SOFT RESUMMATION: AN INTRODUCTION

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SUMMARY

LECTURE I: FACTORIZATION

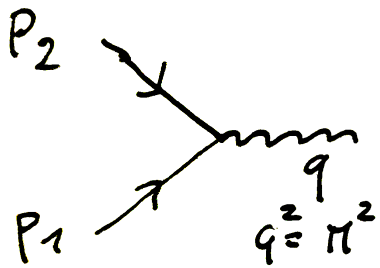
- THE COLLINEAR LIMIT: WEIZSÄCKER-WILLIAMS
- THE INFRARED LIMIT: EIKONAL
- SINGULARITIES
- COLLINEAR FACTORIZATION
- INFRARED CANCELLATION
- MULTIPLE EMISSION
- PHASE SPACE FACTORIZATION

LECTURE II: RENORMALIZATION GROUP IMPROVEMENT

- RENORMALIZATION BASICS
- RG INVARIANCE: PHYSICAL OBSERVABLES
- RG EQUATION: THE PARTONIC CROSS-SECTION
- SOFT LOGS AND SOFT RESUMMATION
- SOFT FACTORIZATION & THE SOFT SCALE
- THE RG EQUATION AND ITS SOLUTION
- THE RESUMMED CROSS SECTION: MATCHING TO FIXED ORDER AND LOG COUNTING

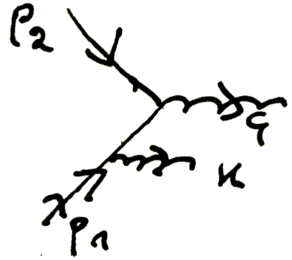
FACTORIZATION

COLLINEAR EMISSION

LO: $\sigma_0 \propto \Bigg|$ 

$\Bigg|^2 = \Big| M_0(p_1, p_2) \Big|^2$

NLO

REAL EMISSION: $\sigma_1 \propto \Bigg|$ 

$+ \ldots \Bigg|^2 = \Big| M_1(p_1, p_2) \Big|^2$

SUDAKOV PARAMETRIZATION

$$k = (1-z)p_1 + k_t + \eta p_2;\ p_1^2 = p_2^2 = 0;\ p_1 \cdot k_t = p_2 \cdot k_t = 0;\ p_1 \cdot p_2 \neq 0$$

EXAMPLE $p_1 = (p, 0, 0, p),\ p_2 = (p, 0, 0, -p);\ (p_1 + p_2)^2 = s \Rightarrow p = \frac{\sqrt{s}}{2}$

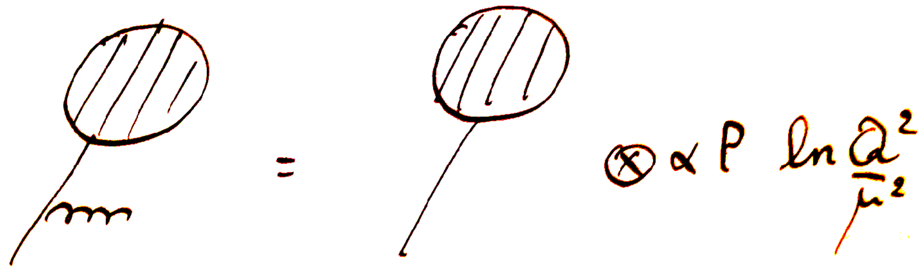
- ON-SHELL $k^2 = 0 \Rightarrow \eta = \frac{k_t^2/s}{1-z}$
- INTERMEDIATE PROPAGATOR DENOMINATOR: $(p_1 - k)^2 = -2p_1 \cdot k = -s\eta = -\frac{k_t^2}{1-z}$
  $\Rightarrow$ ON-SHELL AS $k_t^2 \to 0$ (COLLINEAR)

WEIZSÄCKER-WILLIAMS

- ON SHELL $\not{p}_1 = \sum_r u^s(p_1)\bar{u}^s(p_1) \Rightarrow \frac{i(\not{p}_1 - \not{k})}{(p_1-k)^2} = \frac{\sum_s u^s(p_1-k)\bar{u}^s(p_1-k)}{(p_1-k)^2} + O(k_t^2)$
- AMPLITUDE $M_1 = M_0 \frac{i(\not{p}_1 - \not{k})}{(p_1-k)^2} g_s \gamma^\mu \epsilon_\mu(k) u(p_1)$
- FACTORIZATION $|M_1|^2 = |M_0(p_1 - k)|^2 \frac{1}{k_t^2} \alpha_s P_{qq}^r(z);\ P_{qq}^r(z) = |M(q(p_1) \to q(p_1 - k)g(k))|^2$

UNIVERSALITY

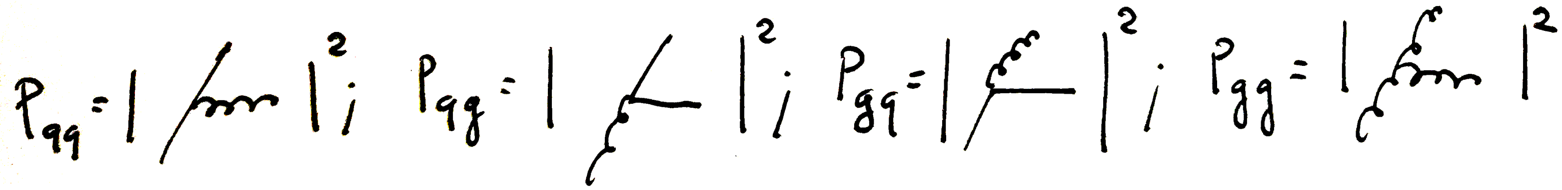
FORM OF LO $\sigma^{(0)}$ NOT USED IN THE ARGUMENT



$$\sigma = \sigma^{(0)}(x) + \alpha_s \frac{1}{k_t^2}\ P(z)\sigma^{(0)}(zp)\left[1 + O(k_t^2)\right]$$

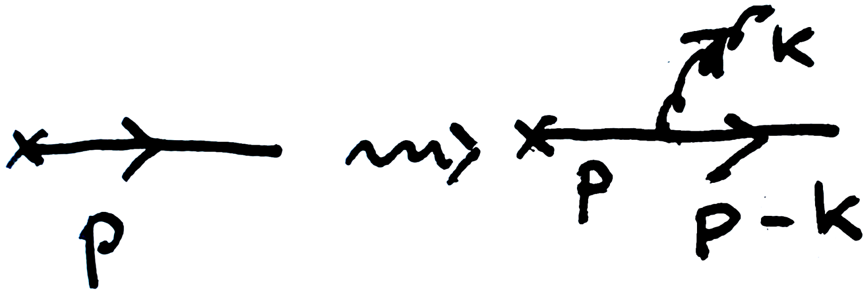
GENERAL COLLINEAR FACTORIZATION

- QUARK CAN RADIATE GLUON, GLUON CAN RADIATE QUARK, GLUON:
- $\sigma_i = \sigma_i^{(0)}(x) + \alpha_s \frac{1}{k_t^2} P_{ij}(z)\sigma_j^{(0)}(zp)$



- ALSO HOLDS FOR MASSIVE FERMIONS

SOFT EMISSION

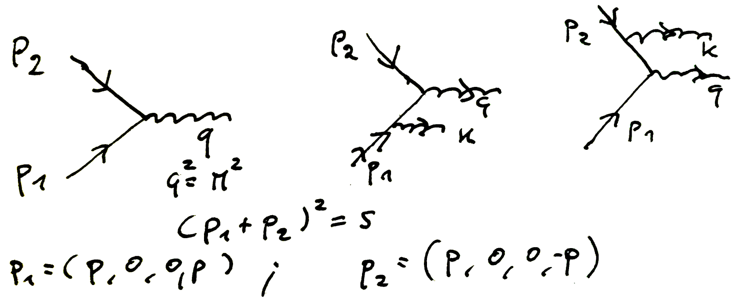


$$u(p) \quad \rightarrow \quad \frac{\not p - \not k + m}{(p+k)^2 - m^2} g_s \gamma^\mu u(p) = g_s \frac{2p^\mu - \gamma^\mu(\not p - m)}{2p \cdot k} u(p) \left[1 + O(k)\right] = u(p) g_s \frac{p^\mu}{p \cdot k} \left[1 + O(k)\right]$$

SOFT GLUON EMISSION $\Rightarrow$ UNIVERSAL EIKONAL FACTOR

- EMISSION IS ALSO COLLINEAR
- COLLINEAR EMISSION MAY BE NON-SOFT
- COLLINEAR EMISSION $\Rightarrow$ UNIVERSAL PARTON-DEPENDENT ALTARELLI-PARISI FACTOR;
- SOFT GLUON EMISSION $\Rightarrow$ UNIVERSAL EIKONAL FACTOR

SINGULARITIES AND LOGS



INTEGRATE OVER EMITTED GLUON MOMENTA

- SUDAKOV PARAMETRIZATION AGAIN: $k = (1-x)\frac{p_1+p_2}{2} + y\frac{p_1-p_2}{2} + k_{\mathrm{T}}$;

  $s = (p_1+p_2)^2$; $k = ((1-x)\frac{\sqrt{s}}{2}, \vec{k}_{\mathrm{T}}, y\frac{\sqrt{s}}{2})$, $y = \pm\sqrt{(1-x)^2 - \frac{4|k_{\mathrm{T}}|^2}{s}}$ (COLL.-ANTICOLL.)

- PHASE SPACE $d\Phi_k = \frac{|k_{\mathrm{T}}|d|k_{\mathrm{T}}|d\phi dk_z}{2E(2\pi)^3}$

  - $\frac{dk_z}{E} = \frac{dE}{|k_z|}$
  - $k_z = y\frac{\sqrt{s}}{2}$; $E = (1-x)\frac{\sqrt{s}}{2}$

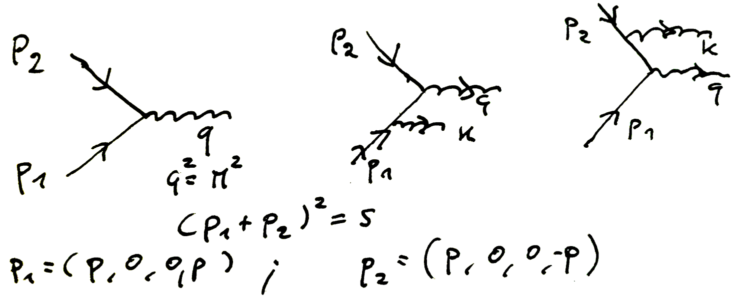
  $$d\Phi_k = \frac{1}{4(4\pi^2)} \frac{dx\, d|k_{\mathrm{T}}|^2}{\sqrt{(1-x)^2 - \frac{4|k_{\mathrm{T}}|^2}{s}}}$$

- AMPLITUDE $M = M_0 g_s \left(\frac{p_1^\mu}{p_1 \cdot k} - \frac{p_2^\mu}{p_2 \cdot k}\right)$; $|M|^2 = -|M_0|^2 g_s^2 \frac{2p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k}$

  - $p_1 \cdot p_2 = \frac{s}{2}$
  - $p_i \cdot k = \frac{s}{4}[(1-x) \pm y]$

  $$|M|^2 = -|M_0|^2 32 g_s^2 \frac{1}{s[(1-x)^2 - y^2]} = -|M_0|^2 8 g_s^2 \frac{1}{|k_{\mathrm{T}}|^2} \text{ (two contribns)}$$

THE SUDAKOV DOUBLE LOG



- PHASE SPACE $d\Phi_k = \frac{1}{4(4\pi^2)} \frac{dx\, d|k_{\mathrm{T}}|^2}{\sqrt{(1-x)^2 - \frac{4|k_{\mathrm{T}}|^2}{s}}}$

- AMPLITUDE $|M|^2 = -|M_0|^2 8 g_s^2 \frac{1}{|k_{\mathrm{T}}|^2}$

- SOFT–COLLINEAR $\int d\Phi_k |M|^2 = -|M_0|^2 \frac{2\alpha_s}{\pi} \int 2 \frac{dx\, d|k_{\mathrm{T}}|^2}{\sqrt{(1-x)^2 - \frac{4|k_{\mathrm{T}}|^2}{s}}} \frac{1}{|k_{\mathrm{T}}|^2}$

  - A DISTRIBUTIONAL IDENTITY

    $$\frac{1}{\sqrt{(1-x)^2 - \frac{4|k_{\mathrm{T}}|^2}{s}}} = \frac{1}{(1-x)_+} - \frac{1}{2}\delta(1-x) \ln \frac{4|k_{\mathrm{T}}|^2}{s} \left[1 + O(|k_{\mathrm{T}}|^2)\right];$$

    + DISTRIBUTION DEF: $\int_0^1 dx \frac{1}{(1-x)_+} f(x) \equiv \int_0^1 dx \frac{f(x) - f(1)}{(1-x)}$

$$\int d\Phi_k |M|^2 = -|M_0|^2 \frac{\alpha}{2\pi} \ln^2 \frac{s}{\mu^2}; \quad \mu \text{ IR CUTOFF}$$

MASS SINGULARITIES & FACTORIZATION

- MASS LOGS

$$|M_1|^2 = \alpha_s \int_0^{k_t^{\max}} \frac{dk_t^2}{k_t^2 - m^2} \int_0^1 dz P_{qq}^r(z) |M_0(zp_1)|^2 = \ln \frac{(k_t^{\max})^2}{m^2} \int_0^1 dz P_{qq}^r(z) |M_0(zp_1)|^2$$

+non log

- (LONGITUDINAL) MOMENTUM CONSERVATION $zs = M^2 \Rightarrow z = \tau \equiv \frac{M^2}{s}$;
$|M_0(p_1)|^2 \propto \delta(1-\tau)$
$|M_0|^2 + |M_1|^2 = \sigma_0 \left(\delta(1-\tau) + \alpha_s P_{qq}^r(\tau) \ln \frac{(k_t^{\max})^2}{m^2} \right)$+non log

- FACTORIZATION:

$$|M_0|^2 + |M_1|^2 = \int_\tau^1 \frac{dz}{z} \left[\delta(1 - \frac{\tau}{z}) + \alpha_s P_{qq}^r \left(\frac{\tau}{z} \right) \ln \frac{(k_t^{\max})^2}{\mu_F^2} \right] \left[\delta(1-z) + \alpha_s P_{qq}^r(z) \ln \frac{\mu_F^2}{m^2} \right]$$

+non log

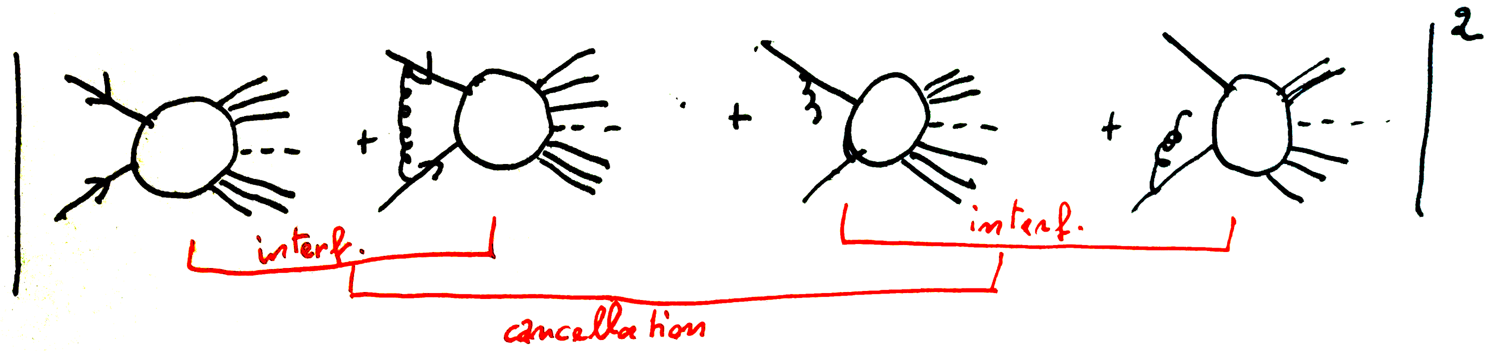
- PARTONIC CROSS SECTION: $\hat{\sigma} = \delta(1-y) + \alpha_s P_{qq}^r(y) \ln \frac{(k_t^{\max})^2}{\mu_F^2}$+non log
  - $y = \frac{M^2}{zs} \Rightarrow$ INCOMING PARTONIC MOMENTUM zp
  - DEFINED AT REFERENCE FACTORIZATION SCALE μ_f^2
  - $m \Rightarrow$ "CONSTITUENT" MASS $m \sim \Lambda$
  - DEPENDENCE m TRADED FOR DEPENDENCE ON $\mu_F \Rightarrow$ NO NP SENSITIVITY
  - SCALE DEPENDENCE UNIVERSAL, GIVEN BY P_{qq}^r

IR SINGULARITIES

- REAL EMISSION PARTONIC CROSS-SECTION: $\hat{\sigma} = \delta(1-y) + \alpha_s P^r_{qq}(y) \ln \frac{(k_t^{\max})^2}{\mu_F^2}$ +non log $\Rightarrow$

- $P^r_{qq}(y) = C_F \frac{1+y^2}{1-y} \Rightarrow$ IR DIVERGENT AS $y \to 1$

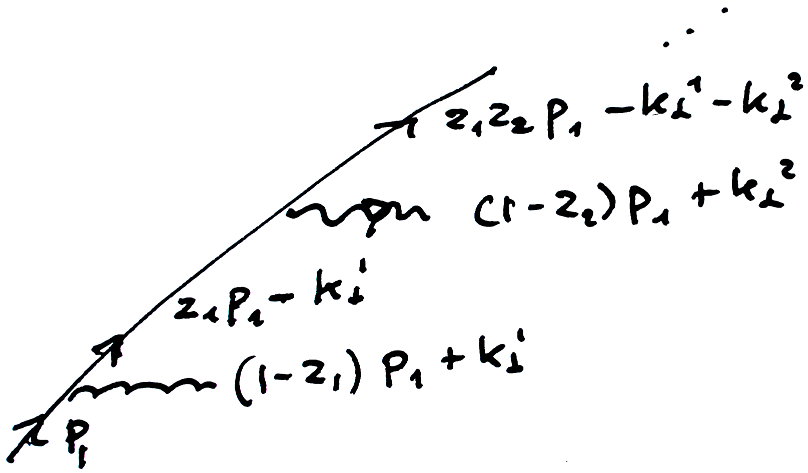
KLN CANCELLATION



- DIVERGENCE CANCELLED BY VIRTUAL CORRECTIONS:
  REAL+VIRTUAL $P_{qq}(y) = C_F \frac{1+y^2}{(1-y)_+} + \frac{3}{2}\delta(1-y)$

- LEFTOVER DOUBLE LOG IN PARTONIC CROSS SECTION: $(k_t^{\max})^2 = M^2 \frac{(1-y)^2}{4y} \Rightarrow$
  $\hat{\sigma} = \delta(1-y) + \alpha_s \left(P_{qq}(y) \ln \frac{M^2}{\mu_f^2} + \left[P^r_{qq}(y) \ln \frac{(1-y)^2}{4y} \right]_+ \right)$
  $\Rightarrow \sigma^{\mathrm{DL}} \equiv \left[\frac{\ln(1-y^2)/4y}{(1-y)} \right]_+$ CONTRIBUTION FROM IR DEP OF $k_t^{\max}$

MULTIPLE EMISSIONS



- TRANSVERSE: $k_t^1 \to k_t^1 + k_t^2 \to \ldots k_t^1 + \cdots + k_t^n$
- LONGITUDINAL: $p_1 \to z_1 p_1 \to z_1 z_2 p_1 \to \ldots z_1 \ldots z_n p_1$
- ORDERED REGIONS $k_t^1 < k_t^2 < \cdots < k_t^n$, $(1 - z_1) < (1 - z_2) < \cdots < (1 - z_n) \Rightarrow$ MULTIPLE LOGS

PHASE SPACE FACTORIZATION
TRANSVERSE

- TOTAL CROSS SECTION: INDEPENDENT $d\vec{k}_t$ INTEGRATION $\Rightarrow$ ORDERED REGION
- TRANVERSE MOMENTUM DISTRIBUTION

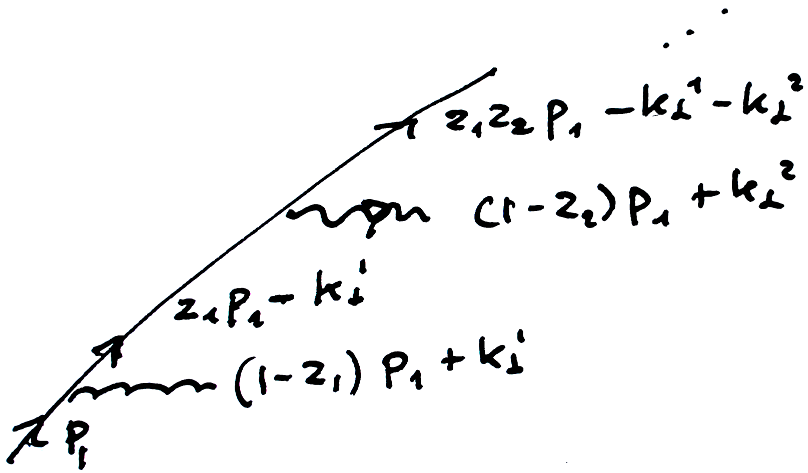
$$\frac{d\sigma}{d\vec{p}_t} = \int d\vec{p}_t d\vec{k}_t^1 \ldots d\vec{k}_t^n \delta(\vec{k}_t^1 + \cdots + \vec{k}_t^n - \vec{p}_t)\sigma(\vec{k}_t^1, \ldots, \vec{k}_t^n)$$

FOURIER

$$d\vec{p}_t d\vec{k}_t^1 \ldots d\vec{k}_t^n \delta(\vec{k}_t^1 + \cdots + \vec{k}_t^n - \vec{p}_t) = d\vec{p}_t \int \frac{d\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{p}_t} d\vec{k}_t^1 e^{-i\vec{b}\cdot\vec{k}_t^1} \ldots d\vec{k}_t^n e^{-i\vec{b}\cdot\vec{k}_t^n}$$

ANGULAR INTEGRAL: $\frac{d\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{p}_t} = \frac{db^2}{4\pi} J_0(bp_t) e^{ibp_t}$

MULTIPLE EMISSIONS



- LONGITUDINAL: $p_1 \to z_1 p_1 \to z_1 z_2 p_1 \to \ldots z_1 \ldots z_n p_1$
- ORDERED REGION $k_t^1 < k_t^2 < \cdots < k_t^n$

PHASE SPACE FACTORIZATION
LONGITUDINAL

- CONVOLUTION SINGLE EMISSION: $\int dz_1 \delta(z_1 - \tau) P_{qq}(z_1) = P_{qq}(\tau)$;
  DOUBLE EMISSION $\int dz_1 dz_2 \delta(z_2 z_1 - \tau) P_{qq}(z_1) P_{qq}(z_2) =$
  $\int_0^1 \frac{dz_2}{z_2} dz_1 \delta\left(z_1 - \frac{\tau}{z_1}\right) P_{qq}(z_1) P_{qq}(z_2) = \int_\tau^1 \frac{dz_2}{z_2} P_{qq}\left(\frac{\tau}{z_2}\right) P_{qq}(z_2) \equiv P_{qq} \otimes P_{qq}\,[\tau]$
- MELLIN $\gamma_{qq}(N) \equiv \int_0^1 x^{N-1} P_{qq}(x)$; $\int_0^1 \tau^{N-1} P_{qq} \otimes P_{qq}\,[\tau] = \gamma_{qq}(N)\gamma_{qq}(N)$
  TURNS CONVOLUTION INTO ORDINARY PRODUCT
- THE PARTONIC CROSS-SECTION $\hat\sigma(N) \equiv \int_0^1 \tau^{N-1} \hat\sigma(\tau) \Rightarrow n$-EMISSION
  $\hat\sigma^{(n)}(N) = \left(\gamma_{qq}(N) \ln \frac{M^2}{\mu_f^2} + \sigma^{\rm DL}(N)\right) \hat\sigma^{(n-1)}(N)$; $\sigma^{\rm DL}(N) \equiv \int_0^1 \tau^{N-1} \sigma^{\rm DL}(N)(\tau)$.
- ALL-ORDER DOUBLE LOGS!

SUMMARY

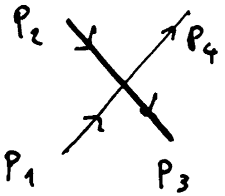
- ALL COLLINEAR RADIATION $\Rightarrow$ FACTORIZES AT THE CROSS-SECTION LEVEL WITH UNIVERSAL PARTON-DEPENDENT SPLITTING FUNCTION
- SOFT RADIATION OF GLUONS $\Rightarrow$ FACTORIZES AT THE AMPLITUDE LEVEL WITH UNIVERSAL EIKONAL FACTOR
- INTEGRATION OVER RADIATED TRANSVERSE MOMENTUM $\Rightarrow$ COLLINEAR LOG SINGULARITY
- INTEGRATION OVER RADIATED ENERGY $\Rightarrow$ SOFT LOG SINGULARITY
- COLLINEAR SINGULARITY CUT OFF BY INTERACTION, FACTORIZED INTO INITIAL CONDITION
  $\Rightarrow$ LEFTOVER COLLINEAR LOG BETWEEN FACTORIZATION & PHYSICAL (HARD) SCALE
- SOFT SINGULARITY CANCELLED BY VIRTUAL CORRECTIONS
  $\Rightarrow$ LEFTOVER SOFT LOG BETWEEN SOFT SCALE & PHYSICAL (HARD) SCALE
- MULTIPLE EMISSION $\Rightarrow$ MULTIPLE FACTORIZATION $\Rightarrow$ MULTIPLE LOGS
- PHASE SPACE FACTORIZATION IN CONJUGATE SPACE

RG IMPROVEMENT

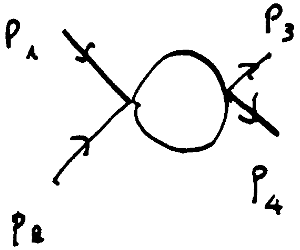
RENORMALIZATION: THE BASIC IDEA

RENORMALIZING THE CHARGE IN ϕ^4 THEORY

$\mathcal{L}_I = -\frac{g}{24}\phi^4$; $\phi\phi \to \phi\phi$ ELASTIC SCATTERING OF MASSIVE SCALAR FIELDS



$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi}\frac{1}{s}; \qquad s = (p_1+p_2)^2$$



+ 2 more

$$\frac{d\sigma}{d\cos\theta} = \frac{g^2}{128\pi}\frac{1}{s}F(s,t)\text{: DIVERGES!};$$
$$t = (p_1-p_3)^2,\ u = (p_1-p_4)^2$$

$$F(s,t) = \lim_{\Lambda\to\infty} 1 + \frac{g}{32\pi}\left(3 + \int_0^1 dx \ln\frac{M^2(s)}{\Lambda^2} + s \to t + s \to u\right); \quad M^2(s) = m^2 - x(1-x)s$$

RENORMALIZATION: EXPRESS A PHYSICAL OBSERVABLE IN TERMS OF OTHER PHYSICAL OBSERVABLES:

WHAT IS THE CHARGE g? DEFINE $g_{\rm phys}$ FROM $\left.\frac{d\sigma}{d\cos\theta}\right|_{s=t=u=\mu_0^2} = \frac{g^2_{\rm phys}}{128\pi}\frac{1}{s}$:

$$\frac{d\sigma}{d\cos\theta} = \frac{g^2_{\rm phys}}{128\pi}\frac{1}{s}F(s,t); \quad F(s,t) = 1 + \frac{g_{\rm phys}}{32\pi}\left(\int_0^1 \ln\frac{M^2(s)}{\mu_r^2} + s \to t + s \to u\right); \quad \mu_r^2 = M^2(\mu_0^2)$$

UV SINGULARITY IS UNIVERSAL $\Rightarrow$ REABSORBED IN DEF. OF THE COUPLING

COUPLING DEPENDS ON THE RENORMALIZATION SCALE μ_r

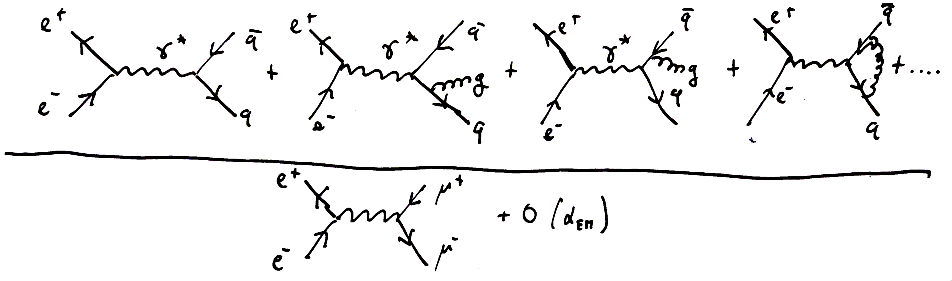
RENORMALIZATION GROUP INVARIANCE

PHYSICAL OBSERVABLES

PHYSICAL RESULTS CANNOT DEPEND ON RENORMALIZATION SCALE μ_r!

THE R RATIO

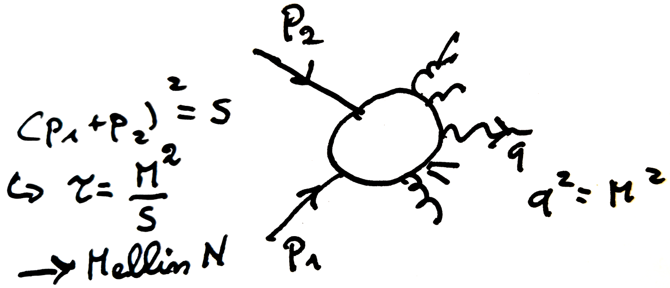
$$R = \frac{\mathrm{e}^+\mathrm{e}^- \to \mathrm{hadrons}}{\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu-}$$



- DIMENSIONAL ANALYSIS: DIMENSIONLESS $R = R\left(\frac{s}{\mu_r^2}, \alpha_s(\mu_r)\right)$
- RG INVARIANCE: $\mu_r \frac{d}{d\mu_r} R = 0$
- SCALE DEPENDENCE OF α_s PERTURBATIVELY COMPUTABLE $\mu_r^2 \frac{d}{d\mu_r^2} \alpha_s(\mu_r) = \beta(\alpha_s(\mu_r))$
- SOLUTION: CHOOSE $\mu_r^2 = s \Rightarrow R = R\,(1, \alpha_s(s))$
- RUNNING COUPLING $\alpha_s(s)$

RENORMALIZATION GROUP INVARIANCE

THE PARTONIC CROSS-SECTION



$$\hat{\sigma} = \sigma^0 C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right); \; C = 1 + O(\alpha_s)$$ DIMENSIONLESS COEFFICIENT FUNCTION

- TOTAL XSECT $\Rightarrow$ PHYSICAL (HARD) SCALE M^2, FACTORIZATION SCALE μ_F & DIMENSIONLESS RATIO $\tau \Leftrightarrow$ MELLIN N+ COUPLING α_s AND ITS SCALE μ_r^2
- DEPENDENCE OF μ_F PERTURBATIVELY COMPUTABLE: ANOMALOUS DIMENSION $\gamma_{qq}(N, \alpha_s(\mu_F^2))$ (Callan-Symanzik equation)
  $$\mu_F^2 \frac{\partial}{\partial \mu_F^2} C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right) = -\gamma_{qq}(N, \alpha_s(\mu_F^2)) C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right)$$
- TWO-STEP SOLUTION:
  - $\partial/\partial \ln \mu_f^2 \to \partial/\partial \ln M^2$: $M^2 \frac{\partial}{\partial M^2} C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right) =$
    $$= \gamma_{qq}(N, \alpha_s(\mu_F^2)) \left[\alpha_s(\mu_r)\right] C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(\mu_r^2), \frac{\mu_r^2}{\mu_F^2}\right)$$
  - $\mu_r^2 = M^2$; $\quad M^2 \frac{\partial}{\partial M^2} C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(M^2)\right) = \gamma_{qq}(N, \alpha_s(M^2)) C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(M^2)\right)$

$$C\left(\frac{M^2}{\mu_F^2}, N, \alpha_s(M^2)\right) = C\left(1, N, \alpha_s(M^2)\right) \exp \int_{\mu_F^2}^{M^2} \frac{d\mu^2}{\mu^2} \gamma_{qq}(N, \alpha_s(\mu_F^2))$$