Residual pseudo-gauge transformations

Zbigniew Drogosz[1], Wojciech Florkowski[1], Mykhailo Hontarenko[1], Radoslaw Ryblewski[2]
[1]Particle Theory Department, Jagiellonian University in Kraków
[2]Institute of Nuclear Physics, Polish Acedemy of Sciences

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Canonical energy-momentum tensor

The canonical tensors follow from the Noether theorem.
The energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \tag{1}$$

The angular momentum current

$$J^\mu{}_{\alpha\beta} = x_\alpha T^\mu{}_\beta - x_\beta T^\mu{}_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^J)} (\tilde{M}_{\alpha\beta})_{JK} \phi^K \equiv x_\alpha T^\mu{}_\beta - x_\beta T^\mu{}_\alpha + S^\mu{}_{\alpha\beta} \tag{2}$$

where J, K are generalized indices depending on type of the field ϕ, $\tilde{M}_{\alpha\beta}$ are appropriate infinitesimal generators of Lorentz transformations, and $S^\mu{}_{\alpha\beta}$ is the canonical spin tensor.

$$\partial_\lambda J^\lambda{}_{\mu\nu} = 0 \implies \partial_\lambda S^{\lambda\mu\nu} = T^{\mu\nu} - T^{\nu\mu} \tag{3}$$

Pseudo-gauge

The currents $T^{\mu\nu}$ and $J^{\lambda\mu\nu}$ are still conserved and the total charges are the same after a *pseudo-gauge transformation (PGT)*

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda G^{\lambda\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda(\Phi^{\lambda\mu\nu} + \Phi^{\nu\mu\lambda} + \Phi^{\mu\nu\lambda}), \tag{4}$$

$$S'^{\lambda\mu\nu} = S^{\lambda\mu\nu} - \Phi^{\lambda\mu\nu} + \partial_\rho Z^{\mu\nu\lambda\rho}, \tag{5}$$

defined by any differentiable tensors

$$\Phi^{\lambda\mu\nu} = -\Phi^{\lambda\nu\mu}, \quad (\text{this implies } G^{\lambda\mu\nu} = -G^{\mu\lambda\nu}), \tag{6}$$

$$Z^{\mu\nu\lambda\rho} = -Z^{\nu\mu\lambda\rho} = -Z^{\mu\nu\rho\lambda}. \tag{7}$$

For example, we can construct a symmetric $T^{\mu\nu}$ with the choice

$$\Phi^{\lambda\mu\nu} = S^{\lambda\mu\nu}, \qquad Z^{\mu\nu\lambda\rho} = 0, \tag{8}$$

This is the *Belinfante pseudo-gauge*.

General Relativity

The EM tensor must be symmetric in GR because the left-hand side of Einstein's equations is symmetric.

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \tag{9}$$

The Belinfante-Rosenfeld EM tensor coincides with the Hilbert EM tensor

$$T_{\mu\nu} = \frac{1}{\sqrt{-\det(g^{\mu\nu})}} \frac{\delta \int \mathrm{d}^4 x \mathcal{L}}{\delta g^{\mu\nu}}, \tag{10}$$

If we try to construct a different symmetric EM tensor,

$$T'^{\mu\nu} = T^{\mu\nu} + \nabla_\lambda G^{\lambda\mu\nu}, \tag{11}$$

$$\begin{aligned} \nabla_\mu T'^{\mu\nu} &= \nabla_\mu \nabla_\lambda G^{\lambda\mu\nu} = -\nabla_\lambda \nabla_\mu G^{\lambda\mu\nu} = \tfrac{1}{2}[\nabla_\mu, \nabla_\lambda] G^{\lambda\mu\nu} \\ &= R^\lambda{}_{\rho\mu\lambda} T^{\rho\mu\nu} + R^\lambda{}_{\rho\lambda\mu} G^{\lambda\rho\nu} + R^\nu{}_{\rho\mu\lambda} G^{\lambda\mu\rho} = R^\nu{}_{\rho\mu\lambda} G^{\lambda\mu\rho} \end{aligned} \tag{12}$$

So, in GR, outside of special cases, there is no pseudo-gauge invariance, and the Belinfante EM tensor appears to be *the* physical one.

Einstein-Cartan gravity

In Einstein-Cartan theory, spacetime has both curvature and torsion

$$Q^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}, \tag{13}$$

and, in general, $R_{[\mu\nu]} \neq 0$. The field equations become

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{14}$$

$$Q^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{\mu} Q^{\sigma}{}_{\nu\sigma} - \delta^{\lambda}_{\nu} Q^{\sigma}{}_{\mu\sigma} = \frac{8\pi G}{c^4} S^{\lambda}{}_{\mu\nu} \tag{15}$$

Here, it is the canonical energy-momentum tensor that is the "correct" one. The spin tensor is defined by the variation of the action with respect to torsion. [short review: [1] Andrzej Trautman, *Einstein-Cartan Theory*, arXiv:gr-qc/0606062]

This theory is equal to GR in vacuum. So far, there have been no measurements precise enough to decide which of them describes reality better.

Examples from QTF: choices where the spin part of the angular momentum for non-interacting particles is conserved

- GLW (de Groot-van Leeuwen-van Weert) pseudo-gauge. For a Dirac field:

$$\hat{\Phi}^{\lambda,\mu\nu} = \frac{i}{4m}\bar{\psi}(\sigma^{\lambda\mu}\overleftrightarrow{\partial}^{\nu} - \sigma^{\lambda\nu}\overleftrightarrow{\partial}^{\mu})\psi, \tag{16}$$

$$\hat{Z}^{\mu\nu\lambda\rho} = 0, \tag{17}$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$.

- HW (Hilgevoord-Wouthuysen) pseudo-gauge

$$\hat{\Phi}^{\lambda,\mu\nu} = \hat{M}^{[\mu\nu]\lambda} - g^{\lambda[\mu}\hat{M}_{\rho}{}^{\nu]\rho}, \tag{18}$$

$$\hat{Z}^{\mu\nu\lambda\rho} = -\frac{1}{8m}\bar{\psi}(\sigma^{\mu\nu}\sigma^{\lambda\rho} + \sigma^{\lambda\rho}\sigma^{\mu\nu})\psi, \tag{19}$$

where

$$\hat{M}^{\lambda\mu\nu} \equiv \frac{i}{4m}\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^{\lambda}\psi. \tag{20}$$

In relativistic (spin) hydrodynamics, without considering gravity, do all pseudo-gauge choices lead to the same physics?

For example, [2] suggests that EM tensors connected by PGTs have the same physics content, but the entropy current has to be redefined, [3] says that there are many equivalent spin hydrodynamics descriptions connected by PGTs, [4] says that local thermodynamic relations are changed by PGTs. In [5], we describe improved thermodynamic relations, although without discussing PGTs.

[2] Fukushima, K., Pu, S. (2021). *Spin hydrodynamics and symmetric energy-momentum tensors–A current induced by the spin vorticity–.* Physics Letters B, **817**, 136346.
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We decomposed the EM tensor into components either parallel or orthogonal to the flow vector u^μ,

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu - \mathcal{P}\Delta^{\mu\nu} + 2\mathcal{Q}^{(\mu} u^{\nu)} + \mathcal{T}^{\mu\nu} + 2\mathcal{H}^{[\mu} u^{\nu]} + \mathcal{F}^{\mu\nu}, \tag{21}$$

where the first four parts are symmetric, and the last two antisymmetric. In [6], we started from the most general PGT and decomposed it in an analogous way,

$$\Phi^{\lambda\mu\nu} = u^\lambda \mathcal{S}^{\mu\nu} + (u^\nu \Delta^{\lambda\mu} - u^\mu \Delta^{\lambda\nu}) I \tag{22}$$

$$+ (u^\nu I_{(s)}^{\langle\lambda\mu\rangle} - u^\mu I_{(s)}^{\langle\lambda\nu\rangle}) + (u^\nu I_{(a)}^{\lambda\mu} - u^\mu I_{(a)}^{\lambda\nu}) + \Phi^{\langle\lambda\rangle\langle\mu\rangle\langle\nu\rangle}, \tag{23}$$

where

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu, \qquad \Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\mu{}_\beta \Delta^\nu{}_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}),$$
$$A^{\langle\alpha\beta\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}, \quad H^{\alpha\beta\langle\gamma\rangle\delta\dots} \equiv \Delta^\gamma{}_\rho H^{\alpha\beta\rho\delta\dots}, \tag{24}$$
$$\mathcal{S}^{\mu\nu} = u_\alpha \Phi^{\alpha\mu\nu},\ I = \frac{1}{3} I^\mu{}_\mu,\ I^{\lambda\nu} = -u_\rho \Phi^{\langle\lambda\rangle\rho\nu}.$$

[6] ZD, Florkowski, W., Hontarenko, M., Ryblewski, R. (2025). *Dynamical constraints on pseudo-gauge transformations.* Physics Letters B, **861**, 139244.

The energy-momentum tensor transforms in a complicated way.

$$\mathcal{E}' = \mathcal{E} + I\theta - \theta_F - \theta_w + I_{(s)}^{\langle\alpha\beta\rangle}\sigma_{\alpha\beta} - I_{(a)}^{\alpha\beta}\Omega_{\alpha\beta}\,, \tag{25}$$

$$\mathcal{P}' = \mathcal{P} - \frac{1}{3}\left(2I\theta + 3DI + F\cdot a + \theta_w - I_{(s)}^{\langle\alpha\beta\rangle}\sigma_{\alpha\beta} + I_{(a)}^{\alpha\beta}\Omega_{\alpha\beta} + \partial_\alpha \Phi^{\langle\beta\rangle}{}_{\langle\beta\rangle}{}^{\langle\alpha\rangle}\right)\,, \tag{26}$$

$$\begin{aligned}\mathcal{Q}'^{\mu} = {}& \mathcal{Q}^{\mu} + \frac{1}{2}\left[-Ia^{\mu} + DF^{\mu} + \theta F^{\mu} - 2D_F u^{\mu} - \nabla^{\mu} I + (F\cdot a)\,u^{\mu}\right] \\ & + u_\nu DI_{(s)}^{\langle\mu\nu\rangle} - \frac{1}{2}u^{\mu}\left(I_{(s)}^{\langle\alpha\beta\rangle}\sigma_{\alpha\beta} - I_{(a)}^{\alpha\beta}\Omega_{\alpha\beta}\right) - \frac{1}{2}\partial_\lambda\left(I_{(s)}^{\langle\mu\lambda\rangle} + I_{(a)}^{\mu\lambda}\right) \\ & - \frac{1}{2}\left(\Phi^{\langle\mu\rangle\langle\nu\rangle\langle\lambda\rangle} + \Phi^{\langle\nu\rangle\langle\mu\rangle\langle\lambda\rangle}\right)\partial_\lambda u_\nu + \frac{1}{2}\epsilon^{\langle\mu\rangle}{}_{\alpha\beta\gamma}(W^{\alpha}\partial^{\beta}u^{\gamma} - u^{\alpha}\partial^{\beta}W^{\gamma})\,,\end{aligned} \tag{27}$$

$$\begin{aligned}\mathcal{T}'^{\mu\nu} = {}& \mathcal{T}^{\mu\nu} - I\sigma^{\mu\nu} + F^{\langle\mu}a^{\nu\rangle} + DI_{(s)}^{\langle\mu\nu\rangle} + I_{(s)}^{\langle\mu\nu\rangle}\theta - I_{(s)}^{\langle\langle\mu\lambda\rangle}\nabla_\lambda u^{\nu\rangle} \\ & + 2u^{(\mu}I_{(s)}^{\langle\nu)\lambda\rangle}a_\lambda + I_{(a)}^{\lambda\langle\mu}\nabla_\lambda u^{\nu\rangle} + \epsilon^{\alpha\beta\lambda\langle\mu}W_\alpha u_\beta \partial_\lambda u^{\nu\rangle} + \partial_\lambda \Phi^{\langle\mu\nu\rangle\langle\lambda\rangle}\,,\end{aligned} \tag{28}$$

with

$$\begin{gathered}\partial_\mu = u_\mu D + \nabla_\mu, \quad D \equiv u_\nu\partial^\nu, \quad \nabla_\mu \equiv \Delta_{\mu\nu}\partial^\nu, \quad \theta = \nabla\cdot u, \\ \nabla\cdot F = \theta_F, \quad \partial_\mu H^{\mu\nu} = h^\nu, \quad Du^\nu = a^\nu, \quad F^\mu\partial_\mu = D_F, \quad H^{\lambda\nu}\partial_\lambda = D_H^\nu, \\ \theta_w = u\cdot h = \epsilon^{\mu\nu\alpha\beta}W_\mu u_\nu \nabla_\alpha u_\beta, \quad \sigma_{\alpha\beta} = \partial_{\langle\alpha}u_{\beta\rangle}, \quad \Omega_{\alpha\beta} \equiv \partial_{\langle[\alpha}u_{\beta]\rangle}.\end{gathered} \tag{29}$$

Simplifying assumptions

The simplifying requirement that symmetric EM tensors transform into symmetric ones (Symmetric-To-Symmetric condition) leads to

$$\partial_\lambda \Phi^{\lambda\mu\nu} = 0, \tag{30}$$

which can be rewritten as

$$\begin{aligned}\partial_\lambda \Phi^{\lambda\mu\nu} &= \theta \mathcal{S}^{\mu\nu} + DH^{\mu\nu} + 2\left[F^{[\mu} a^{\nu]} + u^{[\nu} DF^{\mu]} + u^{[\nu}\partial^{\mu]} I + I\partial^{[\mu} u^{\nu]}\right] \\ &+ \partial_\lambda \Phi^{\langle\lambda\rangle\langle\mu\rangle\langle\nu\rangle} + u^\nu \partial_\lambda \left(I_{(s)}^{\langle\lambda\mu\rangle} + I_{(a)}^{\lambda\mu}\right) - u^\mu \partial_\lambda \left(I_{(s)}^{\langle\lambda\nu\rangle} + I_{(a)}^{\lambda\nu}\right) \\ &+ \left(I_{(s)}^{\langle\lambda\mu\rangle} + I_{(a)}^{\lambda\mu}\right)\partial_\lambda u^\nu - \left(I_{(s)}^{\langle\lambda\nu\rangle} + I_{(a)}^{\lambda\nu}\right)\partial_\lambda u^\mu = 0.\end{aligned} \tag{31}$$

Now, if we assume that the PGT is built out of hydrodynamic variables T, μ, and u^μ, we are left with the form

$$\Phi^{\lambda\mu\nu} = (u^\nu \Delta^{\lambda\mu} - u^\mu \Delta^{\lambda\nu}) I. \tag{32}$$

Then, the Symmetric-To-Symmetric (STS) condition gives

$$\partial_\lambda \Phi^{\lambda\mu\nu} = (u^\nu \partial^\mu - u^\mu \partial^\nu) I + I(\partial^\mu u^\nu - \partial^\nu u^\mu) = 0, \tag{33}$$

which is a system of six equations with only one unknown scalar function I. In general, there are no solutions, unless there are additional symmetries.

- We could consider additional variables, for example, the fields of spin hydrodynamics

Additional symmetries: the boost-invariant expansion

For the one-dimensional Bjorken expansion, the equations have a solution and we are left with *residual pseudo-gauge transformations*

$$\mathcal{E}' = \mathcal{E} + I\theta, \qquad \mathcal{P}' = \mathcal{P} - DI - \frac{2}{3}I\theta, \qquad \mathcal{Q}'^{\mu} = \mathcal{Q}^{\mu}, \qquad \mathcal{T}'^{\mu\nu} = \mathcal{T}^{\mu\nu} - I\sigma^{\mu\nu}. \tag{34}$$

We consider the case where baryon potential $\mu = 0$ and interpret the change of energy density $\mathcal{E} \to \mathcal{E}'$ as a change of the effective temperature, $T \to T'$

$$\mathcal{E}_{\rm eq}(T') = \mathcal{E}_{\rm eq}(T) + I(T)\theta, \tag{35}$$

where the function $\mathcal{E}_{\rm eq}(T)$ is externally given and defines the equation of state. I and the expansion scalar $\theta \equiv \nabla \cdot u$ should be treated as functions of T (or the proper time τ, with $\theta = 1/\tau$).

Similarly, the pressure change

$$\mathcal{P}_{\rm eq}(T') + \Pi'(T') = \mathcal{P}_{\rm eq}(T) + \Pi(T) - DI - \frac{2}{3} I\theta, \tag{36}$$

where $\Pi(T)$ and $\Pi'(T')$ define the bulk pressure.
So, the PGT changes the EM tensor from

$$T^{\mu\nu} = \mathcal{E}(T) u^\mu u^\nu - (\mathcal{P}_{\rm eq}(T) + \Pi(T))\, \Delta^{\mu\nu} + \mathcal{T}^{\mu\nu}, \tag{37}$$

to

$$T'^{\,\mu\nu} = \mathcal{E}(T') u^\mu u^\nu - \left(\mathcal{P}_{\rm eq}(T') + \Pi'(T')\right) \Delta^{\mu\nu} + \mathcal{T}'^{\,\mu\nu}. \tag{38}$$

We express the bulk pressure and the shear tensor in terms of the bulk and shear viscosity coefficients.

$$\Pi(T) = -\zeta(T)\theta, \quad \Pi'(T') = -\zeta'(T')\theta, \tag{39}$$
$$\mathcal{T}^{\mu\nu} = 2\eta(T)\sigma^{\mu\nu}, \quad \mathcal{T}'^{\,\mu\nu} = 2\eta'(T')\sigma^{\mu\nu}, \tag{40}$$

with $\sigma_{\alpha\beta} = \partial_{\langle\alpha} u_{\beta\rangle}$.
The primed functions ζ', η' are different from unprimed $\implies$ dependence of the transport coefficients on PGT.

In our setup, the hydrodynamic equations are reduced to

$$D\mathcal{E} = -(\mathcal{E} + \mathcal{P} + \Pi)\theta + \sigma_{\alpha\beta}\mathcal{T}^{\alpha\beta}, \tag{41}$$

which before the PGT has the form

$$\frac{\mathrm{d}\mathcal{E}_{\mathrm{eq}}(T)}{\mathrm{d}\tau} + \frac{\mathcal{E}_{\mathrm{eq}}(T) + \mathcal{P}_{\mathrm{eq}}(T)}{\tau} - \left[\frac{4}{3}\eta(T) + \zeta(T)\right]\theta^2 = 0, \tag{42}$$

and after the PGT has the form

$$\frac{\mathrm{d}\mathcal{E}_{\mathrm{eq}}(T')}{\mathrm{d}\tau} + \frac{\mathcal{E}_{\mathrm{eq}}(T') + \mathcal{P}_{\mathrm{eq}}(T')}{\tau} - \left[\frac{4}{3}\eta'(T') + \zeta'(T')\right]\theta^2 = 0. \tag{43}$$

We rewrite the second equation renaming T' to T and obtain

$$\frac{4}{3}\left[\eta'(T) - \eta(T)\right] + \zeta'(T) - \zeta(T) = 0. \tag{44}$$

Now, from Eq. (34),

$$\eta'(T') = \eta(T) - \frac{I(T)}{2}, \tag{45}$$

so the PGT corresponds to a change of the kinetic coefficients.

In the non-conformal case, it is possible to construct and solve numerically a differential equation for $I(\tau)$,

$$\frac{dI}{d\tau}+\frac{I}{3\tau}+A_{\rm eq}(T')-A_{\rm eq}(T)=\frac{4}{3\tau}\left[\eta(T')-\eta(T)\right]+\frac{1}{\tau}\left[\zeta(T')-\zeta(T)\right], \quad (46)$$

where $A_{\rm eq}(T)$ describes the trace anomaly, $A_{\rm eq}(T)=\mathcal{P}_{\rm eq}(T)-(1/3)\mathcal{E}_{\rm eq}(T)$.

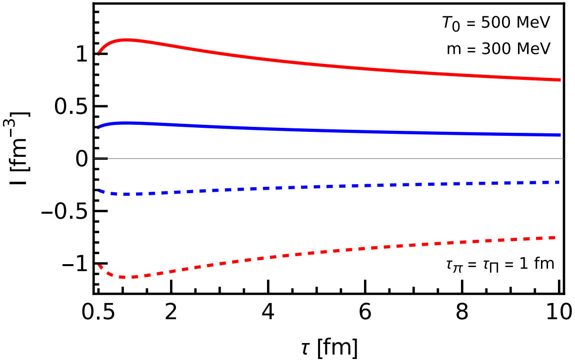

Figure: The function $I(\tau)$ for various initial conditions.

Summary and outlook

- The are various pseudo-gauges used in theoretical physics, and this is a hot and contentious topic in relativistic (spin) hydrodynamics.
- In the simple case we considered, possible PGTs are very restricted, and they correspond to redefinition of kinetic coefficients, keeping their certain combination constant. We can show that entropy production is unchanged too.
- In the future, we can consider a more general pseudogauge, built with additional fields.
- Check whether entropy production is the same with suitable redefinition of temperature and entropy (maybe are least for a subset of pseudogauges).
- Revisit Zubarev's approach.

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