The Drell – Yan Structure Functions in k_T Factorization

Jan Ferdyan

Jagiellonian University

Zakopane, 19 June 2025

Outline

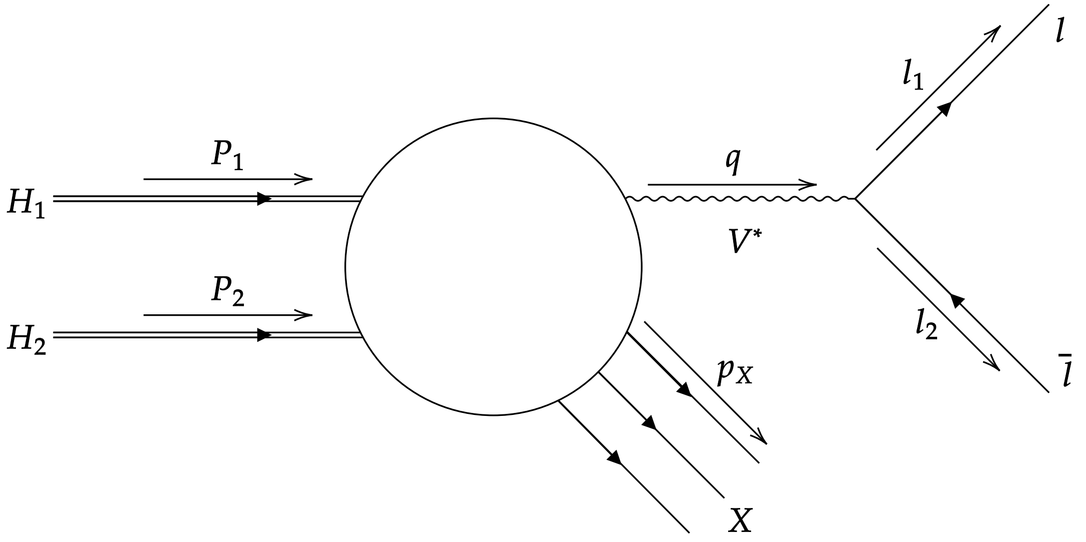
- The Drell – Yan process
- Structure Functions
- Factorization
- Lam – Tung relation
- Amplitudes from the spinor - helicity formalism
- Results

THE DRELL – YAN PROCESS AS A PROBE OF THE HADRONS STRUCTURE

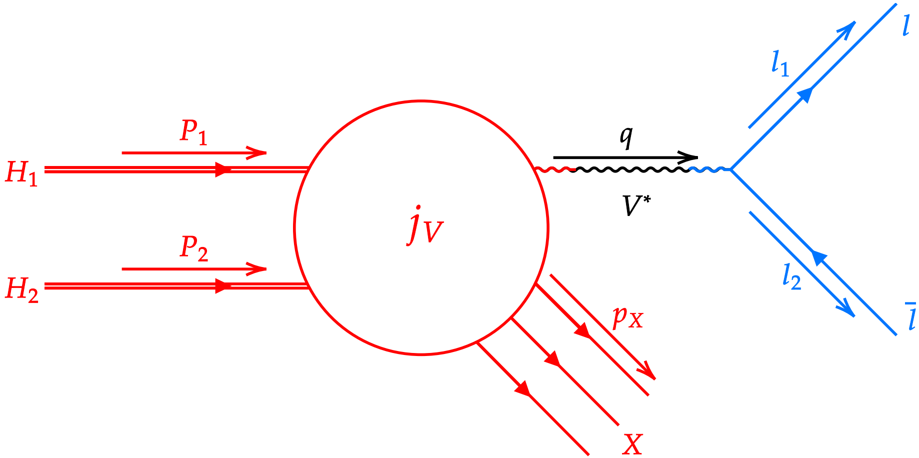
Definition 0.1 (The Drell – Yan process)

$$\textit{Hadron} + \textit{Hadron} \longrightarrow \textit{Electroweak Boson} + X \longrightarrow \textit{Lepton} + \textit{Antilepton} + X$$

$$H_1(P_1) + H_2(P_2) \longrightarrow V^*(q) + X(p_X) \longrightarrow l(l_1) + \bar{l}(l_2) + X(p_X)$$



The Drell – Yan process as a probe of the hadrons structure



$$\mathcal{A}\left(H_1H_2 \to l^+l^-X\right) = \bar{u}(l_1)ie\Gamma_V^{\mu}v(l_2)D_V(q)_{\mu\nu}\,\langle X|iej_V^{\nu}|H_1H_2\rangle$$

The differential cross section: $$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}\Omega\mathrm{d}Y\mathrm{d}^2q_T} = \frac{|\mathcal{A}|^2}{128\pi S} = \frac{\alpha_e^2}{8S}\left|D_V\left(M^2\right)\right|^2 W_{\mu\nu}L^{\mu\nu}$$

$L^{\mu\nu}$ - Standard QED calculation

$W_{\mu\nu}$ - Sensitive for an internal hadron structure

STRUCTURE FUNCTIONS ARE MEASURABLE SOURCE OF INFORMATION ABOUT HADRONS

The lepton angular decomposition can be written as

$$\frac{d\sigma}{dM^2 d\Omega dY d^2 q_T} \propto \Big[\Big(1-\cos^2\vartheta\Big) W_L + \Big(1+\cos^2\vartheta\Big) W_T + \sin^2\vartheta\cos(2\phi) W_{TT} + \sin(2\vartheta)\cos\phi W_{LT}$$
$$+ 2\cos\vartheta W_P + 2\sin\vartheta\cos\phi W_A + \sin^2\vartheta\sin(2\phi) W_7 + \sin(2\vartheta)\sin\phi W_8 + 2\sin\vartheta\sin\phi W_9 \Big].$$

Structure Functions are measurable source of information about hadrons

The lepton angular decomposition can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}\Omega\mathrm{d}Y\mathrm{d}^2q_T} \propto \Bigg[\overbrace{\left(1-\cos^2\vartheta\right)W_L + \left(1+\cos^2\vartheta\right)W_T + \sin^2\vartheta\cos(2\phi)W_{TT} + \sin(2\vartheta)\cos\phi W_{LT}}^{\text{parity conserving}}$$

$$+ \underbrace{2\cos\vartheta W_P + 2\sin\vartheta\cos\phi W_A + \sin^2\vartheta\sin(2\phi)W_7 + \sin(2\vartheta)\sin\phi W_8 + 2\sin\vartheta\sin\phi W_9}_{\text{parity violating}} \Bigg].$$

Structure Functions are measurable source of information about hadrons

The lepton angular decomposition can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}\Omega\mathrm{d}Y\mathrm{d}^2q_T} \propto \Bigg[\overbrace{\left(1-\cos^2\vartheta\right)W_L + \left(1+\cos^2\vartheta\right)W_T + \sin^2\vartheta\cos(2\phi)W_{TT} + \sin(2\vartheta)\cos\phi W_{LT}}^{\text{parity conserving}}$$
$$+ \underbrace{2\cos\vartheta W_P + 2\sin\vartheta\cos\phi W_A + \sin^2\vartheta\sin(2\phi)W_7 + \sin(2\vartheta)\sin\phi W_8 + 2\sin\vartheta\sin\phi W_9}_{\text{parity violating}}\Bigg].$$

ATLAS collaboration used the parametrization:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}Y\mathrm{d}^2q_T\mathrm{d}\Omega} = \frac{3}{16\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}Y\mathrm{d}^2q_T}\Bigg[1+\cos^2\vartheta + \frac{1}{2}A_0\left(1-3\cos^2\vartheta\right) + A_1\sin(2\vartheta)\cos\phi$$
$$+\frac{1}{2}A_2\sin^2\vartheta\cos(2\phi) + A_3\sin\vartheta\cos\phi + A_4\cos\vartheta$$
$$+A_5\sin^2\vartheta\sin(2\phi) + A_6\sin(2\vartheta)\sin\phi + A_7\sin\vartheta\sin\phi\Bigg],$$

Structure Functions are measurable source of information about hadrons

The lepton angular decomposition can be written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}\Omega\mathrm{d}Y\mathrm{d}^2q_T} \propto \Big[\overbrace{\left(1-\cos^2\vartheta\right)W_L + \left(1+\cos^2\vartheta\right)W_T + \sin^2\vartheta\cos(2\phi)W_{TT} + \sin(2\vartheta)\cos\phi W_{LT}}^{\text{parity conserving}}$$
$$+ \underbrace{2\cos\vartheta W_P + 2\sin\vartheta\cos\phi W_A + \sin^2\vartheta\sin(2\phi)W_7 + \sin(2\vartheta)\sin\phi W_8 + 2\sin\vartheta\sin\phi W_9}_{\text{parity violating}} \Big].$$

ATLAS collaboration used the parametrization:

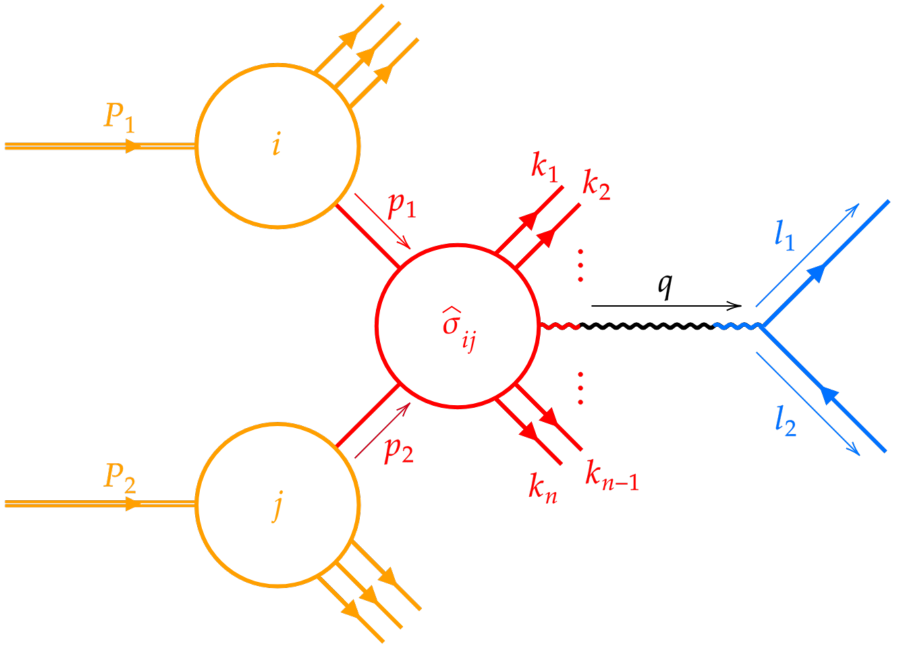
$$\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}Y\mathrm{d}^2q_T\mathrm{d}\Omega} = \frac{3}{16\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}M^2\mathrm{d}Y\mathrm{d}^2q_T}\Big[1+\cos^2\vartheta + \frac{1}{2}A_0\left(1-3\cos^2\vartheta\right) + A_1\sin(2\vartheta)\cos\phi$$
$$+\frac{1}{2}A_2\sin^2\vartheta\cos(2\phi) + A_3\sin\vartheta\cos\phi + A_4\cos\vartheta$$
$$+A_5\sin^2\vartheta\sin(2\phi) + A_6\sin(2\vartheta)\sin\phi + A_7\sin\vartheta\sin\phi\Big],$$

where $A_0 = \frac{2W_L}{W_L+2W_T}$, $A_1 = \frac{2W_{LT}}{W_L+2W_T}$, $A_2 = \frac{4W_{TT}}{W_L+2W_T}$, $A_3 = \frac{4W_A}{W_L+2W_T}$, $A_4 = \frac{2W_P}{W_L+2W_T}$, etc.

THE STRUCTURE FUNCTIONS CAN BE CALCULATED IN QCD

Theorem 1 (Factorization Theorem)

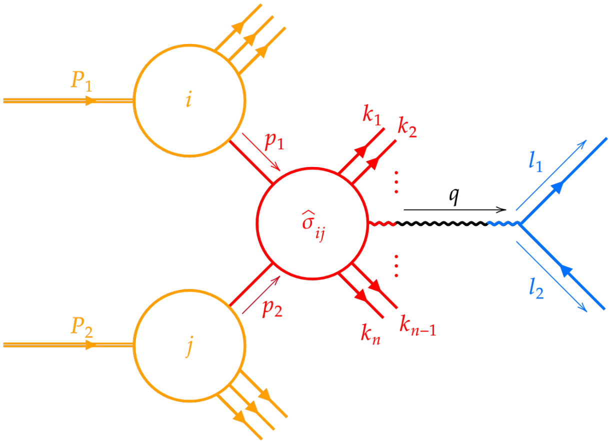
Hadronic cross section = (Parton Distributions) ⊗ (Partonic cross section within the perturbative QCD)



The Structure Functions can be calculated in QCD

Theorem 1 (Factorization Theorem)

Hadronic cross section = (Parton Distributions) ⊗ (Partonic cross section within the perturbative QCD)



$$d\sigma \propto \begin{cases} L_{(rr')} \sum_{i,j} \int_0^1 dx_1 f_i\left(x_1, \mu_F^2\right) \int_0^1 dx_2 f_j\left(x_2, \mu_F^2\right) d\hat{\sigma}_{ij}^{(rr')} & \text{Collinear factorization} \\ L_{(rr')} \sum_{i,j} \int_0^1 \frac{dx_1}{x_1} \int \frac{d^2 p_{1T}}{\pi} \mathcal{F}_i\left(x_1, \mathbf{p}_{1T}^2, \mu_F^2\right) \int_0^1 \frac{dx_2}{x_2} \int \frac{d^2 p_{2T}}{\pi} \mathcal{F}_j\left(x_2, \mathbf{p}_{2T}^2, \mu_F^2\right) d\hat{\sigma}_{ij}^{(rr')} & k_T \text{ factorization} \end{cases}$$

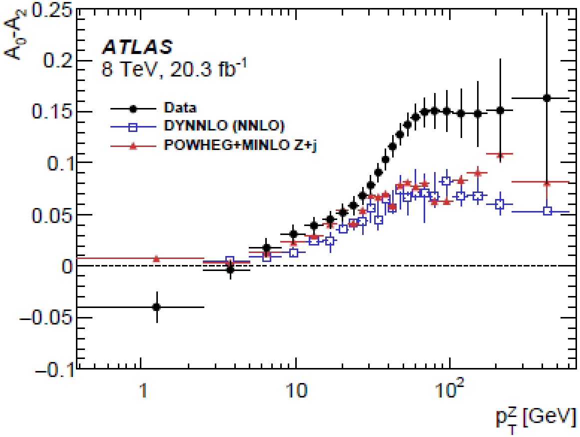
The Lam – Tung relation breaking is not completely explained

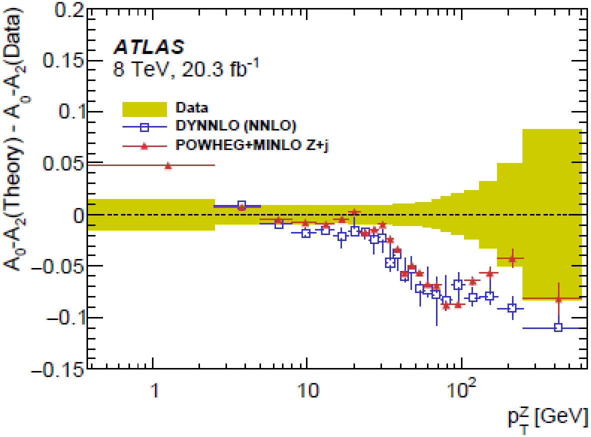
- The Lam – Tung relation is of the form:

$$\boxed{W_L - 2W_{TT} = 0 \qquad \Longleftrightarrow \qquad A_0 - A_2 = 0.}$$

- Fulfilled up to the NLO (for V+jet) in collinear perturbative QCD

- Breaking of Lam – Tung relation may be traced back to a difference between partonic and hadronic collision planes $\longrightarrow$ sensitivity to partons' transverse momenta



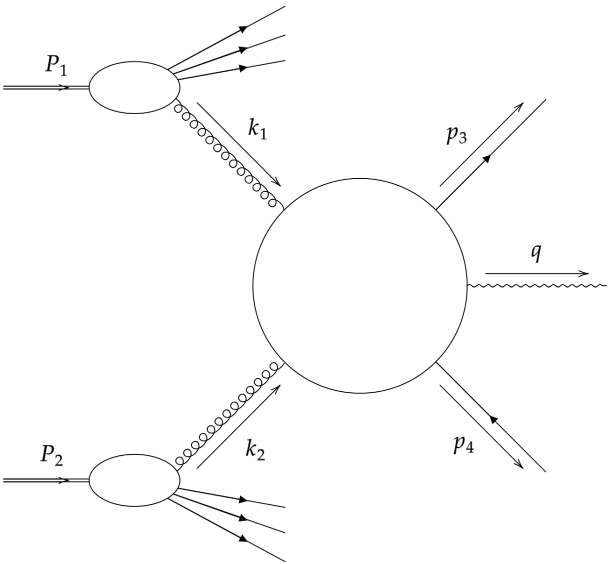


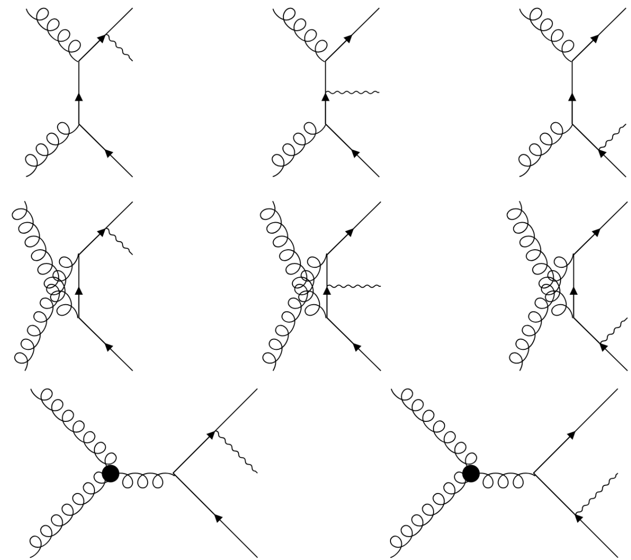
Transverse momentum factorization as a possible solution

- We want to take into account the effects of partons transverse momenta $\Longrightarrow$ k_T **factorization**

Transverse momentum factorization as a possible solution

- We want to take into account the effects of partons transverse momenta $\Longrightarrow$ k_T **factorization**
- The leading contribution comes from the gluons

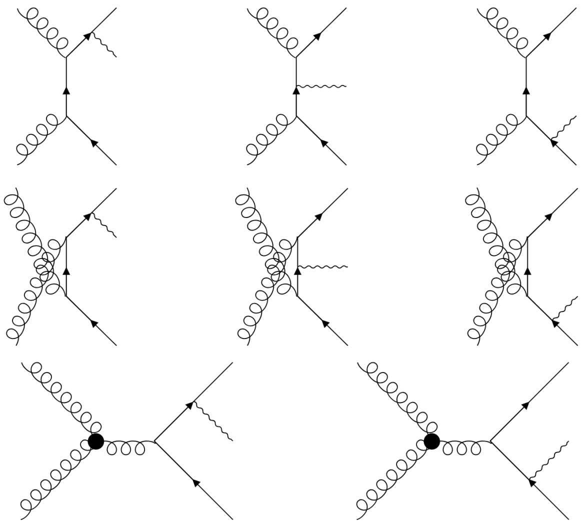




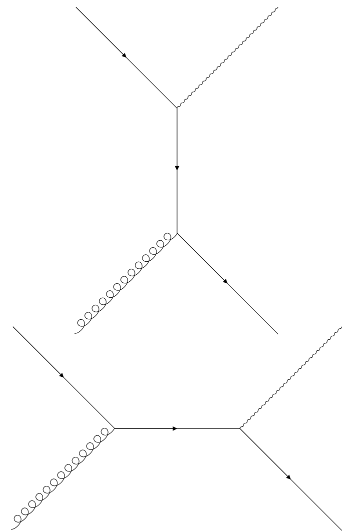
Transverse momentum factorization as a possible solution

- We want to take into account the effects of partons transverse momenta $\Longrightarrow$ k_T **factorization**
- The leading contribution comes from the gluons

$$g^* g^* \longrightarrow q\bar{q}V^*$$

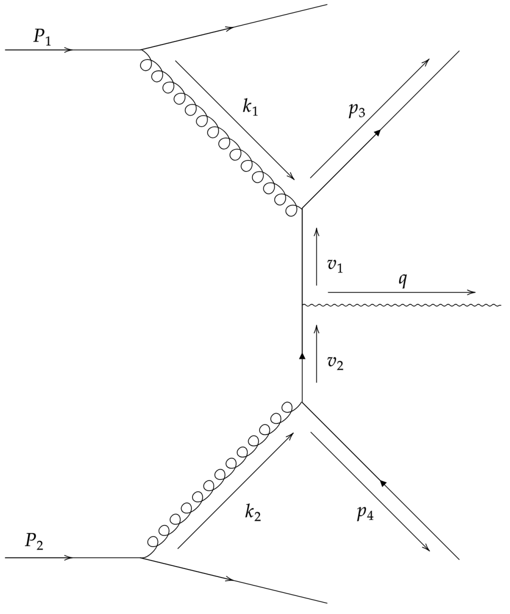


$$q_{\mathrm{val}} g^* \longrightarrow qV^*$$



Scattering Amplitudes from the Spinor - Helicity Formalism

Example diagram:



From the expression for full diagram

$$\mathcal{A}^{ab,\mu}_{ij,\sigma_3\sigma_4} = -ig^2 \left(T^a T^b\right)_{ij} A^\mu_{\sigma_3\sigma_4},$$

we can extract the spinorial part

$$A^\mu_{\sigma_3\sigma_4} = \overline{u}_{\sigma_3}(p_3) \not{P}_1 \frac{\not{v}_1}{v_1^2} \Gamma^\mu_V \frac{\not{v}_2}{v_2^2} \not{P}_2 v_{\sigma_4}(p_4) = \boxed{?},$$

where

$$\Gamma^\mu_V = (v + a\gamma_5)\gamma^\mu,$$

which can be divided into left and right chiral parts

$$A^\mu_{\sigma_3\sigma_4} = (v+a) R^\mu_{\sigma_3\sigma_4} + (v-a) L^\mu_{\sigma_3\sigma_4}$$

The Spinor - Helicity Formalism

We introduce the spinorial notation

$$p_{A\dot{A}} := p_\mu \sigma^\mu_{A\dot{A}}, \qquad p^{\dot{A}A} := p^\mu \overline{\sigma}^{\dot{A}A}_\mu \qquad \Longrightarrow \qquad \not{p} = \begin{bmatrix} 0 & p_{A\dot{A}} \\ p^{\dot{A}A} & 0 \end{bmatrix}$$

Dirac bispinors u_σ and v_σ are defined as

$$\overline{u}_+(p) = [p| := \begin{bmatrix} \lambda(p)^A & 0 \end{bmatrix}, \qquad \overline{u}_-(p) = \langle p| := \begin{bmatrix} 0 & \overline{\lambda(p)}_{\dot{A}} \end{bmatrix},$$
$$v_+(p) = |p] := \begin{bmatrix} \lambda(p)_A \\ 0 \end{bmatrix}, \qquad v_-(p) = |p\rangle := \begin{bmatrix} 0 \\ \overline{\lambda(p)}^{\dot{A}} \end{bmatrix}.$$

in order to satisfy the Dirac equation and such that they have specified helicity (chirality)

$$P_+ |p\rangle = |p\rangle, \quad P_+|p] = 0, \quad P_- |p\rangle = 0, \quad P_-|p] = |p], \qquad \text{where } P_\pm := \frac{\mathbb{1} \pm \gamma_5}{2}. \tag{1}$$

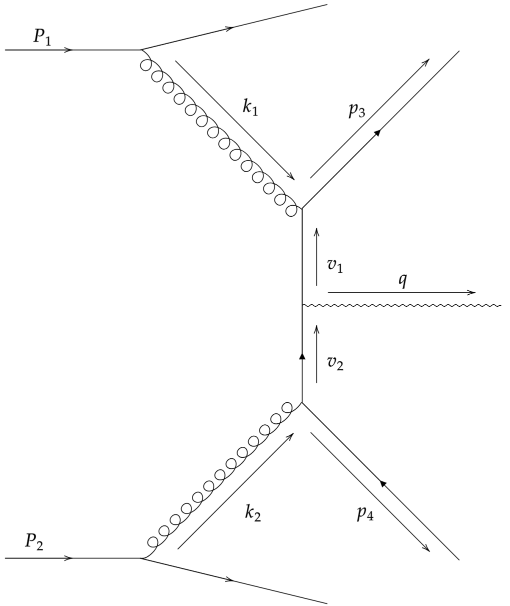
A lot of products vanish automatically

$$\langle p_1 | \gamma^{\mu_1} \dots \gamma^{\mu_n} | p_2] = [p_1 | \gamma^{\mu_1} \dots \gamma^{\mu_n} | p_2\rangle = 0 = \langle p_1|\gamma^{\mu_1} \dots \gamma^{\mu_{n+1}}|p_2\rangle = [p_1 | \gamma^{\mu_1} \dots \gamma^{\mu_{n+1}} | p_2], \tag{2}$$

if n is even.

Scattering Amplitudes from the Spinor - Helicity Formalism

Example diagram:



From the above identities

$$(1) \quad \Longrightarrow \quad R^{\mu}_{+-} = 0 = L^{\mu}_{-+},$$
$$(2) \quad \Longrightarrow \quad R^{\mu}_{++} = R^{\mu}_{--} = 0 = L^{\mu}_{++} = L^{\mu}_{--}.$$

$\Longrightarrow$ We are left with two non-vanishing terms: R^{μ}_{-+} and L^{μ}_{+-}.

$$R^{\dot{A}A}_{-+} = \langle p_3 | \not{P}_1 \frac{\not{v}_1}{v_1^2} P_+ \gamma^{\dot{A}A} \frac{\not{v}_2}{v_2^2} \not{P}_2 | p_4]$$
$$= \boxed{-\frac{2S}{v_3^2 v_2^2 \sqrt{p_3^+ p_4^+}} p_4^{\dot{0}0} p_3^{\dot{1}0} v_3^{\dot{A}1} v_2^{\dot{0}A}}$$

Symmetries Simplify Calculation

Spinorial products satisfy the following identities

$$[p_1 \,|\, \Gamma_\pm \,|\, p_2\rangle = \overline{\langle p_1 \,|\, \Gamma_\mp \,|\, p_2]}, \qquad [p_1 \,|\, \Gamma_\pm \,|\, p_2] = -\overline{\langle p_1 | \Gamma_\mp | p_2\rangle}.$$

where $\Gamma_\pm$ – any composition of the gamma matrices and one of the projections $P_\pm$.

Symmetries Simplify Calculation

Spinorial products satisfy the following identities

$$[p_1\,|\,\Gamma_\pm\,|\,p_2\rangle = \overline{\langle p_1\,|\,\Gamma_\mp\,|\,p_2]}, \qquad [p_1\,|\,\Gamma_\pm\,|\,p_2] = -\overline{\langle p_1|\Gamma_\mp|p_2\rangle}.$$

where $\Gamma_\pm$ – any composition of the gamma matrices and one of the projections $P_\pm$.

This leads to the relations between left and right chiral amplitudes

$$R^\mu_{-+} = \overline{L}^\mu_{+-}, \quad L^\mu_{-+} = \overline{R}^\mu_{+-}, \quad R^\mu_{--} = -\overline{L}^\mu_{++}, \quad L^\mu_{--} = -\overline{R}^\mu_{++},$$

Symmetries Simplify Calculation

Spinorial products satisfy the following identities

$$[p_1 \,|\, \Gamma_\pm \,|\, p_2\rangle = \overline{\langle p_1 \,|\, \Gamma_\mp \,|\, p_2]}, \qquad [p_1 \,|\, \Gamma_\pm \,|\, p_2] = -\overline{\langle p_1 | \Gamma_\mp | p_2\rangle}.$$

where $\Gamma_\pm$ – any composition of the gamma matrices and one of the projections $P_\pm$.

This leads to the relations between left and right chiral amplitudes

$$R^\mu_{-+} = \overline{L}^\mu_{+-}, \quad L^\mu_{-+} = \overline{R}^\mu_{+-}, \quad R^\mu_{--} = -\overline{L}^\mu_{++}, \quad L^\mu_{--} = -\overline{R}^\mu_{++},$$

$\Longrightarrow$ 1 independent expression for each diagram

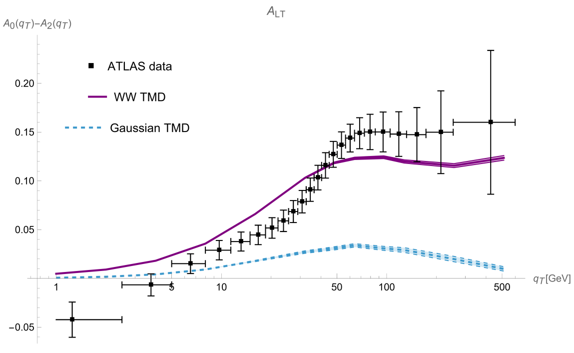
Results for the Lam – Tung Relation

- The "Weizsäcker – Williams" model (in analogy with photon flux)

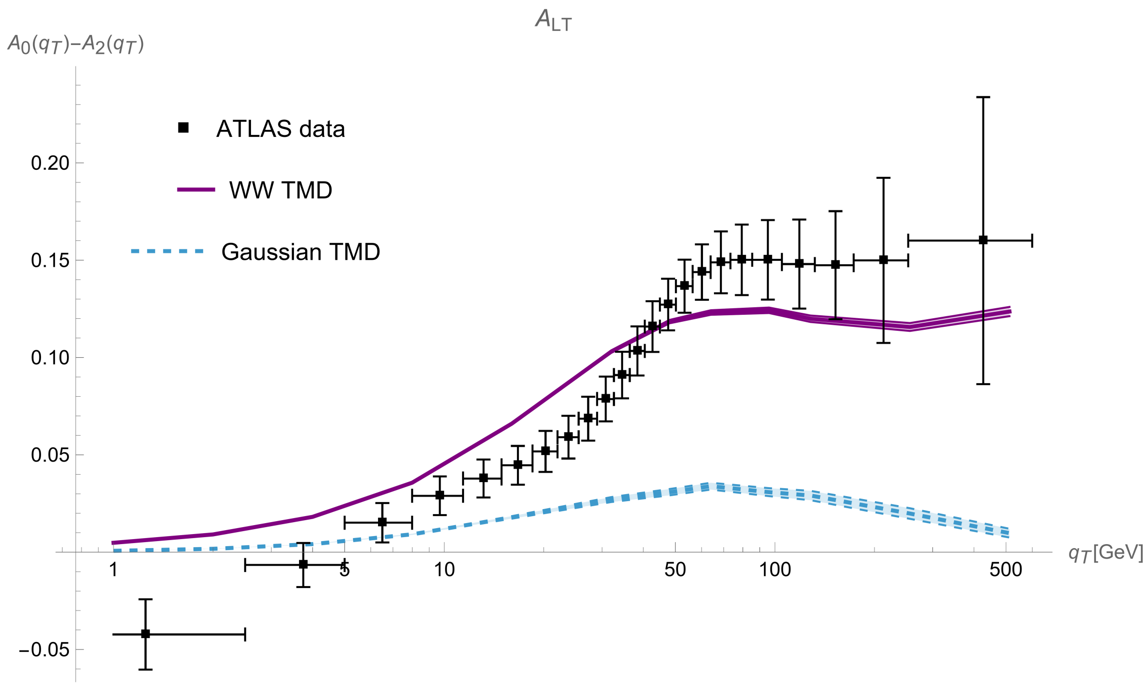
$$\mathcal{F}_{WW}\left(x,k_T^2\right) = \frac{N_1}{k_0^2}(1-x)^7 x^{-\lambda b} \times \begin{cases} 1 & k_T^2 < k_0^2 \\ \left(\frac{k_0^2}{k_T^2}\right)^b & k_T^2 \geq k_0^2 \end{cases} \qquad - \qquad \text{Wide in } k_T$$

- The Gaussian model

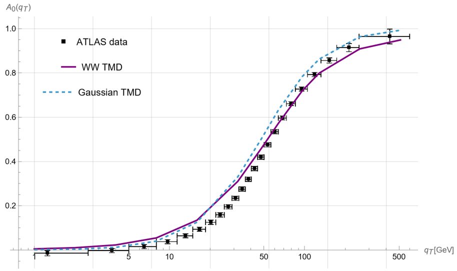
$$\mathcal{F}_G\left(x,k_T^2\right) = N_2(1-x)^7 \exp\left[-\left(\frac{x}{x_0}\right)^\lambda \frac{k_T^2}{k_0^2}\right] \qquad - \qquad \text{Narrow in } k_T$$

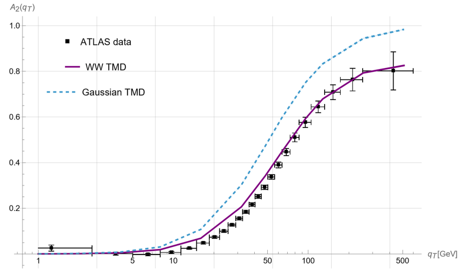


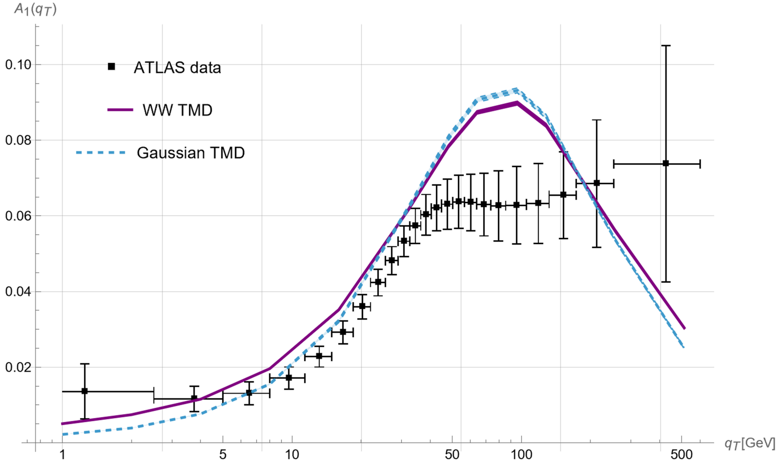
Results for the Lam – Tung Relation



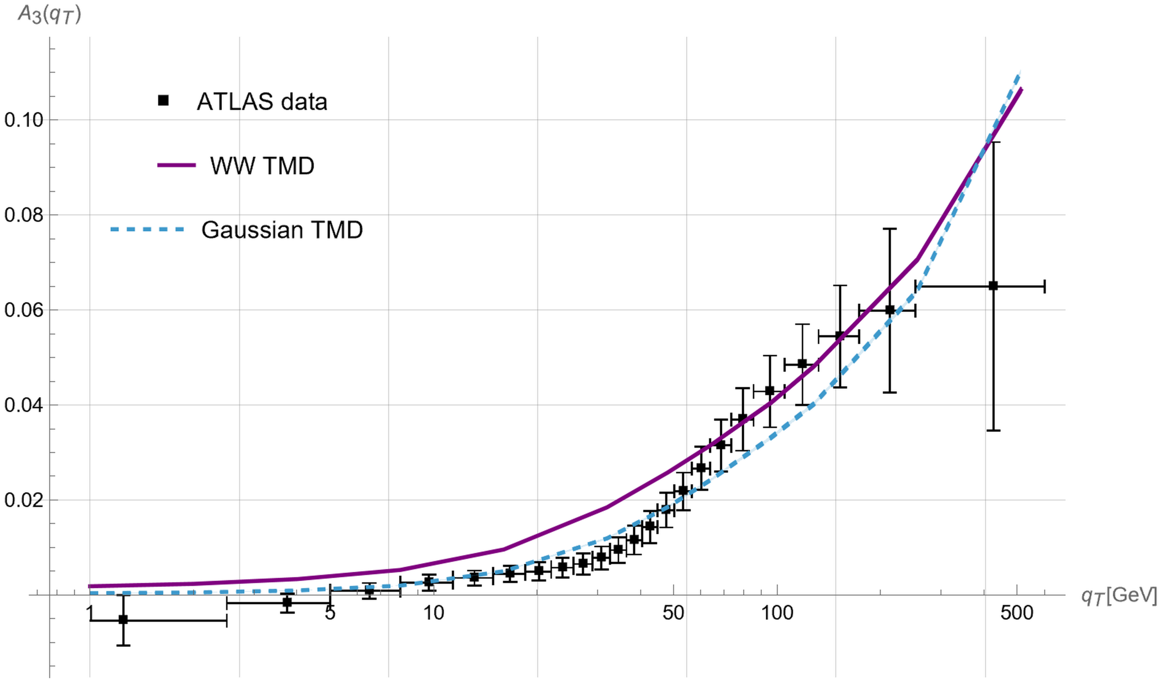
Results for the Parity Conserving Structure Functions



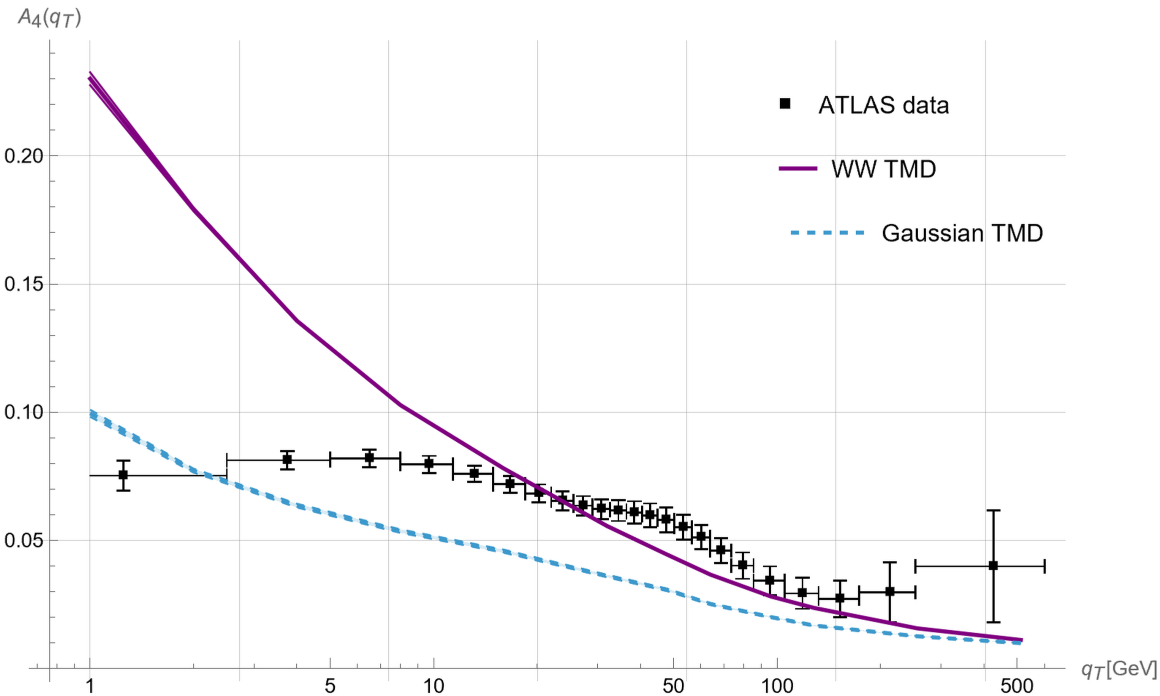




Results for the Parity Violating Structure Functions



Results for the Parity Violating Structure Functions



SUMMARY

- The Drell – Yan process is an excellent probe of internal hadron structure: nine structure functions, three kinematical variables dependence (q_T, Y, M^2).
- NNLO calculations exist in pQCD that do not fully explain Lam – Tung relation breaking
- Lam – Tung relation breaking is sensitive to partons' k_T. The wide in k_T Weizsäcker – Williams TMD model gives reasonable description of the data, and also of the other structure functions.
- We continue program of constraining gluon TMD with Drell – Yan observables.

Bibliography

(1) T. Muta, World Scientific, 2010, ISBN 978-981-279-353-9

(2) G. Aad et al. [ATLAS Collaboration], JHEP 1406 (2014) 112

(3) L. Motyka, M. Sadzikowski, T. Stebel, JHEP 1505 (2015) 087

(4) J. Ferdyan and B. Ruba, Acta Phys. Polon. B **55** (2024) no.9, 9-A2

Project financed by IDUB UJ

Part I

Additional Material

The Drell – Yan Helicity Structure Functions

Useful definition of structure functions can be done in terms of coordinate vectors, where we define (X, Y, Z) such that Z and X are in the hadron scattering plane, Y is perpendicular to this plane and all the spacelike coordinates are orthogonal to the time direction specified by $T^\mu = q^\mu/M$

$$
\begin{aligned}
W^{\mu\nu} =& \tilde{g}^{\mu\nu}\left(W_T + W_{TT}\right) - 2X^\mu X^\nu W_{TT} + Z^\mu Z^\nu (W_L - W_T - W_{TT}) - \left(X^\mu Z^\nu + Z^\mu X^\nu\right)\sqrt{2} W_{LT} \\
& + i\varepsilon^{\mu\nu\rho\sigma} X_\rho T_\sigma \frac{\sqrt{2}}{c_l} W_A + i\varepsilon^{\mu\nu\rho\sigma} Z_\rho T_\sigma \frac{1}{2c_l} W_P + i\left(Z^\mu X^\nu - X^\mu Z^\nu\right)\frac{\sqrt{2}}{c_l} W_9 \\
& - \left(X^\mu Y^\nu + Y^\mu X^\nu\right) W_7 - \left(Z^\mu Y^\nu + Y^\mu Z^\nu\right)\sqrt{2} W_8 .
\end{aligned}
$$

where helicity structure functions are given by helicity amplitudes

$$
W_{rr'} = \varepsilon_\mu^{(r)} W^{\mu\nu} \bar{\varepsilon}_\nu^{(r')},
$$

as

$$
W_L = W_{00}, \quad W_T = \frac{1}{2}\left(W_{++} + W_{--}\right), \quad W_{LT} = \frac{1}{4}\left(W_{+0} + W_{0+} - W_{-0} - W_{0-}\right),
$$

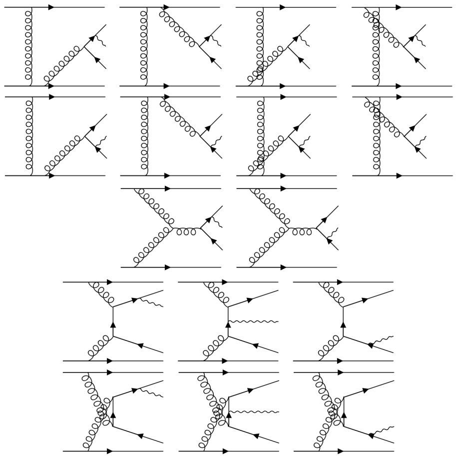
$$
W_{TT} = \frac{1}{2}\left(W_{+-} + W_{-+}\right), \quad W_A = \frac{c_l}{4}\left(W_{+0} + W_{0+} + W_{-0} + W_{0-}\right), \quad W_P = c_l\left(W_{++} - W_{--}\right).
$$

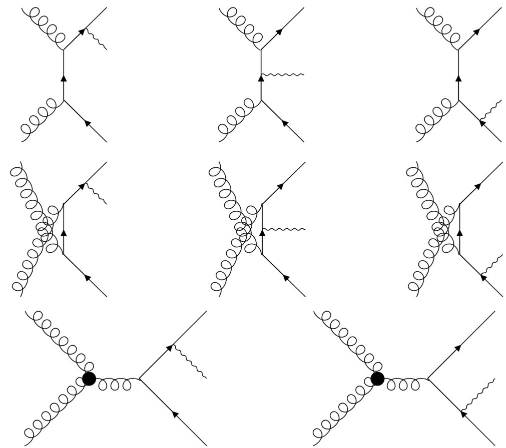
In order to define these structure functions uniquely, one has to choose the polarization vectors

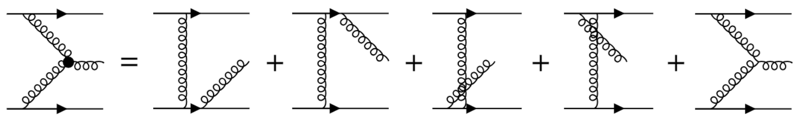
$$
\varepsilon^\mu_{(0)} = Z^\mu, \quad \varepsilon^\mu_{(\pm)} = \mp\frac{1}{\sqrt{2}}\left(X^\mu \pm iY^\mu\right).
$$

Leading Diagrams

Lipatov Effective Vertex







$$\Gamma^{\mu\nu\rho} = \frac{2}{S}\left[\left(x_1 + \frac{2\mathbf{k}_{1T}^2}{x_2 S}\right)P_1^\mu + \left(x_2 + \frac{2\mathbf{k}_{2T}^2}{x_1 S}\right)P_2^\mu - (k_{1T} - k_{2T})^\mu\right]P_1^\nu P_2^\rho$$

Relations between collinear and high energy channels

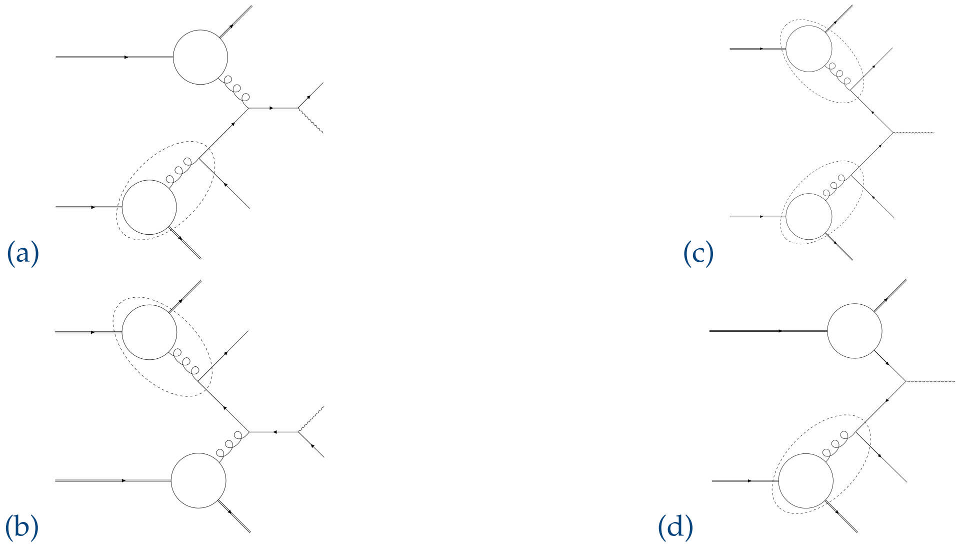
The $q_{\text{val}} g^* \to qV$ channel

- contains $g \to \bar{q}$ LO DGLAP splitting for $q_{\text{val}}\bar{q} \to V$ matrix element and the NLO $qg \to qV$ contribution.
- neglects the NLO loop corrections for $q\bar{q} \to V$.

The $g^* g^* \to q\bar{q}V$ channel

- contains $g \to q_{\text{sea}}$, $g \to \bar{q}$ NLO DGLAP splittings for the $q_{\text{sea}}\bar{q} \to V$ matrix element
- and $g \to q_{\text{sea}}$, $g \to \bar{q}$ LO DGLAP splittings for $q_{\text{sea}}g \to qV$ and $\bar{q}g \to \bar{q}V$ respectively.
- contains the leading contribution to the NNLO collinear $gg \to q\bar{q}V$ matrix element.
- neglects the loop corrections for $q_{\text{sea}}g \to qV$, $\bar{q}g \to \bar{q}V$ and $q\bar{q} \to V$.

Relations between collinear and high energy channels



(a) $g \to q_{\text{sea}}$ splitting

(b) $g \to \bar{q}$ splitting

(c) $g \to q_{\text{sea}}$ and $g \to \bar{q}$ splittings

(d) $g \to \bar{q}$ splitting

Proof of the Bispisnor Products Identy

The charge conjugation acts in the following way

$$\mathcal{C}\psi = \mathcal{C}\begin{bmatrix}\eta_A \\ \overline{\chi}^{\dot{A}}\end{bmatrix} = \begin{bmatrix}\chi_A \\ \overline{\eta}^{\dot{A}}\end{bmatrix} = \overline{\psi}^T, \qquad \text{satisfies:} \qquad \mathcal{C}^2 = \mathbb{1}, \quad \mathcal{C}\gamma^\mu\mathcal{C} = \gamma^\mu, \quad \mathcal{C}\gamma_5\mathcal{C} = \gamma_5,$$

and is anti-linear. Now we will define another anti-linear operator

$$\mathcal{Q} := \gamma_5\mathcal{C}, \qquad \text{which satisfies} \qquad \mathcal{Q}^2 = -\mathbb{1}, \quad \mathcal{Q}\gamma^\mu\mathcal{Q}^{-1} = \gamma^\mu, \quad \mathcal{Q}\gamma_5\mathcal{Q}^{-1} = -\gamma_5.$$

For any two bispinors we have that

$$\overline{\mathcal{Q}\psi_1}\mathcal{Q}\psi_2 = \overline{\overline{\psi_1}\psi_2}.$$

It can be directly shown that $\mathcal{Q}|p] = |p\rangle$, $\mathcal{Q}\,|p\rangle = -|p]$, and in consequence

$$[p_3\,|\,\Gamma\,|\,p_4\rangle = \overline{|p_3\rangle}\Gamma\,|p_4\rangle = \overline{\mathcal{Q}|p_3]}\Gamma\mathcal{Q}|p_4] = \overline{\mathcal{Q}|p_3]}\mathcal{Q}\Gamma|p_4] = \overline{\overline{|p_3]}\Gamma|p_4]} = \overline{\langle p_3\,|\,\Gamma\,|\,p_4]}, \qquad [p_3\,|\,\Gamma\,|\,p_4] = -\overline{\langle p_3|\Gamma|p_4\rangle}.$$

$\mathcal{Q}$ exchanges the projections $\mathcal{Q}P_\pm\mathcal{Q}^{-1} = P_\mp$, so

$$[p_3\,|\,\Gamma_\pm\,|\,p_4\rangle = \overline{\mathcal{Q}|p_3]}\Gamma_\pm\mathcal{Q}|p_4] = \overline{\mathcal{Q}|p_3]}\mathcal{Q}\Gamma_\mp|p_4] = \overline{\langle p_3\,|\,\Gamma_\mp\,|\,p_4]}, \qquad [p_3\,|\,\Gamma_\pm\,|\,p_4] = -\overline{\langle p_3|\Gamma_\mp|p_4\rangle}.$$

THE AMPLITUDES SQUARED

Full amplitude $\mathcal{A}^\mu$ is given by the sum of all diagrams.

- For the $q_{\text{val}} g^* \to q V^*$ there are two diagrams with the same color structure which gives the amplitude
$$\mathcal{A}^\mu = \mathcal{A}_1^\mu + \mathcal{A}_2^\mu = -igT_{ij}^a \left(A_1^\mu + A_2^\mu\right),$$
so the amplitude square is
$$\mathcal{M}^{\mu\nu} = \frac{1}{N\left(N^2-1\right)} \sum_{i,j,a} \mathcal{A}^\mu \overline{\mathcal{A}}^\nu = \frac{g^2}{2N} A^\mu \overline{A}^\nu.$$

- For the $g^* g^* \to q\bar{q} V^*$ the amplitude can be decomposed into symmetric (S) and antisymmetric (A) parts in the adjoint color indices
$$\mathcal{A}^\mu := \sum_{n=1}^{8} \mathcal{A}_n^\mu = \mathcal{A}_S^\mu + \mathcal{A}_A^\mu, \qquad \text{where} \qquad \mathcal{A}_S^\mu := -ig^2 \left(\frac{1}{N}\delta^{ab}\delta_{ij} + d^{abc}T_{ij}^c\right) A_S^\mu, \quad \mathcal{A}_A^\mu := g^2 f^{abc} T_{ij}^c A_A^\mu.$$
In the amplitude squared, the symmetric and antisymmetric parts in the adjoint color indices do not interfere:
$$\mathcal{M}^{\mu\nu} = \frac{1}{\left(N^2-1\right)^2} \sum_{i,j,a,b} \mathcal{A}^\mu \overline{\mathcal{A}}^\nu = \frac{g^4\left(N^2-2\right)}{2N\left(N^2-1\right)} A_S^\mu \overline{A}_S^\nu + \frac{g^4 N}{2\left(N^2-1\right)} A_A^\mu \overline{A}_A^\nu = \mathcal{M}_S^{\mu\nu} + \mathcal{M}_A^{\mu\nu}.$$

The Amplitudes Squared

Comparison with The Trace Method

The spinorial matrix $\mathfrak{A}_n^\mu$ appearing in the amplitude A_n^μ is defined by:

$$A_n^\mu = \overline{u}_{\sigma_3}(p_3)\mathfrak{A}_n^\mu v_{\sigma_4}(p_4).$$

In an analogous way we define the matrices $\mathfrak{A}_S^\mu, \mathfrak{A}_A^\mu$.
The amplitudes squared can be computed as traces

$$\sum_{\sigma_3,\sigma_4} \mathcal{M}_S^{\mu\nu} = \frac{g^4\left(N^2-2\right)}{2N\left(N^2-1\right)} \operatorname{Tr}\left[\left(\not{p}_3 + m_3\right)\mathfrak{A}_S^\mu\left(\not{p}_4 - m_4\right)\mathfrak{A}_S^{\dagger\nu}\right],$$
$$\sum_{\sigma_3,\sigma_4} \mathcal{M}_A^{\mu\nu} = \frac{g^4 N}{2\left(N^2-1\right)} \operatorname{Tr}\left[\left(\not{p}_3 + m_3\right)\mathfrak{A}_A^\mu\left(\not{p}_4 - m_4\right)\mathfrak{A}_A^{\dagger\nu}\right].$$

Using the trace formulas one can check numerically the helicity structure functions of the form

$$\mathcal{M}_{rr'} = \varepsilon_\mu^{(r)}\mathcal{M}^{\mu\nu}\overline{\varepsilon}_\nu^{(r')},$$

where $r, r' \in \{+,-,0\}$ are basis polarizations of V^*.

The Cutted Cross Sections

From the obtained squared amplitudes one can derive the polarized cross sections by integrating over the phase space with the parton distributions.

- $q_{\text{val}} g^* \to q V^*$

$$\frac{\mathrm{d}\sigma^{(q_{\text{val}} g^*)}_{r_1 r_2}}{\mathrm{d}M^2 \mathrm{d}Y \mathrm{d}^2 q_T} = \sum_f \int \mathrm{d}x_q \wp_{f,\text{val}}(x_q, \mu_F) \int \frac{\mathrm{d}^2 k_T}{\pi k_T^2} \mathcal{F}\left(x_g, k_T^2, \mu_F\right) \frac{2}{(8\pi)^2 x_q (1 - x_F) S^2} \mathcal{M}^{(q_{\text{val}} g^*),f}_{r_1 r_2},$$

- $g^* g^* \to q \bar{q} V^*$

$$\frac{\mathrm{d}\sigma^{(g^* g^*)}_{r_1 r_2}}{\mathrm{d}M^2 \mathrm{d}Y \mathrm{d}^2 q_T} = \int \mathrm{d}x_1 \int \frac{\mathrm{d}^2 k_{1T}}{\pi k_{1T}^2} \mathcal{F}\left(x_1, k_{1T}^2, \mu_F\right) \int \mathrm{d}x_2 \int \frac{\mathrm{d}^2 k_{2T}}{\pi k_{2T}^2} \mathcal{F}\left(x_2, k_{2T}^2, \mu_F\right) \frac{(2\pi)^4}{2S} \mathcal{M}^{(g^* g^*)}_{r_1 r_2} \frac{\mathrm{d}z \mathrm{d}\phi_\kappa}{8(2\pi)^9}.$$

Gluon TMD models

- The Weizsäcker – Williams model

$$\mathcal{F}_{WW}\left(x,k_T^2\right)=\frac{N_1}{k_0^2}(1-x)^7x^{-\lambda b}\times\begin{cases}1 & k_T^2<k_0^2\\ \left(\frac{k_0^2}{k_T^2}\right)^b & k_T^2\geq k_0^2\end{cases}.$$

- Modified Weizsäcker – Williams model (for $b=1$)

$$\mathcal{F}'_{WW}\left(x,k_T^2,\mu^2\right)=\frac{xf_g\left(x,\mu^2\right)}{k_0^2\left[1+\log\left(\frac{\mu^2}{k_0^2}\right)\right]}\times\begin{cases}1 & k_T^2<k_0^2\\ \frac{k_0^2}{k_T^2} & k_T^2\geq k_0^2\end{cases}.$$

- The Gaussian model

$$\mathcal{F}_G\left(x,k_T^2\right)=N_2(1-x)^7\exp\left[-\left(\frac{x}{x_0}\right)^\lambda\frac{k_T^2}{k_0^2}\right].$$

- The KMR model

$$\mathcal{F}_{KMR}\left(x,k_T^2,\mu^2\right)=\frac{\partial}{\partial Q^2}\left[xf_g(x,Q^2)T_g(Q,\mu)\right]_{Q^2=k_T^2},$$

where

$$T_g(Q,\mu)=\exp\left[-\int_{Q^2}^{\mu^2}\frac{\mathrm{d}p^2}{p^2}\frac{\alpha_s(p^2)}{2\pi}\int_0^{\mu/(p+\mu)}\mathrm{d}zz\left(P_{gg}(z)+\sum_q P_{qg}(z)\right)\right].$$