SOFT RESUMMATION:
AN INTRODUCTION

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SUMMARY

LECTURE II: RENORMALIZATION GROUP IMPROVEMENT

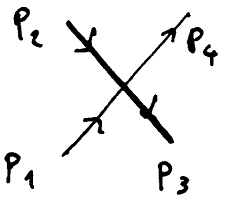
- RENORMALIZATION BASICS
- RG INVARIANCE: PHYSICAL OBSERVABLES
- RG EQUATION: THE PARTONIC CROSS-SECTION
- SOFT LOGS AND SOFT RESUMMATION
- SOFT FACTORIZATION & THE SOFT SCALE
- THE RG EQUATION AND ITS SOLUTION
- THE RESUMMED CROSS SECTION: MATCHING TO FIXED ORDER AND LOG COUNTING

RG IMPROVEMENT

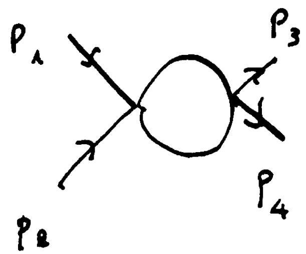
RENORMALIZATION: THE BASIC IDEA

RENORMALIZING THE CHARGE IN ϕ^4 THEORY

$\mathcal{L}_I = -\frac{g}{24}\phi^4$; $\phi\phi \to \phi\phi$ ELASTIC SCATTERING OF MASSIVE SCALAR FIELDS



$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi}\frac{1}{s}; \qquad s = (p_1 + p_2)^2$$



+ 2 more

$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi}\frac{1}{s}F(s,t):\ \text{DIVERGES!};$$
$$t = (p_1 - p_3)^2,\ u = (p_1 - p_4)^2$$

$$F(s,t) = \lim_{\Lambda\to\infty} 1 + \frac{g}{32\pi}\left(3 + \int_0^1 dx \ln\frac{M^2(s)}{\Lambda^2} + s \to t + s \to u\right); \quad M^2(s) = m^2 - x(1-x)s$$

RENORMALIZATION: EXPRESS A PHYSICAL OBSERVABLE IN TERMS OF OTHER PHYSICAL OBSERVABLES:

WHAT IS THE CHARGE g? DEFINE g_{phys} FROM $\left.\frac{d\sigma}{d\cos\theta}\right|_{s=t=u=\mu_0^2} = \frac{g_{\text{phys}}^2}{128\pi}\frac{1}{s}$:

$$\frac{d\sigma}{d\cos\theta} = \frac{g_{\text{phys}}^2}{128\pi}\frac{1}{s}F(s,t); \quad F(s,t) = 1 + \frac{g_{\text{phys}}}{32\pi}\left(\int_0^1 \ln\frac{M^2(s)}{\mu_r^2} + s \to t + s \to u\right); \quad \mu_r^2 = M^2(\mu_0^2)$$

UV SINGULARITY IS UNIVERSAL $\Rightarrow$ REABSORBED IN DEF. OF THE COUPLING

COUPLING DEPENDS ON THE RENORMALIZATION SCALE μ_r

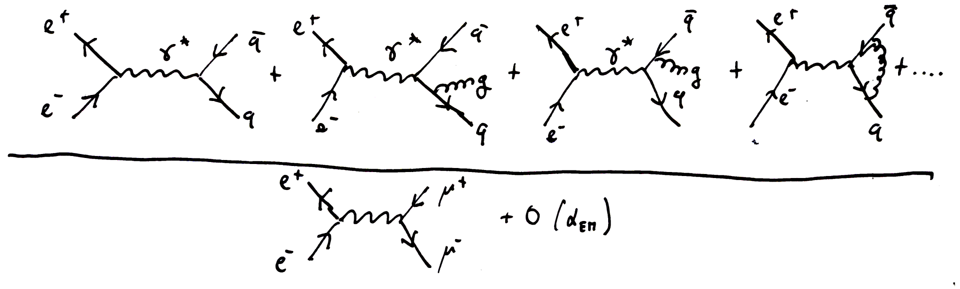
RENORMALIZATION GROUP INVARIANCE

PHYSICAL OBSERVABLES

PHYSICAL RESULTS CANNOT DEPEND ON RENORMALIZATION SCALE μ_r!

THE R RATIO

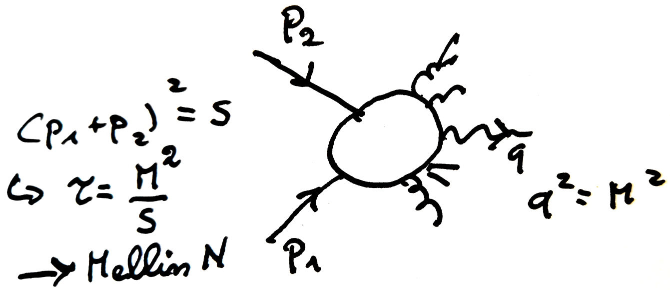
$$R = \frac{\mathrm{e}^{+}\mathrm{e}^{-} \to \mathrm{hadrons}}{\mathrm{e}^{+}\mathrm{e}^{-} \to \mu^{+}\mu-}$$



- DIMENSIONAL ANALYSIS: DIMENSIONLESS $R = R\left(\frac{s}{\mu_r^2}, \alpha_s(\mu_r)\right)$
- RG INVARIANCE: $\mu_r \frac{d}{d\mu_r} R = 0$
- SCALE DEPENDENCE OF α_s PERTURBATIVELY COMPUTABLE
  $\mu_r^2 \frac{d}{d\mu_r^2} \alpha_s(\mu_r) = \beta(\alpha_s(\mu_r)) = -\beta_0 \alpha_2^2 + O(\alpha^3)$
- SOLUTION: CHOOSE $\mu_r^2 = s \Rightarrow R = R\,(1, \alpha_s(s))$
- RUNNING COUPLING $\alpha_s(s) = \frac{\alpha_s(\mu^2)}{1+\beta_0 \alpha_s(\mu^2) \ln \frac{s}{\mu^2}} \left(1 + O\left(\alpha_s\right)\right)$

RENORMALIZATION GROUP INVARIANCE

THE PARTONIC CROSS-SECTION



$$\hat{\sigma} = \sigma^0 C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right);\ C = 1 + O(\alpha_s)$$ DIMENSIONLESS COEFFICIENT FUNCTION

- TOTAL XSECT $\Rightarrow$ PHYSICAL (HARD) SCALE M^2, FACTORIZATION SCALE μ_F & DIMENSIONLESS RATIO $\tau \Leftrightarrow$ MELLIN N+ COUPLING α_s AND ITS SCALE μ_r^2
- DEPENDENCE OF μ_F PERTURBATIVELY COMPUTABLE: ANOMALOUS DIMENSION $\gamma_{qq}(N, \alpha_s(\mu_F^2))$ (Callan-Symanzik equation)
$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right) = -\gamma_{qq}(N, \alpha_s(\mu_F^2)) C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right)$$
- TWO-STEP SOLUTION:
  - $\partial/\partial \ln \mu_f^2 \to \partial/\partial \ln M^2$: $M^2 \frac{\partial}{\partial M^2} C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right) =$
  $$= \gamma_{qq}(N, \alpha_s(\mu_F^2))\,[\alpha_s(\mu_r)]\, C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right)$$
  - $\mu_r^2 = M^2$; $\quad M^2 \frac{\partial}{\partial M^2} C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(M^2)\right) = \gamma_{qq}(N, \alpha_s(M^2)) C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(M^2)\right)$

$$C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(M^2)\right) = C\left(1, N, \alpha_s(M^2)\right) \exp \int_{\mu_F^2}^{M^2} \frac{d\mu^2}{\mu^2} \gamma_{qq}(N, \alpha_s(\mu_F^2))$$

RENORMALIZATION GROUP INVARIANCE

THE PARTONIC CROSS-SECTION

(Callan-Symanzik equation)

$$\mu_F^2 \frac{d}{d\mu_F^2} C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right) = \gamma_{qq}(N, \alpha_s(\mu_F^2)) C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right)$$

EQUIVALENT FORMS

- THE EQUATION $\mu_r^2 = \mu_f^2 = \mu^2; \quad \mu^2 \frac{d}{d\mu^2} \ln C\left(\frac{M^2}{\mu^2}, N, \alpha_s(\mu^2)\right) = \gamma_{qq}(N, \alpha_s(\mu^2))$
- THE SOLUTION

$$\ln C\left(\frac{M^2}{\mu^2}, N, \alpha_s(M^2)\right) = \ln C\left(1, N, \alpha_s(M^2)\right) + \int_{\alpha_s(\mu^2)}^{\alpha_s(M^2)} \frac{d\alpha_s}{\beta(\alpha_s)} \gamma_{qq}(N, \alpha_s)$$

RESUMMATION

PERTURBATIVE EXPANSIONS $\beta(\alpha_s) = -\beta_0 \alpha_s^2 + O(\alpha^3)$; $\gamma_{qq}(N, \alpha_s) = \gamma_N^{(0)} \alpha_s + O(\alpha^2)$;

$$\exp - \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha}{\alpha} \frac{\gamma_N^{(0)}}{\beta_0} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{-\frac{\gamma_N^{(0)}}{\beta_0}}$$

$$= \left(1 + \beta_0 \alpha_s(\mu^2) \ln \frac{Q^2}{\mu^2}\right)^{\frac{\gamma_N^{(0)}}{\beta_0}} + O\left[\alpha^2(\mu^2) \ln \frac{Q^2}{\mu^2}\right] \qquad \text{LEADING LOG}$$

$$= 1 + \alpha_s(\mu^2) \gamma_N^{(0)} \ln \frac{Q^2}{\mu^2} + O(\alpha^2(\mu^2)) \qquad \text{LEADING ORDER}$$

THE SOFT LIMIT

SOFT LOGS

- AT EACH PERTURBATIVE ORDER, $\alpha_s \ln \frac{(k_t^{\max})^2}{\mu^2} P_{qq} \Rightarrow$ TWO SOFT LOGS:
  INFRARED $\int_\tau^1 dy \frac{1}{1-y}_+ \sim \ln(1-\tau)$ &
  COLLINEAR $\int_{\mu_f^2}^{(s-M^2)^2/s} \frac{dk_t^2}{k_t^2} \sim \ln\left[\frac{M^2}{\mu_f^2}\frac{(1-\tau)^2}{\tau}\right] \sim \ln(1-\tau)^2$
- $\overline{\text{MS}}$ FACTORIZATION $\Rightarrow$ ONLY $\ln \frac{M^2}{\mu_f^2}$ SUBTRACTED FROM PARTONIC CROSS-SECTION
- $\overline{\text{MS}}$ PARTONIC XSECT CONTAINS TWO EXTRA SOFT LOGS AT EACH EXTRA PERTURBATIVE ORDER

x SPACE VS. N SPACE

$$\int_0^1 dx x^{N-1} \frac{\ln^p(1-x)}{(1-x)_+} = \frac{1}{p+1} \ln^{p+1} \frac{1}{N} + O(\ln^p \frac{1}{N})$$

- MELLIN KERNEL $x^{N-1} \Rightarrow x \to 1 \Leftrightarrow N \to \infty$
- $\ln N \leftrightarrow \ln(1-x)$
- MELLIN OF $\delta(1-x) \Rightarrow$ CONSTANT
- MELLIN OF $x^k \Rightarrow \frac{1}{N+k}$
- SOFT LIMIT: $C^{\text{soft}}(N, \alpha_s) = \lim_{N\to\infty} C(N, \alpha_s)$

RESUMMATION

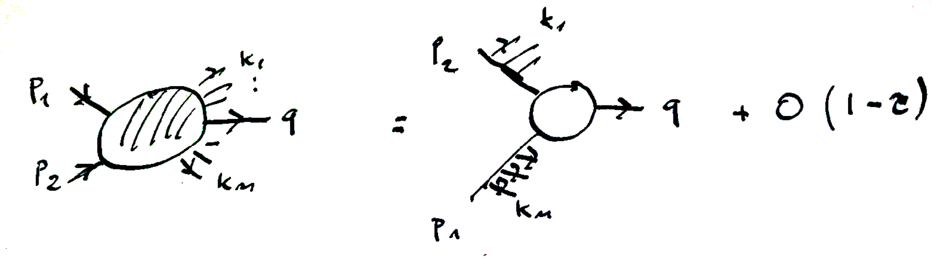
- HIGHEST POWER OF LOG INCREASES WITH PERTURBATIVE ORDER
- THE GOAL: INCLUDE LOG TERMS TO ALL ORDER IN α
- LOG ACCURACY: HOW MANY POWERS OF OF LOG BELOW THE HIGHEST INCLUDED

RESUMMATION FROM RG INVARIANCE

(colorless production: Higgs, Drell Yan)

IS EXTERNAL RADIATION ALL THERE IS?

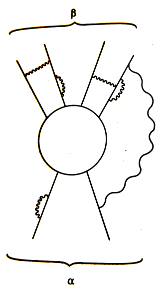
REAL-VIRTUAL FACTORIZATION



; $q^2 = M^2$; $\tau = \frac{M^2}{s}$; $k^2 \le \frac{(s^2 - M^2)^2}{s}$

- YES!: RADIATION FROM INTERNAL LINES POWER-SUPPRESSED IN SOFT LIMIT

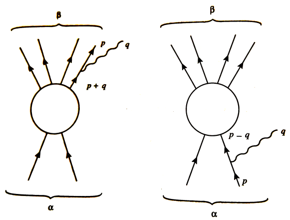
- LOOPS:



$= H(M^2)$

H "HARD" FUNCTION
(LOOP CORRECTIONS TO LEADING-ORDER σ^0)
DIMENSIONLESS, BORN KINEMATICS
$\Rightarrow$ DEPENDS ONLY ON M^2

- REAL:



$= J(M^2, \tau)$

J "JET" FUNCTION(S):
DIMENSIONLESS, NONTRIVIAL KINEMATICS
$\Rightarrow$ CONTAINS τ DEPENDENCE

- $\ln \hat{\sigma}(M^2, \tau) = \ln H(M^2) + \ln J(M^2, \tau)$
  - PARTONIC XSEC FACTORIZES
  - IR SINGS CANCEL BETWEEN H & J

RESUMMATION FROM RG INVARIANCE

THE TWO INGREDIENTS:

- KINEMATICS $\Rightarrow$ SCALE DEPENDENCE
- RG INVARIANCE $\Rightarrow$ ALL-ORDER RESUMMATION

THE SOFT SCALE

- SOFT LIMIT $\Rightarrow$
  - CONSTANTS (IN $H(M^2)$): DEP. ON M^2 (THROUGH $\frac{M^2}{\mu^2}$ & α_s)
  - LOGS (IN $J(M^2,\tau)$): DEP. ON τ THROUGH ARGUMENT OF LOG $\rightarrow (k_t^{\max})^2[M^2,\tau]$
- $J(M^2,\tau)$ DEPENDS ON FIXED COMBINATION OF M^2, τ

determination of $k_t^{\max}$

$p_1+p_2=k'+q$; $q^2=M^2$; $s=(p_1+p_2)^2=(q_0+k_0')^2=\left(\sqrt{M^2+q_t^2+q_z^2}+\sqrt{k'^2+q_t^2+q_z^2}\right)^2$

q_t maximized for fixed s when $k'^2=q_z^2=0 \Rightarrow s=\left(\sqrt{M^2+q_t^2}+\sqrt{q_t^2}\right)^2$

solve for $q_t \Rightarrow (q_t^{\max})^2=M^2\frac{(1-\tau)^2}{4\tau}$

- $J(M^2,\tau)$ DEPENDS ONLY ON THE DIMENSIONFUL VARIABLE $M^2(1-\tau)^2$
- MELLIN SPACE: $\ln[M^2(1-\tau)^2]\Rightarrow \ln\frac{M^2}{N^2}$
- $J(M^2,N)$ DEPENDS ONLY ON THE DIMENSIONFUL VARIABLE $\frac{M^2}{N^2}$

RG INVARIANCE

THE RG EQUATION…

OBSERVABLE (FINITE) $\Leftrightarrow$ RG INVARIANT

- MELLIN-SPACE BARE PARTONIC CROSS-SECTION: (DIVERGENT, REGULARIZED: μ REGULATOR)
$$\ln \hat\sigma^{(0)}\left(M^2,\mu^2,N,\alpha(\mu^2)\right) = \ln H^{(0)}\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right) + \ln J^{(0)}\left(\frac{M^2/N^2}{\mu^2},\alpha(\mu^2)\right)$$

- PHYSICAL ANOMALOUS DIMENSION (FINITE) $\gamma^{\rm phys} = M^2\frac{d}{dM^2}\left(\ln H^{(0)} + \ln J^{(0)}\right) = \gamma^c + \gamma^l$
$$\gamma^c = M^2\frac{d}{dM^2}\ln H^{(0)}\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right); \qquad \gamma^l = M^2\frac{d}{dM^2}\ln J^{(0)}\left(\frac{M^2/N^2}{\mu^2},\alpha(\mu^2)\right)$$
γ^c, γ^l NOT SEPARATELY FINITE! $\Rightarrow$ COLLINEAR FINITE BUT IR DIVERGENT!

- DIFFERENTIATE AGAIN W.R. TO SCALE! $\Rightarrow$ RG INVARIANCE CONDITION
$$\mu^2\frac{d}{d\mu^2}\gamma^l\left(\frac{M^2/N^2}{\mu^2},\alpha(\mu^2)\right) + \mu^2\frac{d}{d\mu^2}\gamma^c\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right) = 0$$

- DERIVATIVE MUST MATCH $\Rightarrow$ CANNOT DEPEND ON $\frac{M^2}{N^2}$ & M^2!:
$$\mu^2\frac{d}{d\mu^2}\gamma^l\left(\frac{M^2/N^2}{\mu^2},\alpha(\mu^2)\right) = -\mu^2\frac{d}{d\mu^2}\gamma^c\left(\frac{M^2}{\mu^2},\alpha(\mu^2)\right) = -g(\alpha(\mu^2))$$
$g(\alpha(\mu^2)) = g_1\alpha_s(\mu^2) + g_2\alpha_s^2(\mu^2)g_1 + \ldots \Rightarrow$ PERTURBATIVE

- LOOKS LIKE A RG EQUATION!

RG INVARIANCE
… AND ITS SOLUTION

$$\mu^2 \frac{d}{d\mu^2}\gamma^l\left(\frac{M^2/N^2}{\mu^2}, \alpha(\mu^2)\right) = -\mu^2 \frac{d}{d\mu^2}\gamma^c\left(\frac{M^2}{\mu^2}, \alpha(\mu^2)\right) = -g(\alpha(\mu^2))$$

- γ^c & $\gamma^l \Rightarrow$ MATCHED SOLUTIONS
  - $$\gamma^c\left(\frac{M^2}{\mu^2}, \alpha(\mu^2)\right) = g_0^c\left(\alpha(M^2)\right) - \int_{\mu^2}^{M^2} \frac{d\lambda^2}{\lambda^2} g\left[\alpha(\lambda^2)\right]$$
  - $$\gamma^l\left(\frac{M^2/N^2}{\mu^2}, \alpha(\mu^2)\right) = g_0^l\left(\alpha(M^2/N^2)\right) + \int_{\mu^2}^{M^2/N^2} \frac{d\lambda^2}{\lambda^2} g\left[\alpha(\lambda^2)\right]$$
- COMBINATION $\Rightarrow$ FINITE $\Rightarrow$ MANIFESTLY RG INVARIANT

$$\gamma^{\rm phys}\left(N, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) = g_0^c(\alpha_s(M^2)) + g_0^l\left(\alpha(M^2/N^2)\right) + \int_{M^2}^{M^2/N^2} \frac{d\lambda^2}{\lambda^2} g(\alpha_s(\lambda^2))$$

$$= g_0^c(\alpha_s(M^2)) + g_0^l\left(\alpha(M^2)\right) + \int_{M^2}^{M^2/N^2} \frac{d\lambda^2}{\lambda^2}\left[g(\alpha_s(\lambda^2)) + \beta(\alpha_s(\lambda^2))\frac{dg_0^l}{d\alpha_s}(\alpha_s(\lambda^2))\right]$$

- THE SOLUTION:

$$\gamma^{\rm phys}\left(N, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) = \bar g_0(\alpha_s(M^2)) + \int_{M^2}^{M^2/N^2} \frac{d\lambda^2}{\lambda^2} \bar g(\alpha_s(\lambda^2))$$

THE RESUMMED CROSS SECTION

- ALL-ORDER LOGS OF N $\Leftrightarrow$ RUNNING OF α_s FROM SCALE M^2 (HARD) TO SCALE $\frac{Q^2}{N^2}$ (SOFT)
- $\gamma^{\rm phys}$ $\Rightarrow$ RESUMMED COEFFICIENT FUNCTION $\Rightarrow$ RESUMMED FINITE JET FUNCTION
  - PHYSICAL ANOMALOUS DIMENSION:
    $M^2\frac{d}{dM^2}\ln J\left(\frac{M^2/N}{\mu^2},\alpha_s(\mu^2)\right) = +\int_{M^2}^{M^2/N^2}\frac{d\lambda^2}{\lambda^2}\bar g(\alpha_s(\lambda^2))$
  - SCALE DEPENDENCE

$$\ln J\left(\frac{M^2/N^2}{\mu^2},\alpha_s(\mu^2)\right)-\ln J\left(\frac{M_0^2/N^2}{\mu^2},\alpha_s(\mu^2)\right)=\int_1^{N^2}\frac{dn}{n}\int_{M_0^2}^{M^2}\frac{dk^2}{k^2}\,\bar g\left(\alpha_s(k^2/n)\right)$$

INCLUDES ALL LOG-ENHANCED TERMS

THE DOUBLE LOGS

- WHICH LOGS ARE COLLINEAR?
  $\ln C\left(\frac{M^2}{\mu_F^2},N,\alpha_s(M^2)\right)=\ln C\left(1,N,\alpha_s(M^2)\right)+\int_{\mu_F^2}^{M^2}\frac{d\mu^2}{\mu^2}\gamma_{qq}(N,\alpha_s(\mu_F^2))$
- SEPARATE “FIXED-SCALE” $\bar g(\alpha_s)=-A(\alpha_s)+\frac{\partial B(\alpha_s(k^2))}{\partial\ln k^2}$:

$$\ln J\left(\frac{M^2/N^2}{\mu^2},\alpha_s(\mu^2)\right)=\int_1^{N^2}\frac{dn}{n}\left[\left(-\int_{\mu^2}^{M^2}\frac{dk^2}{k^2}A(\alpha_s(k^2/n))\right)+B(\alpha_s(M^2/n))\right]$$

- LO $P_{qq}=\alpha_s A_1\frac{1}{(1-x)_+}\Rightarrow\gamma_{qq}=-\alpha_s A_1\ln N$
- ALL-ORDER $\overline{\rm MS}$ $\gamma_{qq}(N,\alpha_s(\mu^2))=-\ln N\sum_n\alpha_s^n(\mu^2)A_k^{(n)}+O\left(N^0\right)+O\left(\frac{1}{N}\right)=A(\alpha_s(k^2))$ (CUSP)

THE RESUMMED CROSS SECTION

MATCHED RESUMMED RESULT

$$\hat{\sigma}\left(N, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) = H\left(\alpha_s(M^2)\right) \exp \int_1^{N^2} \frac{dn}{n} \left[\left(-\int_{\mu^2}^{M^2/n} \frac{dk^2}{k^2} A(\alpha_s(k^2))\right) + B(\alpha_s(M^2/n))\right]$$

HARD FACTORIZATION SCALE $\mu_f^2 = M^2$

$$\hat{\sigma}(N, 1, \alpha_s(M^2)) = H\left(\alpha_s(M^2)\right) \exp \int_{M^2}^{M^2/N^2} \frac{dk^2}{k^2} \left[A(\alpha_s(k^2)) \ln \frac{M^2/N^2}{k^2} + B(\alpha_s(k^2))\right]$$

- DOUBLE LOGS $\Rightarrow$ IR-SINGULAR (CUSP) COLLINEAR ANOMALOUS DIMENSION $A(\alpha_s(k^2))$
- NON-COLLINEAR-SINGULAR SOFT (NNLO, NLO FOR DIS): $B(\alpha_s(k^2))$
- CONSTANTS: $H(\alpha_s(k^2))$
- ACCURATE UP TO $O\left(\frac{1}{N}\right)$
- DIAGONAL QUARK & GLUON CHANNEL RESUMMED:
- NOTE FACTOR 2 FROM TWO INCOMING PARTONS, FACTOR $1/2$ FROM $\ln N$ VS. $\ln N^2$

THE RESUMMED CROSS SECTION

$$\hat{\sigma}(N,1,\alpha_s(M^2)) = H\left(\alpha_s(M^2)\right)\exp\int_{M^2}^{M^2/N^2}\frac{dk^2}{k^2}\left[A(\alpha_s(k^2))\ln\frac{M^2/N^2}{k^2}+B(\alpha_s(k^2))\right]$$

LOG COUNTING

$$\hat{\sigma}(N,\alpha_s) = g_0(\alpha_s)\exp\left[\ln N g_1(\alpha_s \ln N) + g_2(\alpha_s \ln N) + \alpha_s g_3(\alpha_s \ln N) + \ldots\right];$$

$$g_0(\alpha_s) = 1 + \alpha_s g_{0,1} + \alpha_s^2 g_{0,2} + O(\alpha_s^3); \quad g_1(\lambda) = \textstyle\sum_{k=2}^{\infty} g_{1,k}\lambda^k,\ g_i(\lambda) = \sum_{k=1}^{\infty} g_{i,k}\lambda^k \text{ FOR } i \geq 2$$

LOG APPROX.	XSECT ACCURACY	EXP. ACCURACY: $\alpha_s^n L^k$	g_0: α_s^i	g_j ORDER	A: α_s^i	B: α_s^i
LL	$k = 2n$	$k = n + 1$	0	1	1	0
NLL	$2n - 2 \leq k \leq 2n$	$k = n$	1	2	2	1
NNLL	$2n - 4 \leq k \leq 2n$	$k = n - 1$	2	3	3	2

THE RESUMMED CROSS SECTION

MATCHED RESUMMED RESULT

$$\hat{\sigma}(N, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)) = H\left(\alpha_s(M^2)\right) \exp \int_1^{N^2} \frac{dn}{n} \left[\left(- \int_{\mu^2}^{M^2/n} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \right) + B(\alpha_s(M^2/n)) \right]$$

HARD FACTORIZATION SCALE $\mu_f^2 = M^2$

$$C(N, 1, \alpha_s(M^2)) = H\left(\alpha_s(M^2)\right) \exp \int_{M^2}^{M^2/N^2} \frac{dk^2}{k^2} \left[A(\alpha_s(k^2)) \ln \frac{M^2/N^2}{k^2} + B(\alpha_s(k^2)) \right]$$

EQUIVALENT FORM

- IDENTITY:

$$\int_0^1 dx \, \frac{x^{N-1}-1}{1-x} \ln^k(1-x) = -\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{dL^k} \int_0^{1-1/N} \frac{dx}{1-x} \ln^k(1-x) + O(1/N)$$

$$\hat{\sigma}(N, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)) = H[\alpha_s(M^2)] \exp 2 \left[\int_0^1 dx \frac{x^{N-1} - 1}{1-x} \left(\int_{M^2}^{(1-z)^2 M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2) \right) + \bar{B}(\alpha_s((1-x)^2 M^2)) \right]$$

TRANSVERSE MOMENTUM RESUMMATION

PHASE-SPACE FACTORIZATION

LONGITUDINAL $\leftrightarrow$ MELLIN; TRANSVERSE $\leftrightarrow$ FOURIER

$$\frac{d\hat\sigma}{dp_t^2}(\alpha_s, p_t^2) = \frac{M^2}{2\pi}\int d^2b\, e^{-i\vec{p}_t\cdot\vec{b}}\,\Sigma(\alpha_s, b^2) = \int_0^{+\infty} db\, b\, J_0(bq_T)\,\Sigma(\alpha_s, b^2)$$

RESUMMATION

$$\mu_f^2 = M^2$$

$$\frac{d\hat\sigma_{ij}}{dp_t^2}\left(N, p_t, \alpha_s\left(M^2\right), M^2\right) = \sigma_0 \int_0^\infty db\, \frac{b}{2}\, J_0\left(bp_t\right) H_{ij}\left(N, \alpha_s\left(M^2\right)\right) S(M, N, b)$$

- ij PARTONIC SUBCHANNEL
- RESUMMATION $\Rightarrow$ SUDAKOV EXPONENT

$$S(M,b) = \exp\left[-\int_{\frac{1}{b^2}}^{M^2}\frac{dq^2}{q^2}\left[A^{pt}\left(\alpha_s\left(q^2\right)\right)\ln\frac{M^2}{q^2} + B^{pt}\left(\alpha_s\left(q^2\right), N\right)\right]\right]$$

- LEADING LOG $\rightarrow$ LEADING ORDER A; DEFINES A-B SEPARATION
  NOTE BEYOND NNLL A DIFFERS FROM CUSP ANOMALOUS DIMENSION

HARD FUNCTION

$$H_{ij}(\alpha_s) = [C_i(N, \mathbf{b})C_j(N, \mathbf{b}) + G_i(N, \mathbf{b})G_j(N, \mathbf{b})]$$

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SUMMARY

- COLLINEAR LOG RESUMMATION ⇒ ANOMALOUS DIMENSION+ RUNNING COUPLING
- SOFT-COLLINEAR ⇒ DOUBLE LOG
- REAL+VIRTUAL ⇒ EXTERNAL RADIATION & FACTORIZATION
- SOFT LIMIT ⇒ KINEMATIC DEP. THROUGH SOFT SCALE
- SOFT-COLLINEAR CANCELLATION ⇒ MATCHED RG IMPROVEMENT
- FIXED $\mathrm{N}^k\mathrm{L}$ ORDER ⇒ ALL-ORDER $\mathrm{N}^k\mathrm{LL}$ RESUMMATION