Exact Wigner function for chiral spirals

by Sudip Kumar Kar[1]

Collaborators: Samapan Bhadury ,Wojciech Florkowski

and Valeriya Mykhaylova

[1]Institute of Theoretical Physics, Jagiellonian University,

ul. St. Lojasiewicza 11, 30-348 Krakow, Poland

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Outline

- Motivation
- Background
- Chiral spirals and their solution
- Wigner functions
- Twist in polarization
- Breakdown of semiclassical expansion

Motivation

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Letter

Polarization of spin-$1/2$ particles with effective spacetime dependent masses

Samapan Bhadury [a], Arpan Das [b c], Wojciech Florkowski [a], Gowthama K.K. [d], Radoslaw Ryblewski [b]

$$\sigma(t,z) = \frac{M_0}{M(t)}\sigma_0\left(z - v(t-t_0)\right)$$

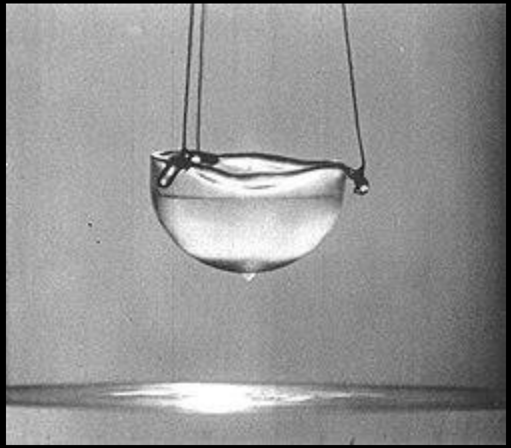
Spin density *Mass(time dependent)*

“ ....This behavior indicates that the spin polarization phenomena may be connected with the effects of chiral symmetry restoration. .....”

- As a starting point, it was natural to explore the multitude of condensates that were available to us from studies about pion condensation.
- Thus, chiral spiral was selected

Historical background

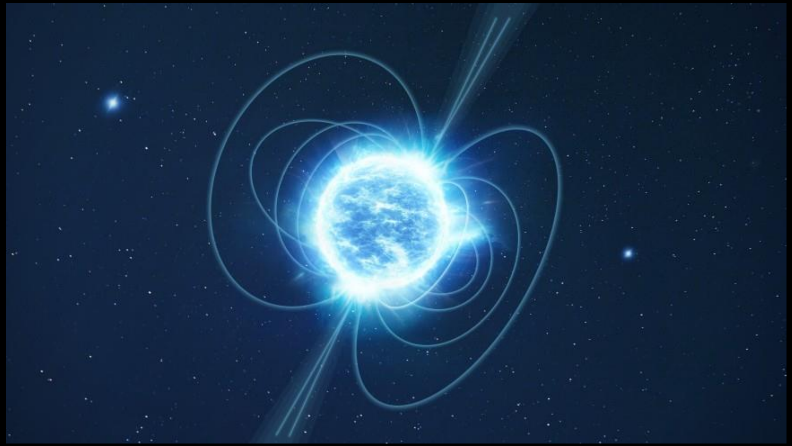
Bose-Einstein condensate



Superfluid He

Superconducting Metals

- A.B. Migdal and Independently R.F. Sawyer and D.J. Scalapino proposed the existence of a Bose-Einstein condensate in pions in 1972.



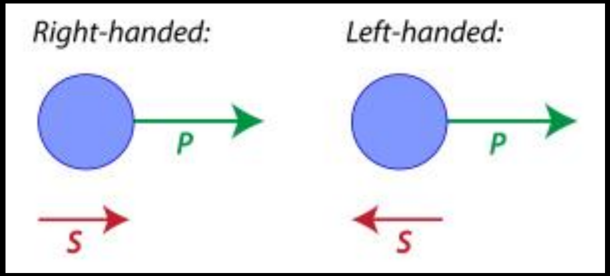
Neutron stars

Image credit : Wikipedia & ESA

Chiral spirals as a potential ground state of matter

- Chiral spirals → Mean fields that are used to study pion condensation. *[V. Schon and M. Thies, arXiv:hep-th/0008175.]*
- Have been studied thoroughly in last ~45 years

A spiral of condensates



Chiral symmetry

For example: NJL model with zero mass

Dirac particles with chiral symmetry

Interaction

Externally given mean field of for scalar and pseudoscalar condensates.

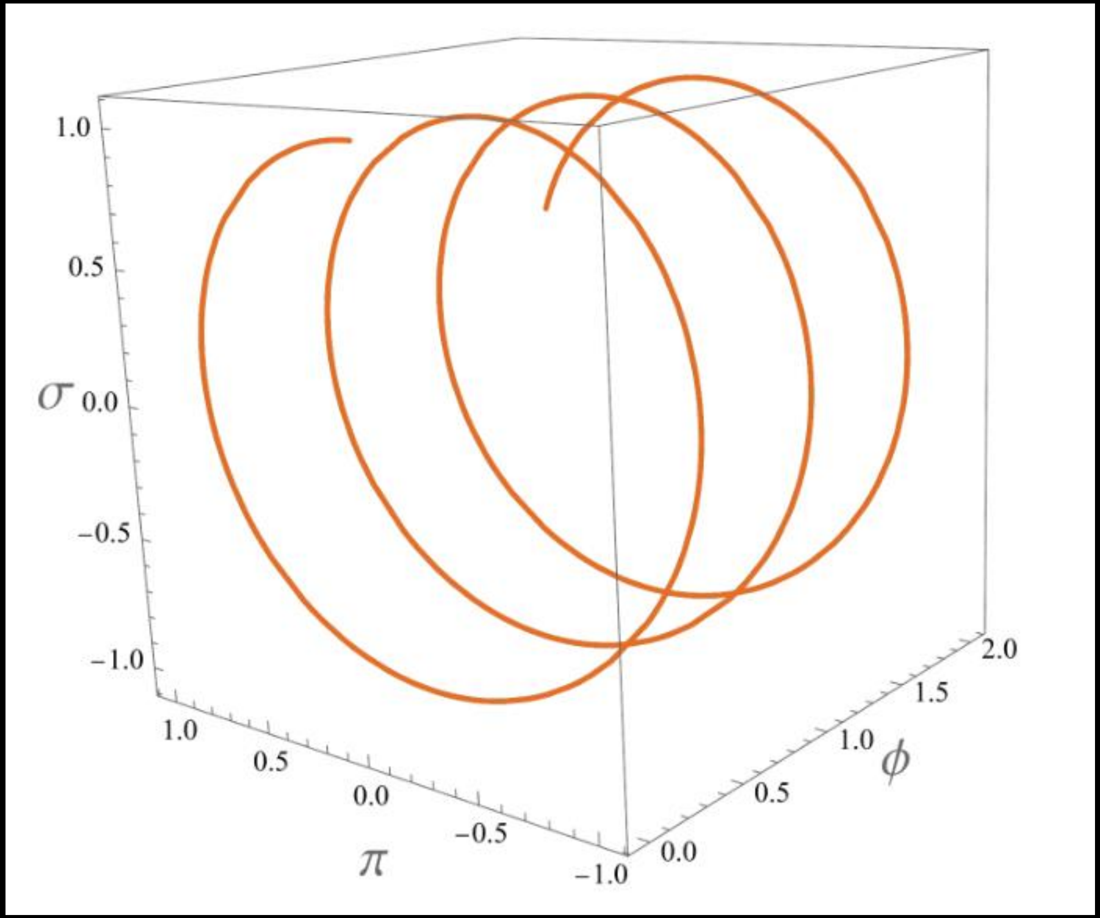
Governed by

$$\left[i\hbar\gamma_\mu\partial^\mu - \sigma - i\gamma_5\pi\right]\psi(x) = 0,$$

- In chiral spirals we assume a periodic form for the mean fields given as

$$\sigma = M\cos(\phi), \qquad \pi = M\sin(\phi).$$

Why Spirals?



We choose the following form for ϕ,

$$\phi = \frac{\mathbf{q} \cdot \mathbf{x}}{\hbar}$$

$\mathbf{q}$ is called the inhomogeneity factor

One ansatz to solve them all

- The forms of the condensate turns our dynamical equation into a Dirac-like equation given as

$$\left[i\hbar\,\gamma_\mu \partial^\mu - M e^{i\gamma_5(\mathbf{q}\cdot\mathbf{x})/\hbar}\right]\psi(x) = 0$$

- Can be solved using the ansatz [F. Dautry and E. M. Nyman, Nuclear Physics A 319, 323 (1979).]

$$\psi_\pm(x) = \exp\left(-\frac{i\gamma_5}{2}\frac{\mathbf{q}\cdot\mathbf{x}}{\hbar}\right)\chi_\pm(\mathbf{p})e^{\mp ip\cdot x/\hbar}$$

- Ansatz → space dependence gone

One ansatz to solve them all

- Now we need to solve the eigensystem

$$\begin{pmatrix} \pm\left(M - \frac{1}{2}\boldsymbol{\tau}\cdot\mathbf{q}\right) & \boldsymbol{\tau}\cdot\mathbf{p} \\ \boldsymbol{\tau}\cdot\mathbf{p} & \mp\left(M + \frac{1}{2}\boldsymbol{\tau}\cdot\mathbf{q}\right) \end{pmatrix}$$

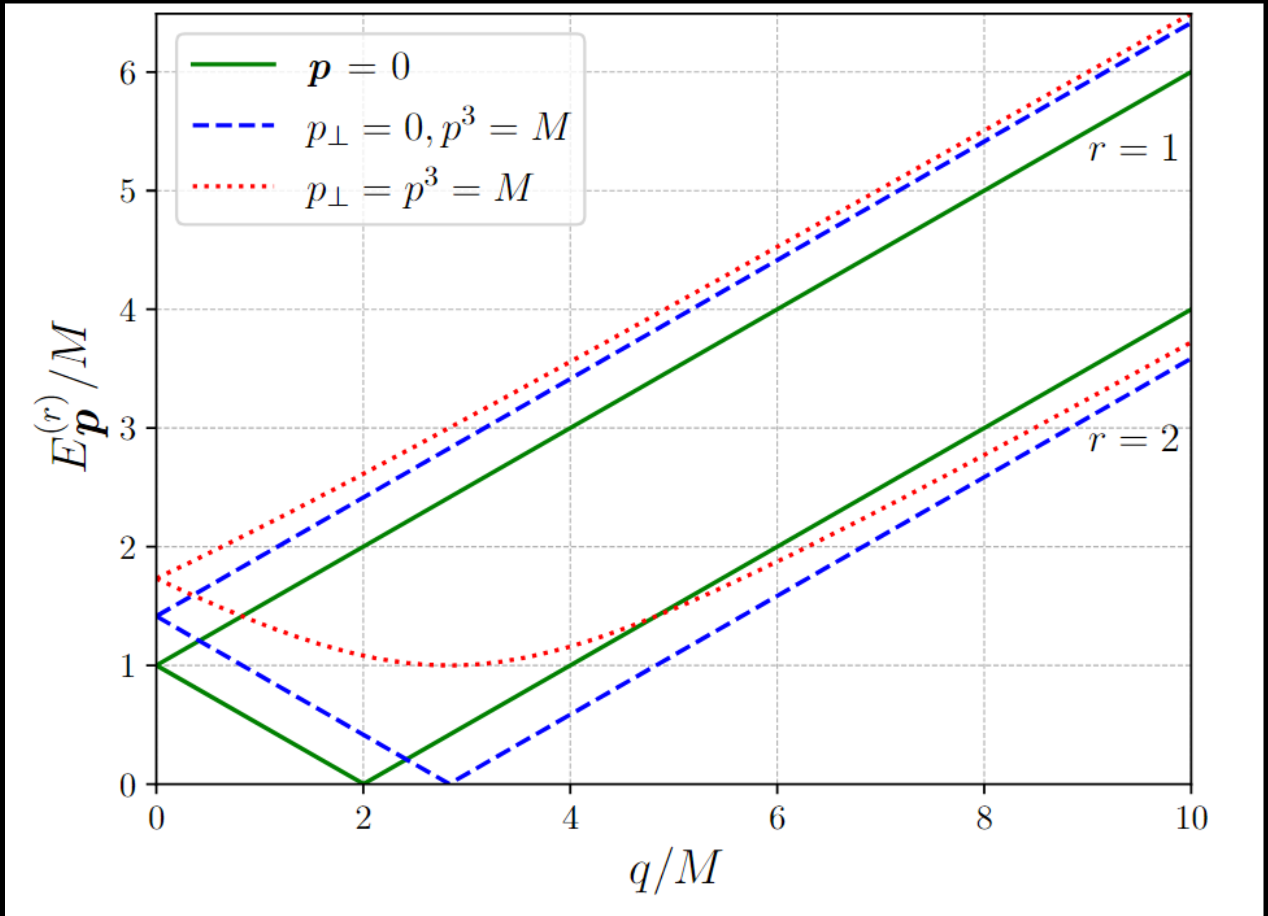
+ : "Positive" energy
- : "Negative" energy
$\boldsymbol{\tau}$: Pauli matrices

- The eigenvalues give the energy spectrum which features a split in energy for different spins ($q > 0$),

$$E_{\mathbf{p}}^{(r)} = \sqrt{\mathbf{p}^2 + q^2/4 + M^2 + (-1)^{r-1} q E_{\mathbf{p}}^{\parallel}} \quad (r = 1{,}2).$$

Where, $E_{\mathbf{p}}^{\parallel} = \sqrt{M^2 + (p^3)^2}$

Energy spectrum of the system



$r = 2 < r = 1$ → Natural inclination towards polarization → Wigner functions → Spinors

Spinors

$$\chi_{\pm}^{(r)}(\mathbf{p}) = N_{\pm}^{(r)} \begin{pmatrix} \dfrac{E_{\mathbf{p}}^{(r)} \pm (-1)^r E_{\mathbf{p}}^{\parallel} \mp \frac{q}{2}}{p^1 + ip^2} \\ \pm \dfrac{p^3}{M + (-1)^r E_{\mathbf{p}}^{\parallel}} \\ \pm \dfrac{E_{\mathbf{p}}^{(r)} \pm (-1)^r E_{\mathbf{p}}^{\parallel} \mp \frac{q}{2}}{p^1 + ip^2} \dfrac{p^3}{M + (-1)^r E_{\mathbf{p}}^{\parallel}} \\ 1 \end{pmatrix}$$

The factor $N_{\pm}^{(r)}$ is chosen such that the system is normalized to

$$\chi_{+}^{(r)\dagger}(\mathbf{p})\chi_{+}^{(r)}(\mathbf{p}) = 2E_{\mathbf{p}}^{(r)}, \quad \chi_{-}^{(r)\dagger}(\mathbf{p})\chi_{-}^{(r)}(\mathbf{p}) = 2E_{\mathbf{p}}^{(r)}.$$

Orthogonality of the spinors is given as

Chiral spiral	Free Dirac case
$\chi_{+}^{(r)\dagger}(\mathbf{p})\,\chi_{+}^{(s)}(\mathbf{p}) = 2E_{\mathbf{p}}^{(r)}\delta^{rs}$	$u^{(r)\dagger}(\mathbf{p})u^{(s)}(\mathbf{p}) = 2E_{\mathbf{p}}\delta^{rs}$
$\chi_{-}^{(r)\dagger}(\mathbf{p})\,\chi_{-}^{(s)}(\mathbf{p}) = 2E_{\mathbf{p}}^{(r)}\delta^{rs}$	$v^{(r)\dagger}(\mathbf{p})v^{(s)}(\mathbf{p}) = 2E_{\mathbf{p}}\delta^{rs}$
$\chi_{+}^{(r)\dagger}(\mathbf{p})\,\chi_{-}^{(s)}(-\mathbf{p}) = 0$	$u^{(r)\dagger}(\mathbf{p})v^{(s)}(-\mathbf{p}) = 0$
$\chi_{-}^{(r)\dagger}(-\mathbf{p})\,\chi_{+}^{(s)}(\mathbf{p}) = 0$	$v^{(r)\dagger}(-\mathbf{p})u^{(s)}(\mathbf{p}) = 0$
$\bar{\chi}_{+}^{(r)}(\mathbf{p})\,\chi_{-}^{(s)}(\mathbf{p}) = 0$	$\bar{u}^{(r)}(\mathbf{p})v^{(s)}(\mathbf{p}) = 0$
$\bar{\chi}_{-}^{(r)}(\mathbf{p})\,\chi_{+}^{(s)}(\mathbf{p}) = 0$	$\bar{v}^{(r)}(\mathbf{p})u^{(s)}(\mathbf{p}) = 0$

The completeness relation reads

$$\sum_{r=1}^{2} \frac{1}{2E_{\mathbf{p}}^{(r)}} \left[\chi_{+,a}^{(r)}(\mathbf{p})\chi_{+,b}^{(r)\dagger}(\mathbf{p}) + \chi_{-,a}^{(r)}(-\mathbf{p})\chi_{-,b}^{(r)\dagger}(-\mathbf{p})\right] = \delta_{ab}.$$

"Spin" of the spinors

- The $\mathbf{p} \to 0$ limit reveals the relationship between the index (r) and the direction of spin polarization.

"Up" $\chi_+^{(2)}(\mathbf{p} \to 0) = e^{-i\phi}\sqrt{\lvert 2M - q\rvert}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\uparrow$	"Down" $\chi_+^{(1)}(\mathbf{p} \to 0) = -\mathrm{sgn}(\cos\theta)\sqrt{2M+q}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\downarrow$
"Up" $\chi_-^{(2)}(\mathbf{p} \to 0) = \sqrt{\lvert 2M - q\rvert}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $\uparrow$	"Down" $\chi_-^{(1)}(\mathbf{p} \to 0) = e^{-i\phi}\mathrm{sgn}(\cos\theta)\sqrt{2M+q}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $\downarrow$

The spinor fields

- The general field is thus given as

$$\psi(x) = \int \frac{d^3p}{(2\pi\hbar)^{3/2}} \sum_{r=1,2} \frac{1}{\sqrt{2E_{\mathbf{p}}^{(r)}}} \left[u^{(r)}(\mathbf{p},\mathbf{x}) b_r(\mathbf{p}) e^{-\frac{i}{\hbar}p\cdot x} + v^{(r)}(\mathbf{p},\mathbf{x}) c_r^*(\mathbf{p}) e^{\frac{i}{\hbar}p\cdot x} \right]$$

- Where we have

$$u^{(r)}(\mathbf{p},\mathbf{x}) = \exp\left(-\frac{i\gamma_5}{2}\frac{\mathbf{q}\cdot\mathbf{x}}{\hbar}\right)\chi_+^{(r)}(\mathbf{p}) \quad v^{(r)}(\mathbf{p},\mathbf{x}) = \exp\left(-\frac{i\gamma_5}{2}\frac{\mathbf{q}\cdot\mathbf{x}}{\hbar}\right)\chi_-^{(r)}(\mathbf{p}).$$

- These fields are actually quantum, check

$$\{\psi_a(t,\mathbf{x}), \psi_b^\dagger(t,\mathbf{y})\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

$$\{\psi_a(t,\mathbf{x}), \psi_b(t,\mathbf{y})\} = \{\psi_a^\dagger(t,\mathbf{x}), \psi_b^\dagger(t,\mathbf{y})\} = 0,$$

- with $b_r(\mathbf{p})$, $c_r^\dagger(\mathbf{p})$ → standard anti-commutation relation

Wigner functions

- Wigner functions → Phase space distribution function in quantum systems
- For spin-1/2 systems, these are given as

$$W_{ab}(x,k) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{ik\cdot y}{\hbar}} \left\langle \bar{\psi}_b\left(x+\frac{y}{2}\right)\psi_a\left(x-\frac{y}{2}\right)\right\rangle$$

The only non zero exp. values

$$\langle b_s^\dagger(\mathbf{p}')b_r(\mathbf{p})\rangle = \delta^{(3)}(\mathbf{p}'-\mathbf{p})f_{\mathbf{p}}(1+\boxed{\boldsymbol{\zeta}_{\mathbf{p}}}\cdot\boldsymbol{\tau})_{sr}$$

$$\langle c_s^\dagger(\mathbf{p}')c_r(\mathbf{p})\rangle = \delta^{(3)}(\mathbf{p}'-\mathbf{p})\,\bar{f}_{\mathbf{p}}\,(1+\boxed{\bar{\boldsymbol{\zeta}}_{\mathbf{p}}}\cdot\boldsymbol{\tau})_{sr},$$

Axial current polarization

Clifford algebra decomposition of the Wigner function

$\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \Sigma_{\mu\nu}\}$
Basis for all 4 × 4 matrices ← Gamma matrices → Clifford algebra
$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1_{4\times 4}$

- Decomposition of Wigner function is thus

$$W_{ab} = \left[\mathcal{F} + i\gamma_5\mathcal{P} + \gamma_\mu\mathcal{V}^\mu + \gamma_\mu\gamma_5\mathcal{A}^\mu + \frac{1}{2}\Sigma_{\mu\nu}\mathcal{S}^{\mu\nu}\right]_{ab}.$$

- Components $\{\mathcal{F}, \mathcal{P}, \mathcal{V}^\mu, \mathcal{A}^\mu, \mathcal{S}^{\mu\nu}\}$ → Trace with W

Scalar Pseudoscalar Vector Axial vector Tensor

- Currents → Integrate the components over 4-momentum

Vector and axial vector currents

- The spatial part of vector current vanishes ($V^i(x)$=0)
- The temporal part is

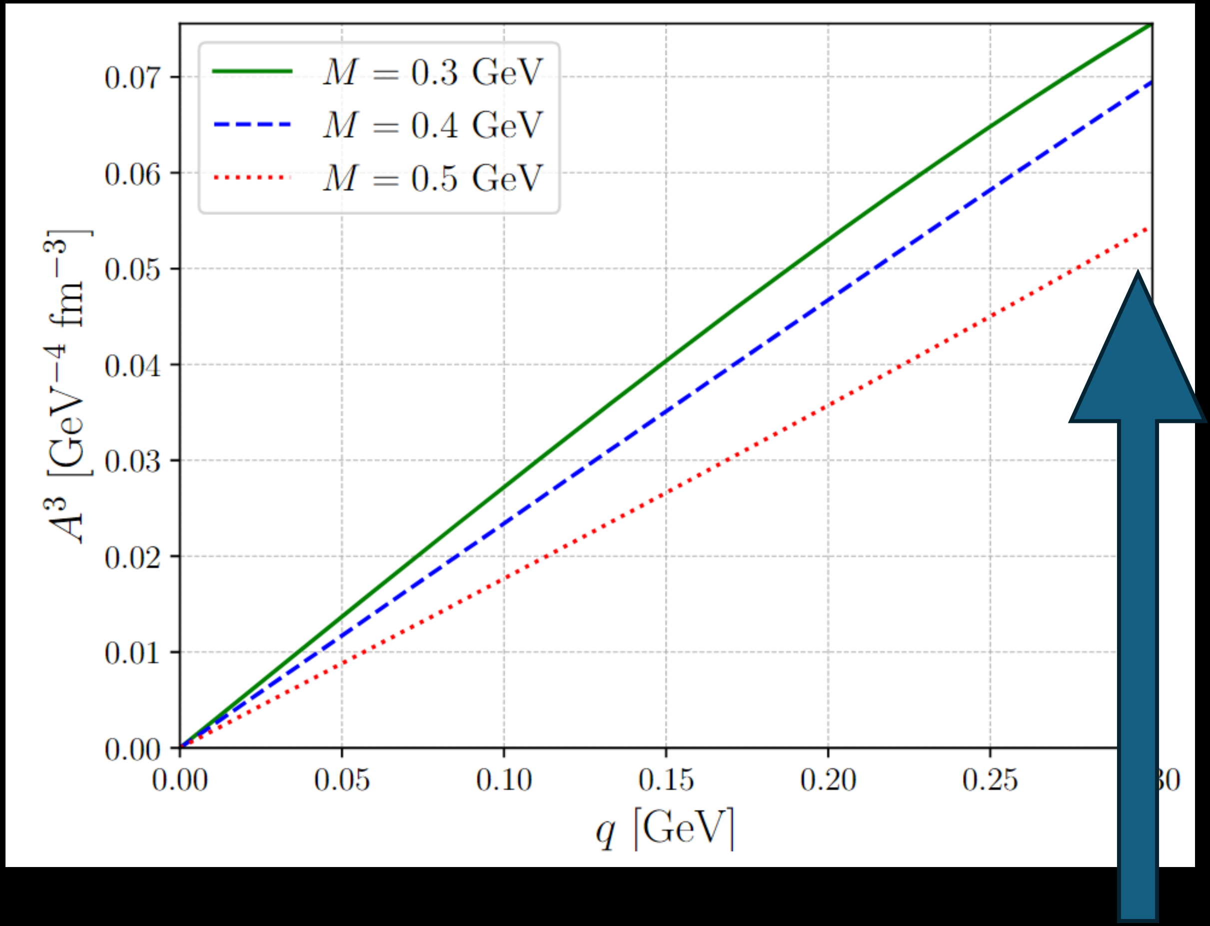
$$V^0(x) = \int d^4k\, \mathcal{V}^0(x,k) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi\hbar)^3} \left(f_{\mathbf{k}}^{(r)} - \bar{f}_{\mathbf{k}}^{(r)} \right).$$

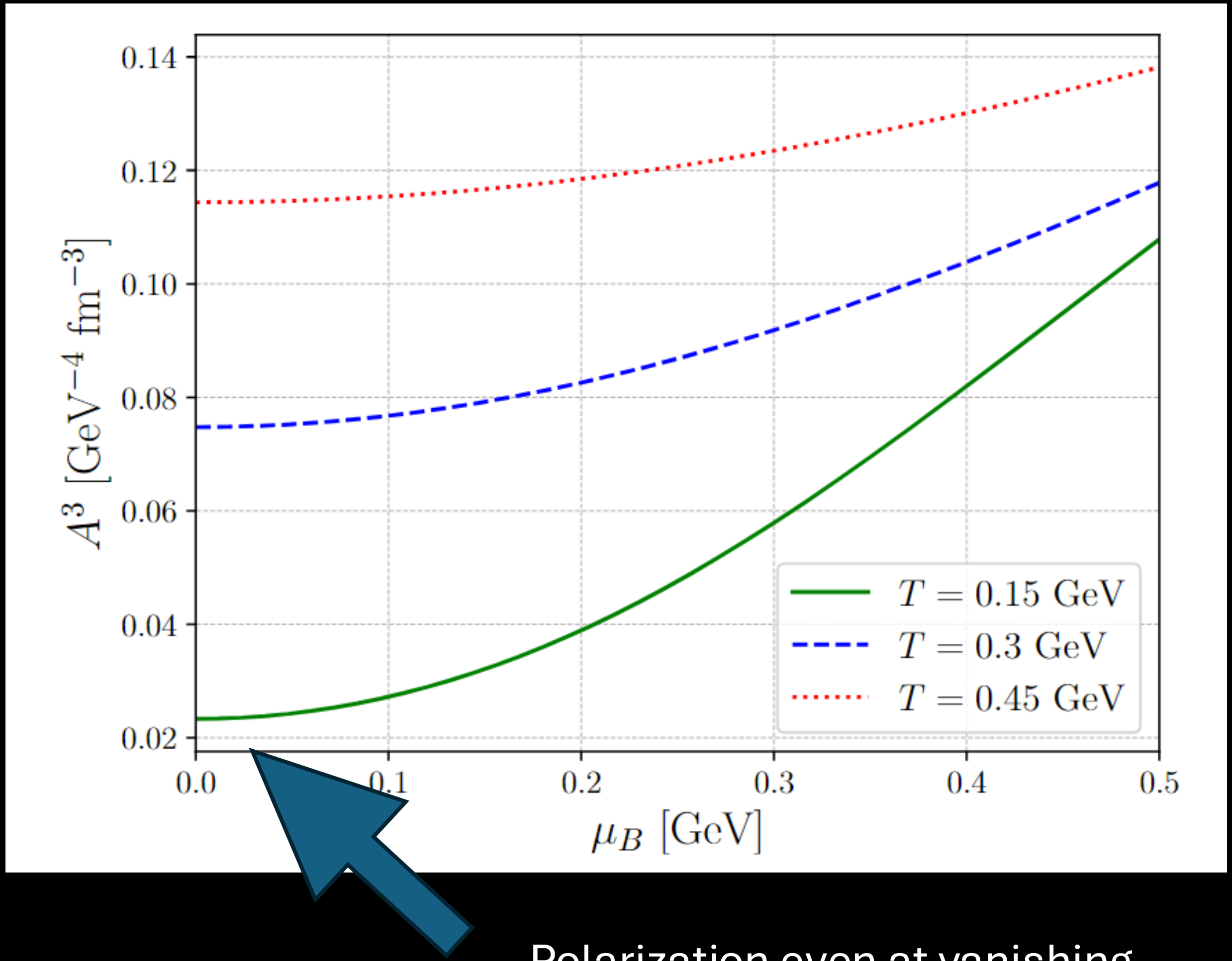
- For the axial vector, $A^0(x)$= $A^1(x)$ = $A^2(x)$ =0

$$A^3(x) = \int d^4k\, \mathcal{A}^3(x,k) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi\hbar)^3} \frac{(-1)^r E_{\mathbf{k}}^{\parallel}}{E_{\mathbf{k}}^{(r)}} \left[1 + \frac{(-1)^{r-1} q}{2E_{\mathbf{k}}^{\parallel}} \right] \left(f_{\mathbf{k}}^{(r)} + \bar{f}_{\mathbf{k}}^{(r)} \right).$$

- This shows good agreement with the formula obtained in [M. Kutschera, W. Broniowski,and A. Kotlorz, Nuclear Physics A 516, 566 (1990).] up to the internal degrees of freedom connected with flavor and color.

Effects of inhomogeneity on axial current





More inhomogeneity = More polarization

Polarization even at vanishing baryon chemical potential

The homogeneous limit

- When the inhomogeneity factor vanishes, the vector looks like

$$\chi_{\pm}^{(r)}(\mathbf{p}, q \to 0) = \sqrt{\frac{(p^1)^2 + (p^2)^2}{2E_{\mathbf{p}}^{\parallel}}} \begin{pmatrix} \dfrac{\sqrt{E(\mathbf{p}) \pm (-1)^r E_{\mathbf{p}}^{\parallel}}\sqrt{E_{\mathbf{p}}^{\parallel} + (-1)^r M}}{p^1 + ip^2} \\ \pm(-1)^r \dfrac{p^3}{\sqrt{E(\mathbf{p}) \pm (-1)^r E_{\mathbf{p}}^{\parallel}}\sqrt{E_{\mathbf{p}}^{\parallel} + (-1)^r M}} \\ \pm(-1)^r \dfrac{p^3 \sqrt{E(\mathbf{p}) \pm (-1)^r E_{\mathbf{p}}^{\parallel}}}{(p^1 + ip^2)\sqrt{E_{\mathbf{p}}^{\parallel} + (-1)^r M}} \\ \dfrac{\sqrt{E_{\mathbf{p}}^{\parallel} + (-1)^r M}}{\sqrt{E(\mathbf{p}) \pm (-1)^r E_{\mathbf{p}}^{\parallel}}} \end{pmatrix}$$

- This looks nothing like the homogeneous case we know!

The homogeneous limit

- The real homogeneous limit is seen by taking linear combination of the limiting vectors

$$\chi_+^{(r)}(\mathbf{p}, \mathbf{q} = 0) = U^{rr'} u_d^{r'}(\mathbf{p}) \qquad \chi_-^{(r)}(\mathbf{p}, \mathbf{q} = 0) = V^{rr'} v_d^{r'}(\mathbf{p})$$

- The matrices are Unitary

$$U^\dagger U = V^\dagger V = 1$$

- The U and V matrices twist the $\boldsymbol{\zeta}_\mathbf{p}$ & $\bar{\boldsymbol{\zeta}}_\mathbf{p}$ vectors even at a vanishing inhomogeneity.

The homogeneous limit

Expected effect

$$\chi_{\pm}^{(r)}(\mathbf{p},\mathbf{q}) \xrightarrow{q\,\to\,0 \text{ limit}} \sqrt{E_p+M}\begin{pmatrix} \varphi^{(r)} \\ \dfrac{\boldsymbol{\tau}\cdot\mathbf{p}}{E_p+M}\,\varphi^{(r)} \end{pmatrix}, \sqrt{E_p+M}\begin{pmatrix} \dfrac{\boldsymbol{\tau}\cdot\mathbf{p}}{E_p+M}\eta^{(r)} \\ \eta^{(r)} \end{pmatrix} \xrightarrow{\text{Polarized along}} (0,0,1)$$

Actual effect

$$\chi_{\pm}^{(r)}(\mathbf{p},\mathbf{q}) \xrightarrow{q\,\to\,0 \text{ limit}} U^{rr'}u_d^{r'}(\mathbf{p}), V^{rr'}v_d^{r'}(\mathbf{p}) \xrightarrow{\text{Polarized along}} -\frac{1}{E_p^{\parallel}(E_{\mathbf{p}}+M)}\left(p^1p^3,\, p^2p^3,\, E_p^{\parallel 2}+E_{\mathbf{p}}M\right)$$

Semiclassical expansion of the Wigner function

- One can obtain the dynamical equations for the Wigner function components,

$$\boxed{K^\mu \mathcal{V}_\mu - \sigma\mathcal{F} + \pi\mathcal{P} = \frac{i\hbar}{2}[(\partial_\mu \pi)(\partial_k^\mu \mathcal{P}) - (\partial_\mu \sigma)(\partial_k^\mu \mathcal{F})],}$$

$$-iK^\mu \mathcal{A}_\mu - \sigma\mathcal{P} - \pi\mathcal{F} = -\frac{i\hbar}{2}[(\partial_\mu \pi)(\partial_k^\mu \mathcal{F}) + (\partial_\mu \sigma)(\partial_k^\mu \mathcal{P})],$$

$$K_\mu \mathcal{F} + iK^\nu \mathcal{S}_{\nu\mu} - \sigma\mathcal{V}_\mu + i\pi\mathcal{A}_\mu = \frac{i\hbar}{2}[i(\partial_\nu \pi)(\partial_k^\nu \mathcal{A}_\mu) - (\partial_\nu \sigma)(\partial_k^\nu \mathcal{V}_\mu)],$$

$$iK^\mu \mathcal{P} - K_\nu \tilde{\mathcal{S}}^{\nu\mu} - \sigma\mathcal{A}^\mu + i\pi\mathcal{V}^\mu = \frac{i\hbar}{2}[i(\partial_\nu \pi)(\partial_k^\nu \mathcal{V}^\mu) - (\partial_\nu \sigma)(\partial_k^\nu \mathcal{A}^\mu)],$$

$$i(K^\mu \mathcal{V}^\nu - K^\nu \mathcal{V}^\mu) - \epsilon^{\mu\nu\tau\sigma} K_\tau \mathcal{A}_\sigma - \pi\tilde{\mathcal{S}}^{\mu\nu} + \sigma\mathcal{S}^{\mu\nu} = \frac{i\hbar}{2}[(\partial_\gamma \sigma)(\partial_k^\gamma \mathcal{S}^{\mu\nu}) - (\partial_\gamma \pi)(\partial_k^\gamma \tilde{\mathcal{S}}^{\mu\nu})].$$

Up to leading order in $\hbar$

$$k_\mu \mathcal{V}^\mu_{(0)} - \sigma_{(0)}\mathcal{F}_{(0)} + \pi_{(0)}\mathcal{P}_{(0)} = 0$$

Put $k = (k^0, 0)$ and $q \neq 0$

Does not satisfy

Solved using the semiclassical expansion:

$$C = C_{(0)} + \hbar C_{(1)} + \hbar^2 C_{(2)} + \ldots$$

Where $C \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}^\mu, \mathcal{A}^\mu, \mathcal{S}^{\mu\nu}\}$

Summary

- Spinors for chiral spirals were obtained.
- A general quantum field was constructed.
- Wigner function and all its components were computed.
- We discovered a twist in polarization in the limit of vanishing inhomogeneity.
- The computed exact functions were shown to disagree with the semiclassical expansion.

Thank You!

Appendix

$$k^0\mathcal{V}^0(k^0,\boldsymbol{k}=0)=\sum_{r=1}^{2}k^0\Bigg\{\left[\delta\left(k^0-E^{(r)}_{\frac{\boldsymbol{q}}{2}}\right)F^{(r)}_{\frac{\boldsymbol{q}}{2}}-\delta\left(k^0+E^{(r)}_{\frac{\boldsymbol{q}}{2}}\right)\left(\bar{F}^{(r)}_{\frac{\boldsymbol{q}}{2}}-1\right)\right]\frac{C_r\left(\frac{\boldsymbol{q}}{2},\frac{q}{2}\right)}{2}$$
$$+\left[\delta\left(k^0-E^{(r)}_{\frac{\boldsymbol{q}}{2}}\right)F^{(r)}_{\frac{\boldsymbol{q}}{2}}-\delta\left(k^0+E^{(r)}_{\frac{\boldsymbol{q}}{2}}\right)\left(\bar{F}^{(r)}_{\frac{\boldsymbol{q}}{2}}-1\right)\right]\frac{C_{r-1}\left(\frac{\boldsymbol{q}}{2},\frac{q}{2}\right)}{2}\Bigg\} \tag{83}$$

and

$$\mathcal{M}(k^0,\boldsymbol{k}=0)=\sum_{r=1}^{2}\frac{M^2}{E^{(r)}_0}\left[F^{(r)}_0\delta\left(k^0-E^{(r)}_0\right)+\left(\bar{F}^{(r)}_0-1\right)\delta\left(k^0+E^{(r)}_0\right)\right]C_r\left(0,\frac{q}{2}\right). \tag{84}$$