?

- The properties of the system can change significantly.
- Criticality in a finite system?
- The CEP and the first-order region might "disappear".

Might be studied in field theoretical models by implementating the finite size effects.

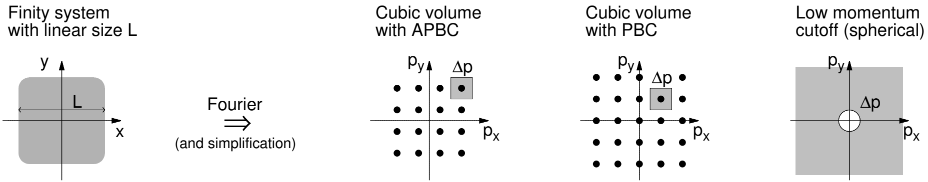
In the thermodynamic limit:

$\mathcal{Z} = \int \mathcal{D}\phi\, e^{iS(\phi)}$, MF: $\phi(x) \to \bar{\phi} + \varphi(x),\ \varphi = 0$, $\mathcal{Z} \Rightarrow U_{\text{eff}}$, minimize $U_{\text{eff}}(\bar{\phi})$

Finite size effects without losing the advantages U_{eff}?

Keep U_{eff}, modify only the momentum space.

- Discretization: $\int dp \to \sum_n$
- Low momentum cutoff: $\int_0^\infty dp \to \int_\lambda^\infty dp$



e.g. for eLSM: Phys.Rev.D **108** (2023) 7, 076010

Volume dependence

There is more than the momentum space!

$$\mathcal{Z} = \int \mathcal{D}\phi e^{iS(\phi)}, \qquad S(\phi) = \int d^4x\, \mathcal{L}(\phi(x), x) \tag{1}$$

Constant, homogenous field + local Lagrangian: $\int d^4x = \mathcal{V}_4 \xrightarrow{\text{fin } T} \beta V$

Integration over single $\mathbb{R}$ valued field ϕ:

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S_E(\phi)} \xrightarrow{\text{scalar field}} \int_{-\infty}^{\infty} d\phi\, e^{-S_E(\phi)} \tag{2}$$

V dependence separates from the potential: $\qquad S_E = \beta V \cdot U(\phi)$

The expectation value:

$$\langle\phi\rangle = \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} d\phi\, \phi\, e^{-\beta V\, U(\phi)} \tag{3}$$

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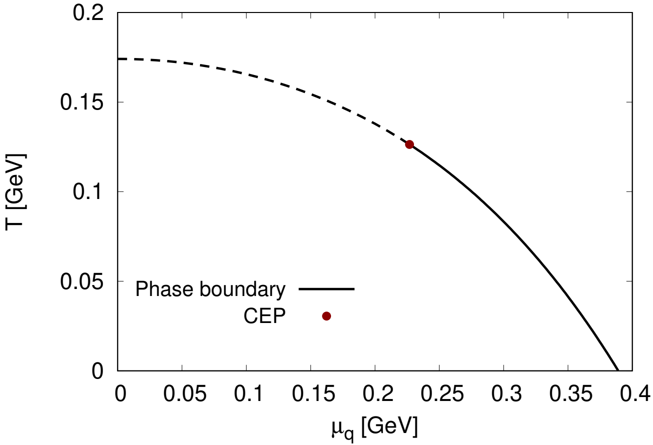
Simple quark-meson type model (classical potential + fermionic thermal fluct.):

$$U(\phi, T, \mu) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - h\phi + 2TN_c \int_L \frac{dp^3}{(2\pi)^3} \left[\log(1 + e^{-\beta(E-\mu)}) + \log(1 + e^{-\beta(E+\mu)})\right] \tag{4}$$

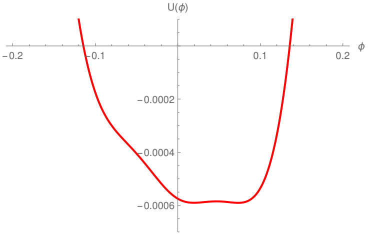
$V \to \infty$, $\dfrac{\partial U(\bar{\phi})}{\partial \bar{\phi}} = 0$ $\Rightarrow$ Crossover at $\mu = 0$, 1st order at $T = 0$, CEP at finite T and μ.

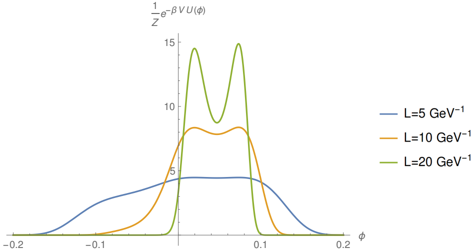


At finite V:

$$\langle\phi\rangle = \int_{-\infty}^{\infty} d\phi\, \phi\, P(\phi, V), \qquad P(\phi, V) = e^{-\beta V\, U(\phi)}/\mathcal{Z} \tag{5}$$

For a fixed T and μ with multiple solutions in MF (1 fm $= 5$ GeV^{-1}):



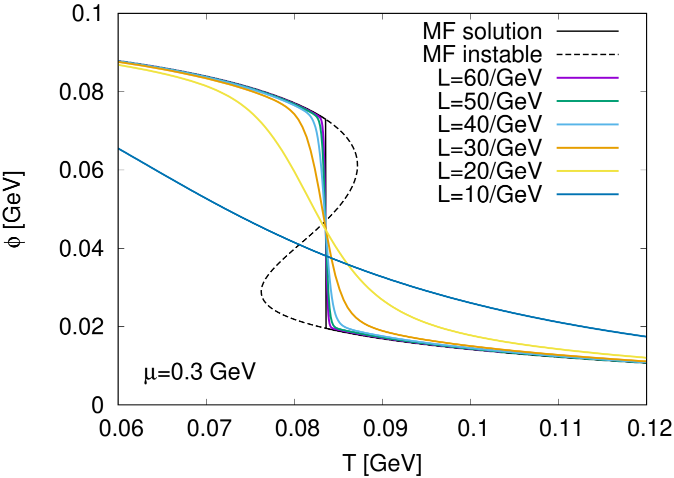


Mean-field:

$$\langle\phi\rangle = \bar{\phi} \Leftarrow \frac{\partial U(\bar{\phi})}{\partial\bar{\phi}} = 0$$

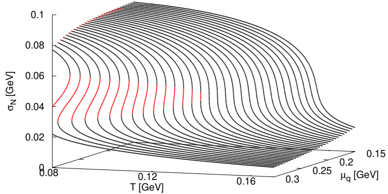
Full V dependence

$$\begin{aligned}\langle\phi\rangle &= \frac{1}{\beta V}\frac{\partial \ln \mathcal{Z}}{\partial h}\\ &= \frac{1}{\mathcal{Z}}\int_{-\infty}^{\infty} d\phi \left(-\frac{\partial U}{\partial h}\right) e^{-\beta V U(\phi)}\\ &= \int_{-\infty}^{\infty} d\phi\, \phi\, P(\phi, V)\end{aligned}$$

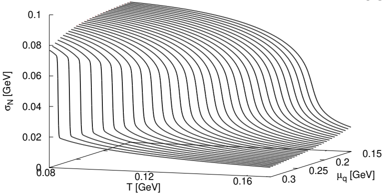


$\langle \phi \rangle$ AT SEVERAL SIZES

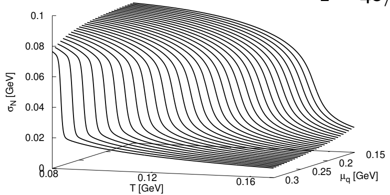
MF



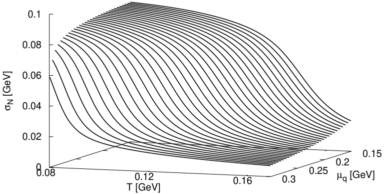
$L = 60/GeV$



$L = 40/GeV$



$L = 20/GeV$



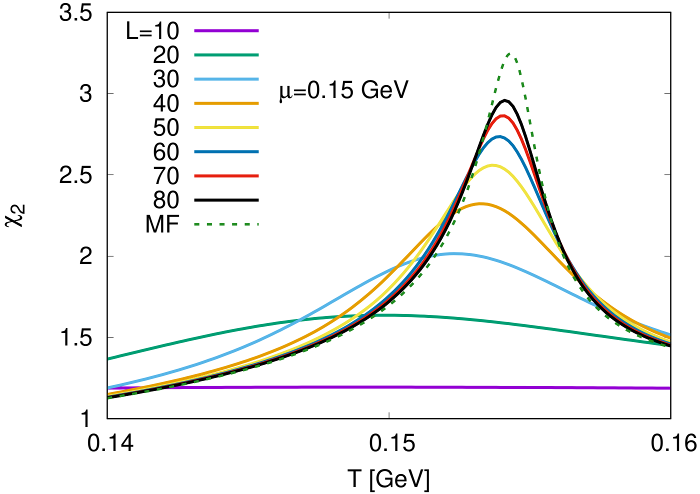
V dependence of the free energy: $\Phi = -T \ln \mathcal{Z}$

Ecpectation value of ϕ and its fluctuations

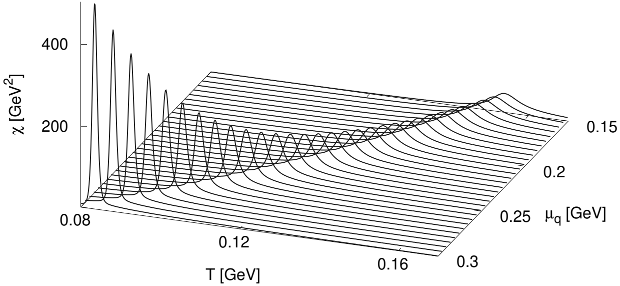
$$\begin{aligned}
\langle\phi\rangle &= \frac{1}{\beta V}\frac{\partial \ln \mathcal{Z}}{\partial h} = -\frac{1}{V}\frac{\partial \Phi}{\partial h}, \\
\chi_2 &= \frac{\partial\langle\phi\rangle}{\partial h} = \frac{1}{\beta V}\left(\frac{1}{\mathcal{Z}}\frac{\partial^2 \mathcal{Z}}{\partial h^2} - \left(\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial h}\right)^2\right) = \beta V\left(\langle\phi^2\rangle - \langle\phi\rangle^2\right) \\
\chi_3 &= \frac{\partial^2\langle\phi\rangle}{\partial h^2}, \qquad \chi_4 = \frac{\partial^3\langle\phi\rangle}{\partial h^3}
\end{aligned} \tag{6}$$

The pressure: $P = -\frac{\partial \Phi}{\partial V}$, The particle number: $\langle N\rangle = -\frac{\partial \Phi}{\partial \mu} = \frac{V}{\mathcal{Z}}\int_{-\infty}^{\infty} d\phi \frac{\partial U}{\partial \mu} e^{-\beta V U}$

Generally: $\langle O\rangle = \mathcal{Z}^{-1}\int d\phi\, O(\phi)\, e^{-\beta V\, U(\phi)}$, for O observables


L=10
20
30
40
50
60
70
80
MF
μ=0.15 GeV
3.5
3
2.5
2
1.5
1
0.14
0.15
0.16
χ_2
T [GeV]

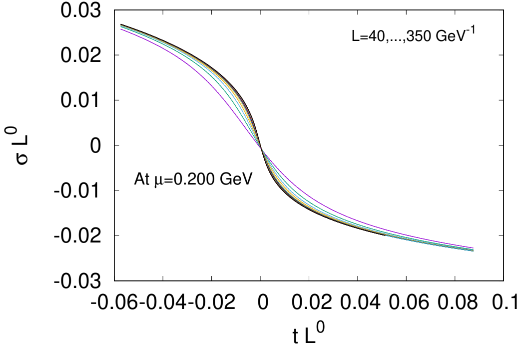
In the 1st order region χ keeps increasing. E.g. at $L = 40/\text{GeV}$:

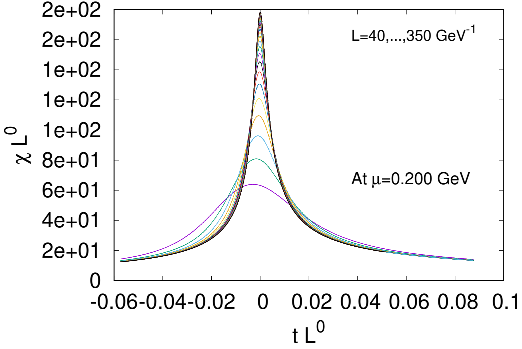


Reason: $\sim V$ scaling at coexistence (see below)

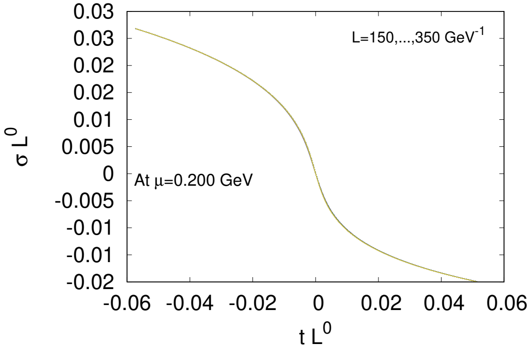
Scaling

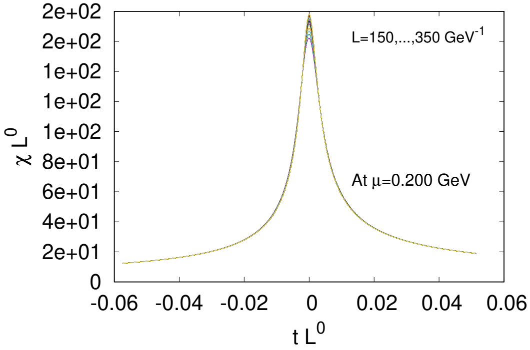
Expectation: $\sigma(tL^{0}) \propto L^{0}, \chi(tL^{0}) \propto L^{0}$



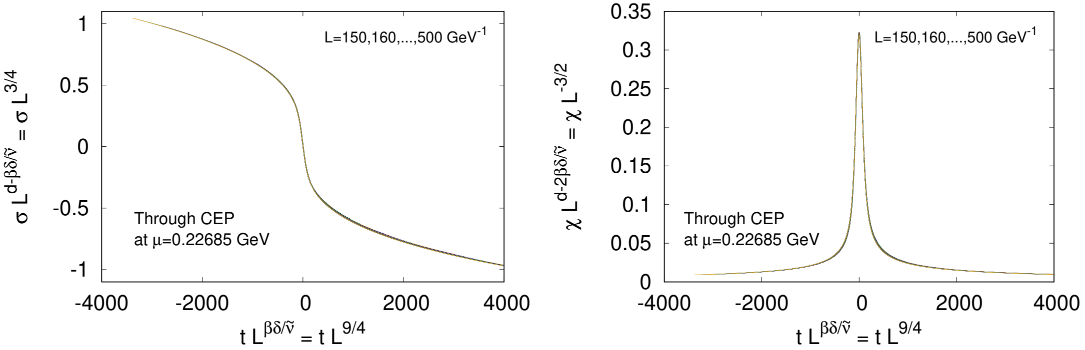


Expectation: $\sigma(tL^{o}) \propto L^{o}, \chi(tL^{o}) \propto L^{o}$

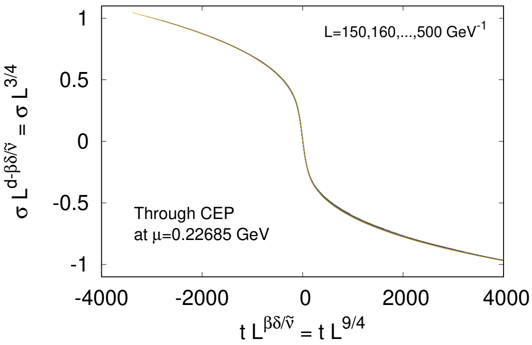


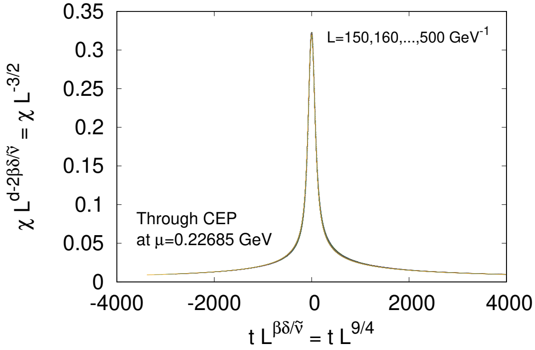


Expectation: $\sigma(tL^{\beta\delta/\tilde{\nu}}) \propto L^{\beta\delta/\tilde{\nu}-d}$, $\chi(tL^{\beta\delta/\tilde{\nu}}) \propto L^{2\beta\delta/\tilde{\nu}-d}$ MF at 3D: hyperscaling violation



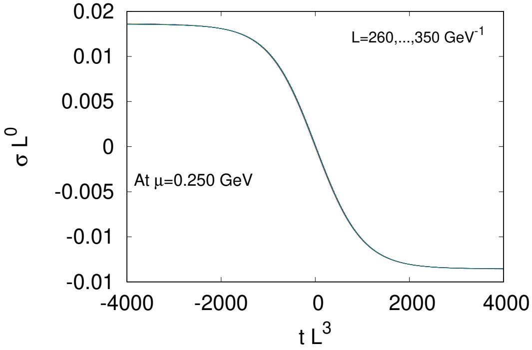
Expectation: $\sigma(tL^{\beta\delta/\tilde{\nu}}) \propto L^{\beta\delta/\tilde{\nu}-d}$, $\chi(tL^{\beta\delta/\tilde{\nu}}) \propto L^{2\beta\delta/\tilde{\nu}-d}$ $\tilde{\nu} = d/(\gamma + 2\beta)$

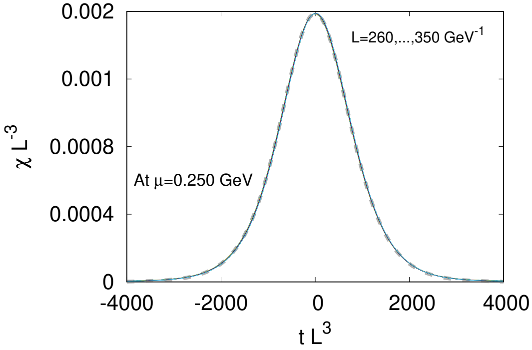




Expectation: $\sigma(tL^3) \propto L^0$, $\chi(tL^3) \propto L^3$

coexistance: $\chi = V\chi_\delta + (\chi_1 + \chi_2)/2$





Using the double Gaussian approximation

$$P(\phi) = \frac{A}{(2\pi\tau)^{1/2}} \left[e^{-((\phi-\sigma_1)^2)-2\chi_1\phi\eta)L^d/(2\tau\chi_1)} + e^{-((\phi-\sigma_2)^2)-2\chi_2\phi\eta)L^d/(2\tau\chi_2)} \right] \quad (7)$$

with A is for the proper normalization $\int d\phi P(\phi) = 1$

Using $\langle\phi\rangle = \int d\phi\, \phi P(\phi)$, $\langle\phi^2\rangle = \int d\phi\, \phi^2 P(\phi)$, and $\chi = V/\tau(\langle\phi^2\rangle - \langle\phi\rangle^2)$ gives

$$\begin{aligned} \langle\phi\rangle &= \lambda_1 U_1 + \lambda_2 U_2 && (8) \\ \chi &= \chi_1 U_1 + \chi_2 U_2 + \frac{L^d}{\tau}(\lambda_1 - \lambda_2)^2 U_1 U_2 && (9) \end{aligned}$$

width introducing $U_i = W_i/(W_1 + W_2)$, $W_i = \chi_i^{1/2} e^{\eta L^d(\chi_i\eta+2\sigma_i)/(2\tau)}$, $\lambda_i = \sigma_i + \eta\chi_i$
With $\sigma_{1/2} = \bar{\sigma} \pm \delta\sigma$, $\chi_{1/2} = \bar{\chi} \pm \delta\chi$, and assuming $\delta\chi$ is small:

$$\chi\big|_{\delta\chi\to 0} = \bar{\chi} + \frac{L^d}{\tau}(\lambda_1 - \lambda_2)^2 \frac{1}{4} \cosh^{-2}\left(\eta L^d(\sigma_1 - \sigma_2)/(2\tau)\right) = \bar{\chi} + \frac{L^d}{\tau}\delta\sigma^2 \cosh^{-2}\left(\eta L^d \delta\sigma/\tau\right) \quad (10)$$

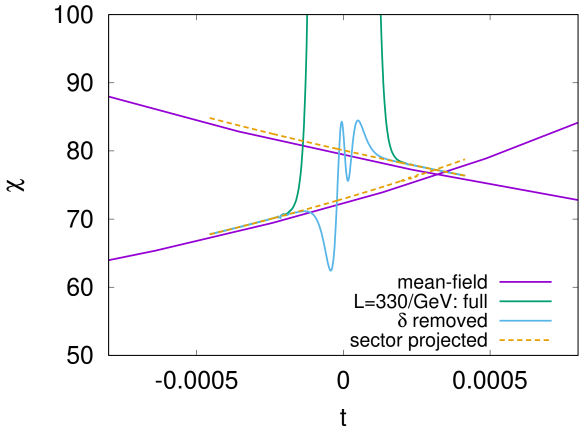
Binder: Lect. Notes Phys. 409, 59 (1992)

Check at $L = 330$ GeV^{-1}

Peak from coexistence is clearly visible

The sector projected curves tend to MF

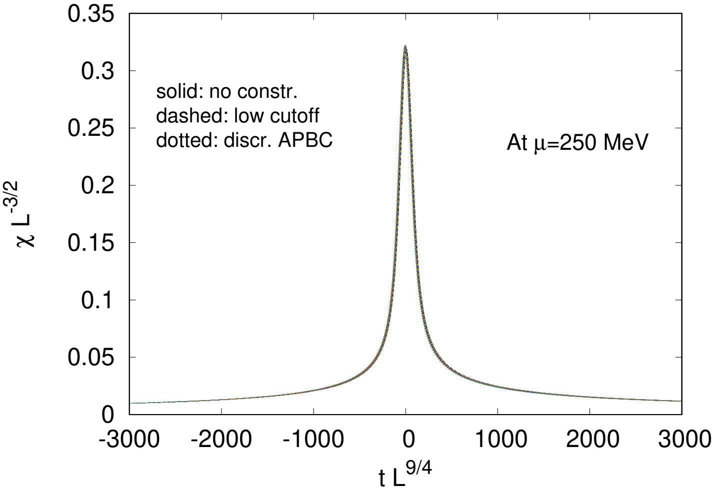
One may fit the χ_δ part and remove it from full (rough approximation)



Contraints:

- low momentum cutoff
- discretization with PBC/APBC

No change in the scaling for large *V*, as expected.



Baryon fluctuations

Cumulants from the grand free energy $\Phi = -T \ln \mathcal{Z}$

$$\kappa_n = -\frac{\partial^n \Phi}{\partial \mu^n} \tag{11}$$

hence (with c_n the central moments)

$$\begin{aligned}
\kappa_1 &= c_1 = \langle N \rangle, \\
\kappa_2 &= c_2 = \langle N^2 \rangle - \langle N \rangle^2, \\
\kappa_3 &= c_3 = \langle N^3 \rangle - 3\langle N^2 \rangle \langle N \rangle + 2\langle N \rangle^3, \\
\kappa_4 &= c_4 - 3c_2^2 = \langle N^4 \rangle - 4\langle N^3 \rangle \langle N \rangle + 12\langle N^2 \rangle \langle N \rangle^2 - 3\langle N^2 \rangle^2 - 6\langle N \rangle^4
\end{aligned}$$

Quantities per volume can be related to those in MF

$$k_n = \kappa_n / V \tag{12}$$

e.g. $k_1 = \langle n \rangle$ is the number density.

Both k_n and the naive "derivative of the pressure" gives the correct MF limit

$$k_n = -\frac{1}{V}\frac{\partial^n \Phi}{\partial \mu^2} \quad \neq \quad \tilde{\chi}_n^q = \frac{\partial^n p}{\partial \mu^2} = -\frac{\partial^n}{\partial \mu^2}\frac{\partial \Phi}{\partial V}$$

$$\searrow \quad V \to \infty \quad \swarrow$$

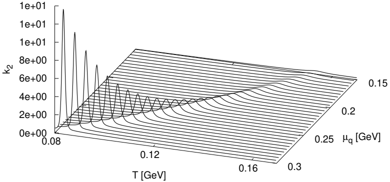
$$\chi_n^{q,\mathrm{MF}} = \frac{\partial^n p^{\mathrm{MF}}}{\partial \mu^2}$$

(The difference decreases as $V \to \infty$)

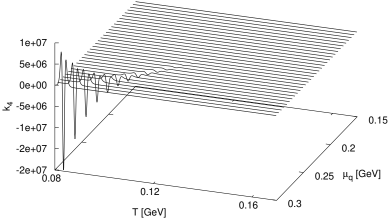
For each n, k_n shows the same structure and behavior as χ_n, the chiral fluctuation.

$L = 40/\text{GeV}$, increase in μ similar to $d\langle\phi\rangle/dh$

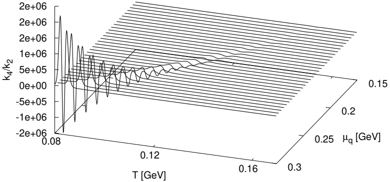
k_2



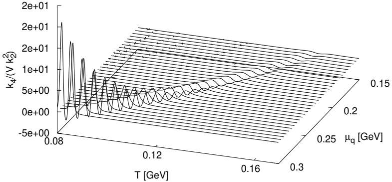
k_4



$\sigma_2\kappa = k_4/k_2$



$k_4/(V\,k_2^2)$

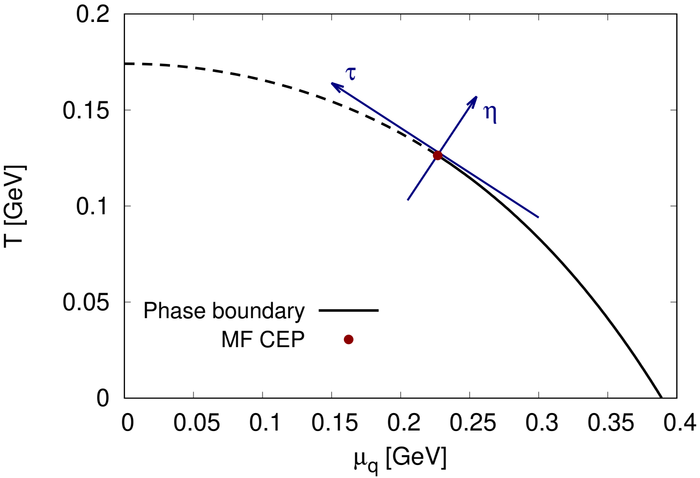


Along the boundary τ direction with $\tau = sgn(T - T_C)\sqrt{(T - T_C)^2 + (\mu - \mu_C)^2}$

The phase boundary
almost straight
in such a small section

Binder cumulant:
$\kappa_B = \frac{\chi_4}{L^d \chi_2^2}$
(sometimes called g_L)

$\kappa_B \propto L^0$, $\tau \propto L^{-1/\tilde{\nu}}$



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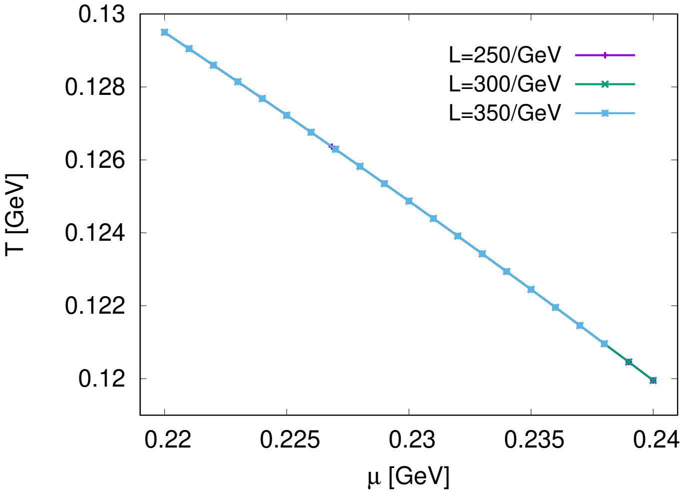

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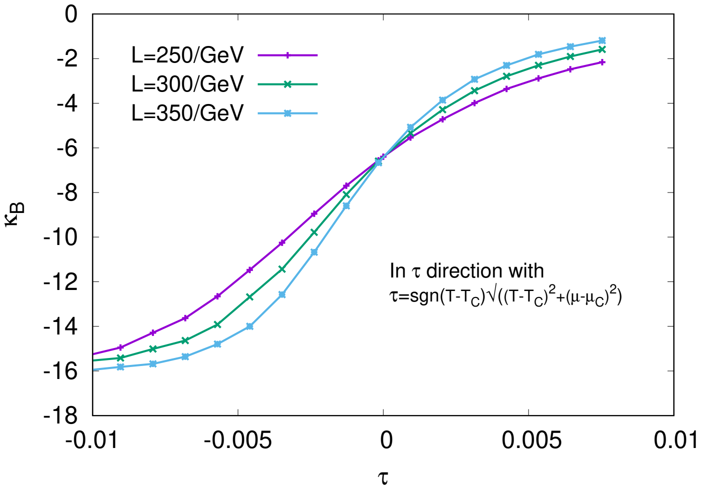

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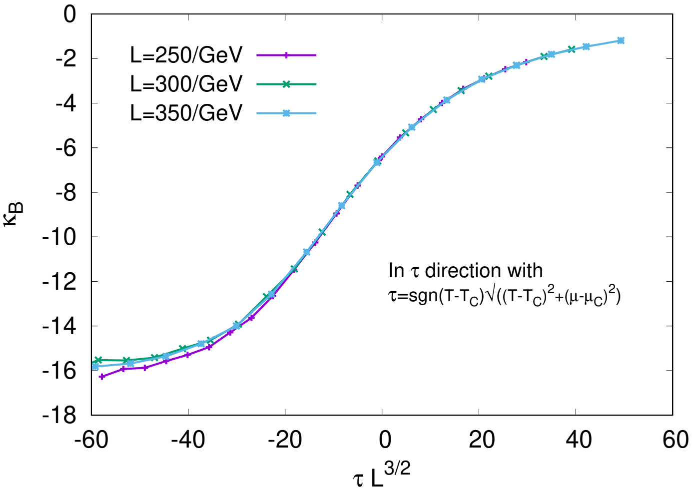
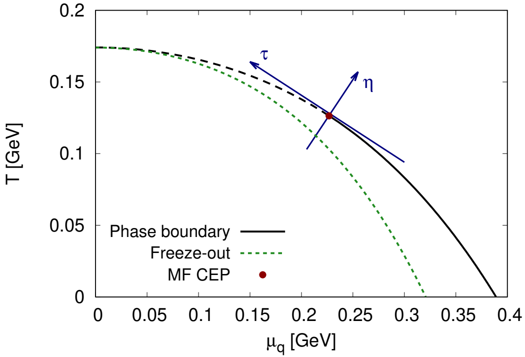
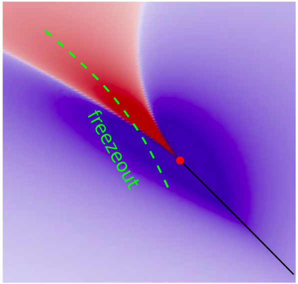


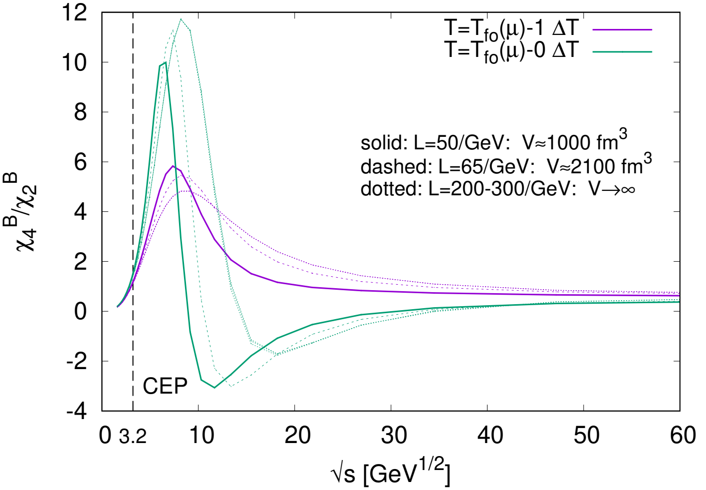


χ_4/χ_2 or k_4/k_2



Phys.Rept. 853 (2020) 1-87

$\mu_B = 3\mu_q$ to $\sqrt{s}$ according to Nature 561 (2018) 7723, 321-330



T=T_fo(μ)-1 ΔT
T=T_fo(μ)-0 ΔT
solid: L=50/GeV: V≈1000 fm^3
dashed: L=65/GeV: V≈2100 fm^3
dotted: L=200-300/GeV: V→∞
CEP
χ4^B/χ2^B
√s [GeV^1/2]
12
10
8
6
4
2
0
-2
-4
0
3.2
10
20
30
40
50
60

- Finite size effects goes beyond the momentum space constraints.
- Physical quantities can be calculated from the V-dependent free-energy.
- Correct scaling can be reproduced – Importance of the coexistence.
- No apparent CEP can be deduced as a maximum in the fluctuations.
- Fluctuations can be studied along the phase boundary and the freeze-out line.

- Other quantities to be calculated.
- Sensitivity to the relative location of CEP and the freeze-out.
- Structure with a disappearing CEP.
- Nontrivial V-dependent potential.

Thank you!