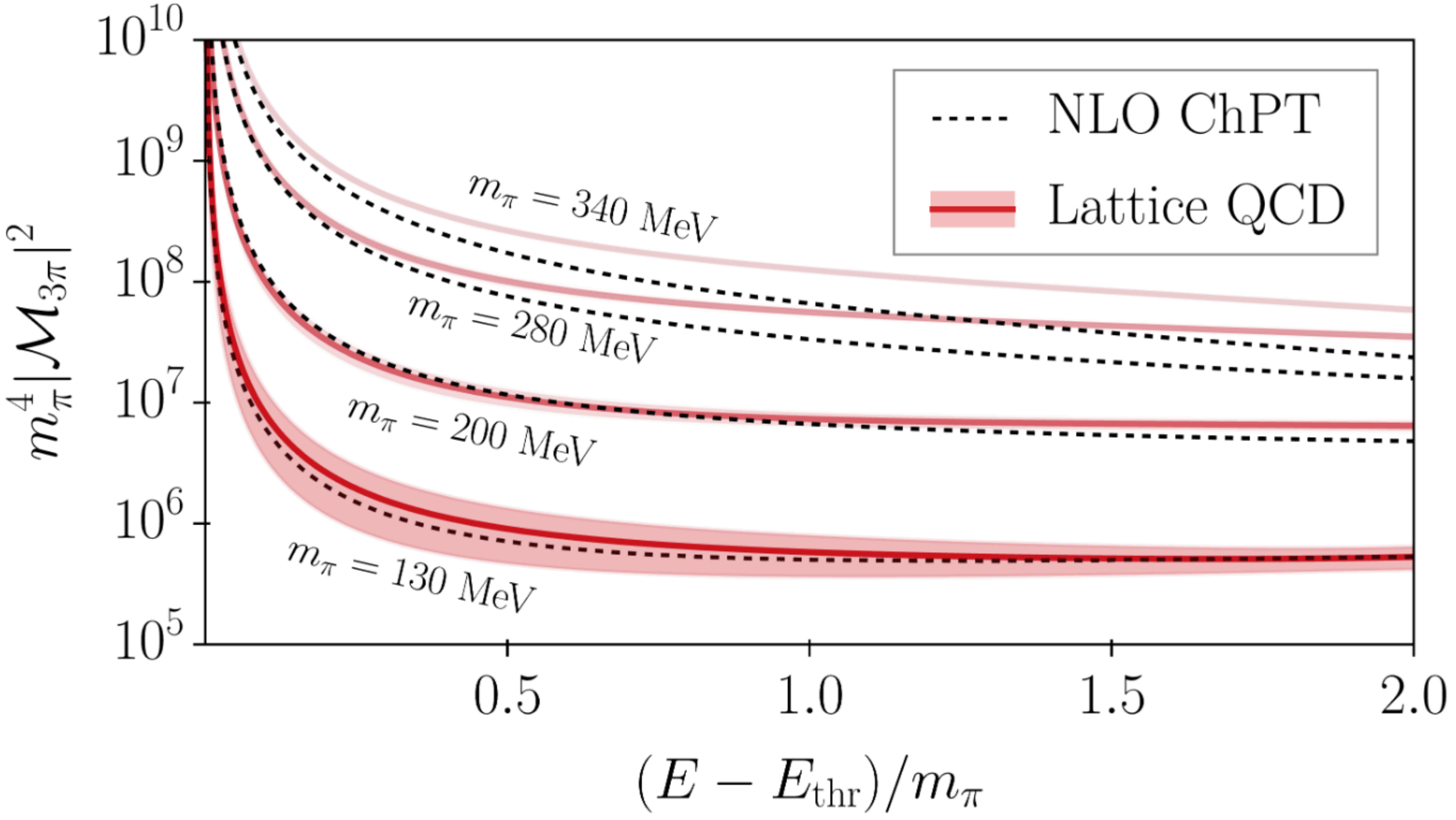
Three-body scattering in Lattice QCD

Sebastian M. Dawid



UNIVERSITY of WASHINGTON

Acknowledgements

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner
"Two- and three-meson scattering amplitudes at physical quark masses from lattice QCD"

All the work described in this talk was done in collaboration with these most kind and honorable people


Zach Draper (UW)


Andrew Hanlon (KSU)


Ben Hörz (Intel)

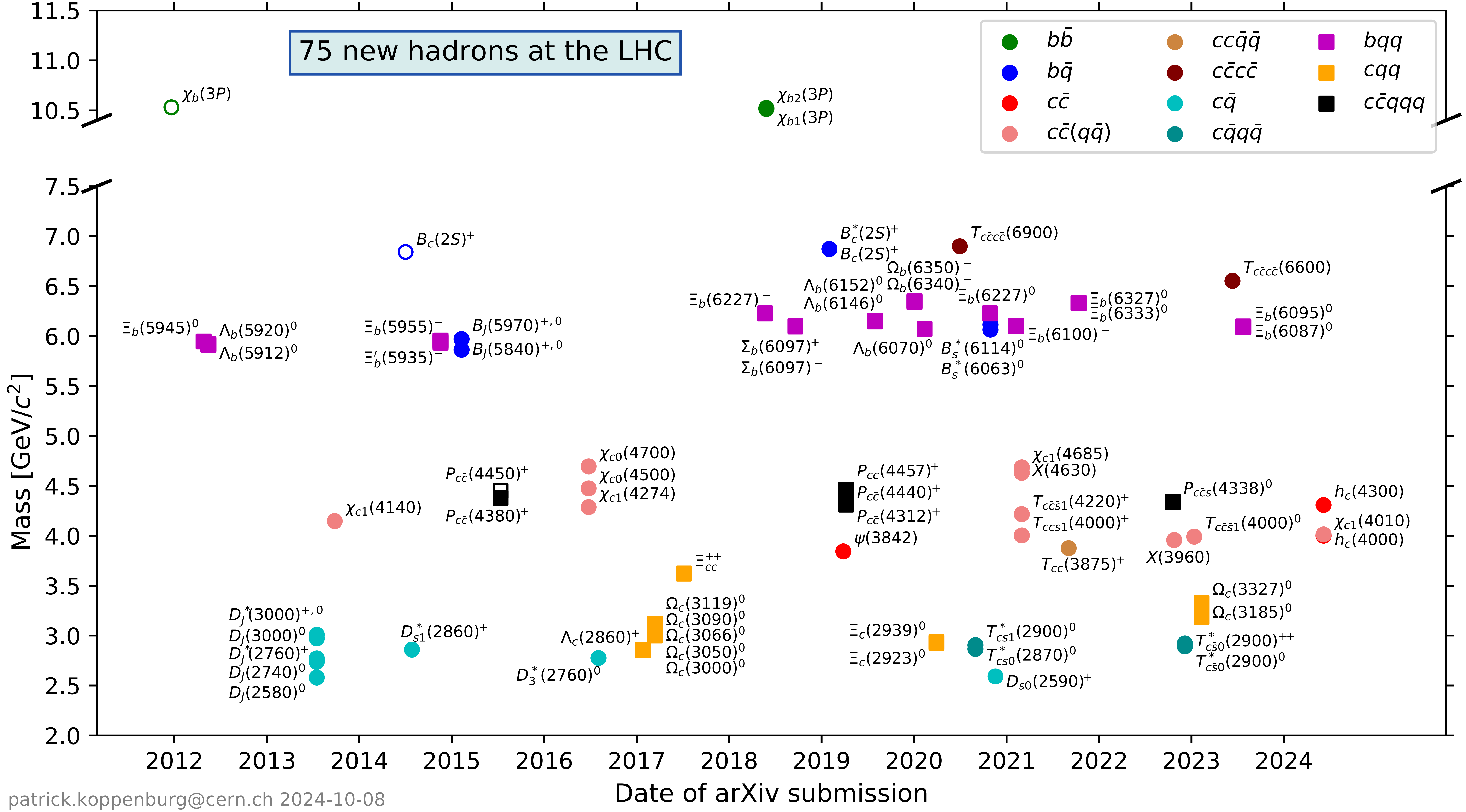

Colin Morningstar (CMU)


Fernando Romero-López (U. Bern)


Stephen Sharpe (UW)


Sarah Skinner (CMU)

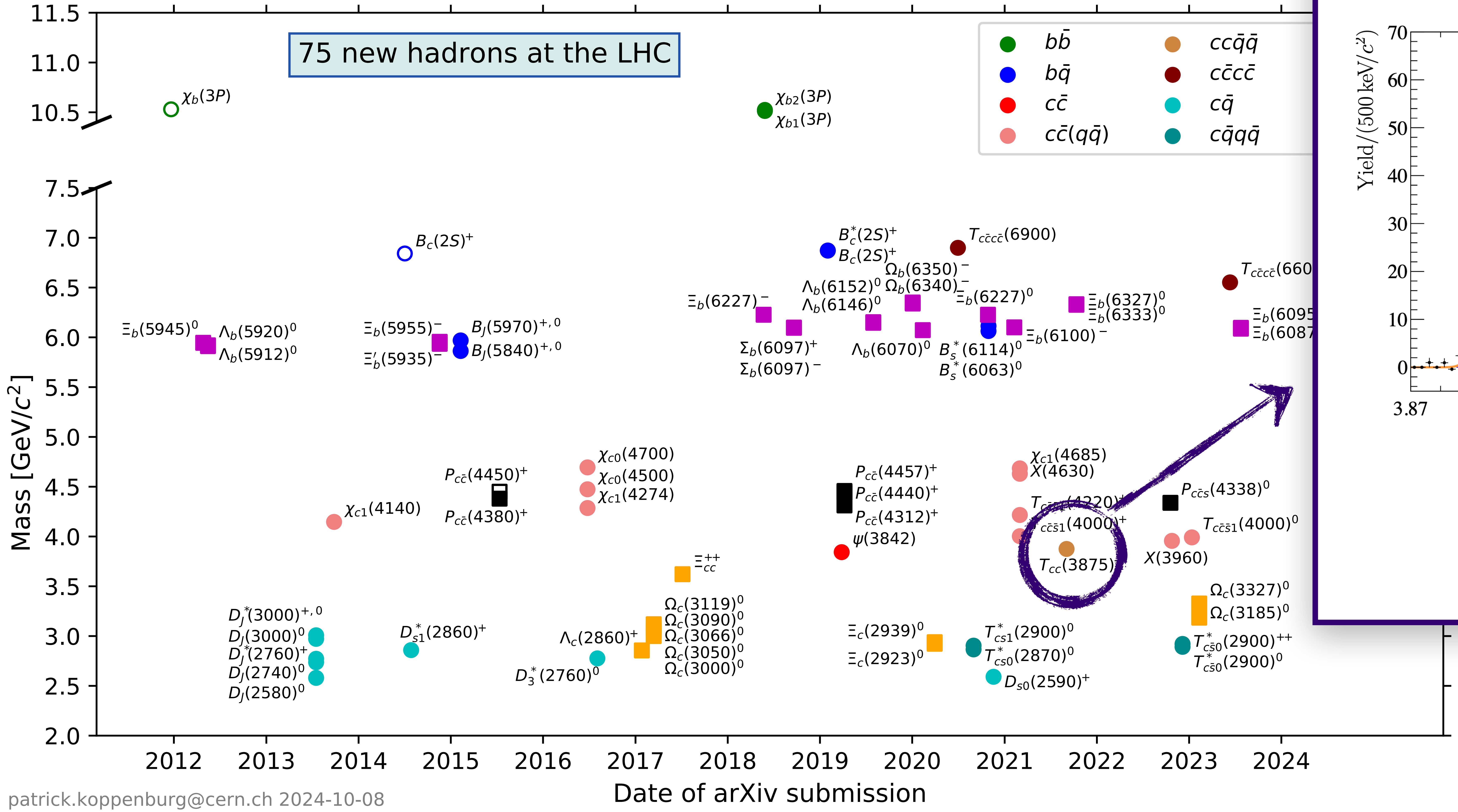
New exotic discoveries



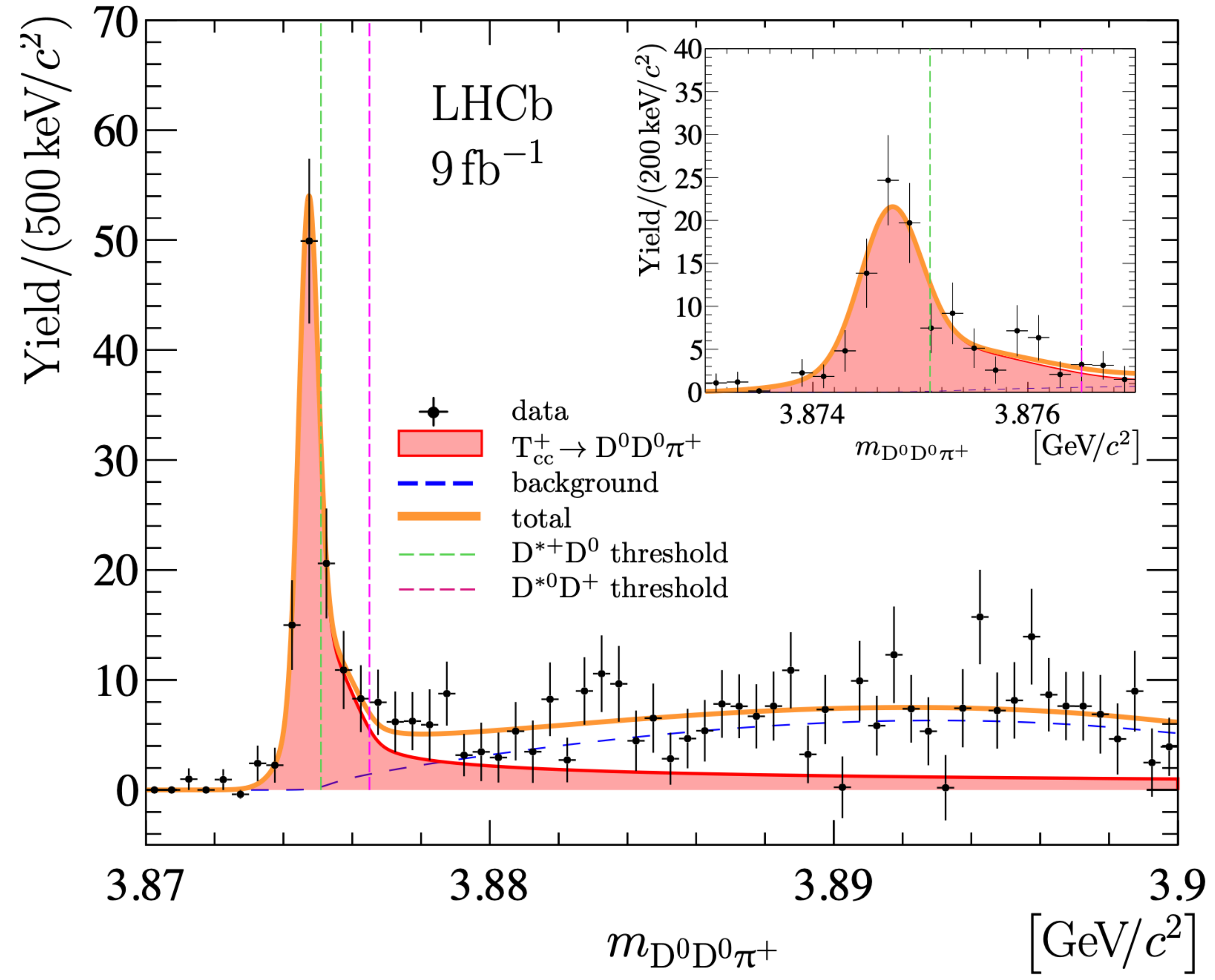
patrick.koppenburg@cern.ch 2024-10-08

LHCb collaboration, Koppenburg, *List of hadrons observed at the LHC*

New exotic discoveries



Doubly charmed tetraquark



Observed in $DD\pi$ decay

[Nature Physics 18 (2022) 751]

LHCb collaboration, Koppenburg, *List of hadrons observed at the LHC*

The three-body program

$$\det_{k,\ell,m_\ell}\left[\mathcal{K}_3^{-1}(E) + F_3(E,\boldsymbol{P},L)\right] = 0$$

Hansen & Sharpe, *Relativistic, model-independent, three-particle quantization condition*

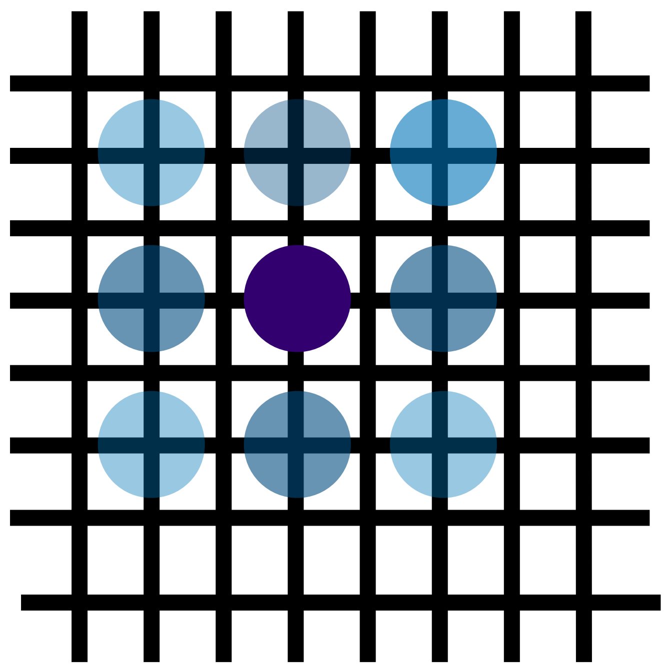
Mai & Döring, *Three-body unitarity in finite volume*

Müller, Pang, Rusetsky & Wu., Relativistic-invariant formulation of the NREFT three-particle quantization condition

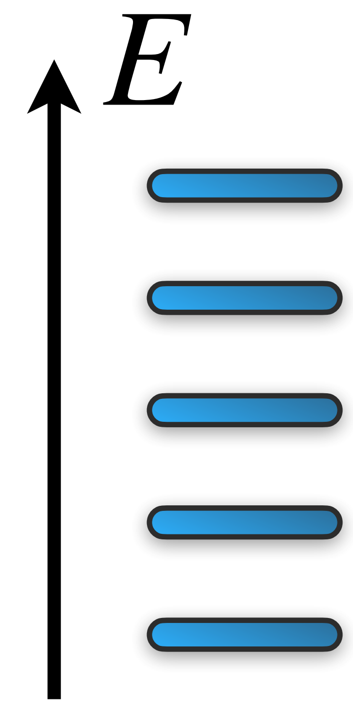
Jackura, Dawid, Fernandez-Ramirez, et al. (JPAC), *Equivalence of three-particle scattering formalisms*

Blanton & Sharpe, *Equivalence of relativistic three-particle quantization conditions*

Finite Volume



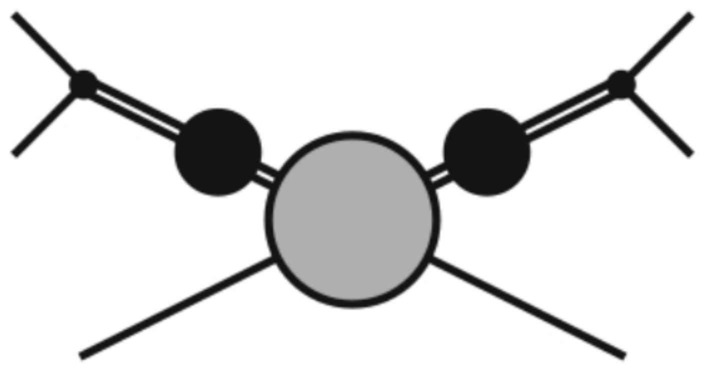
Lattice QCD



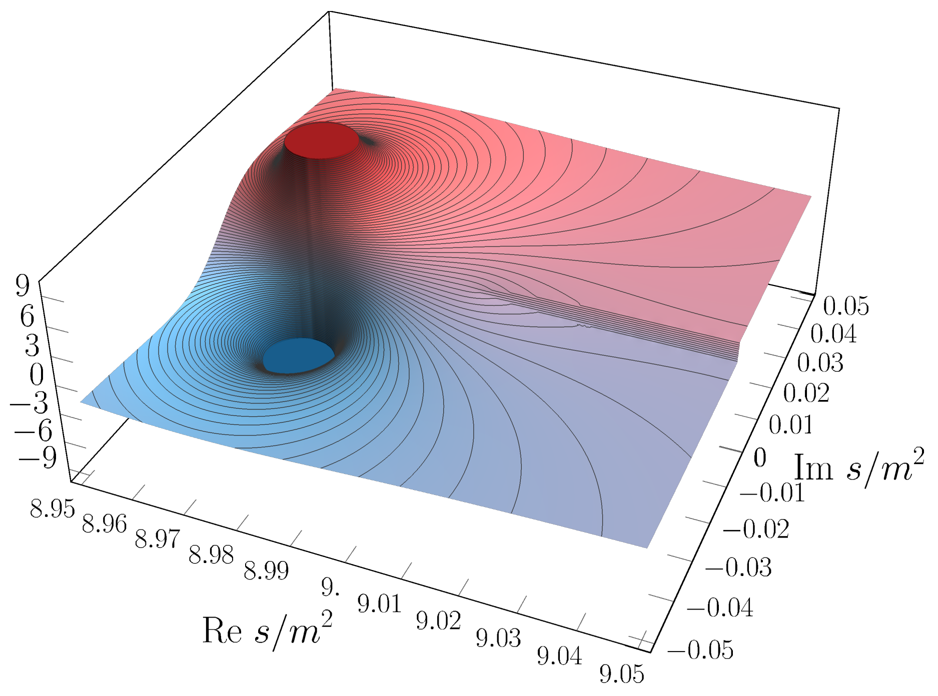
Spectrum

Quantization Condition

Infinite Volume

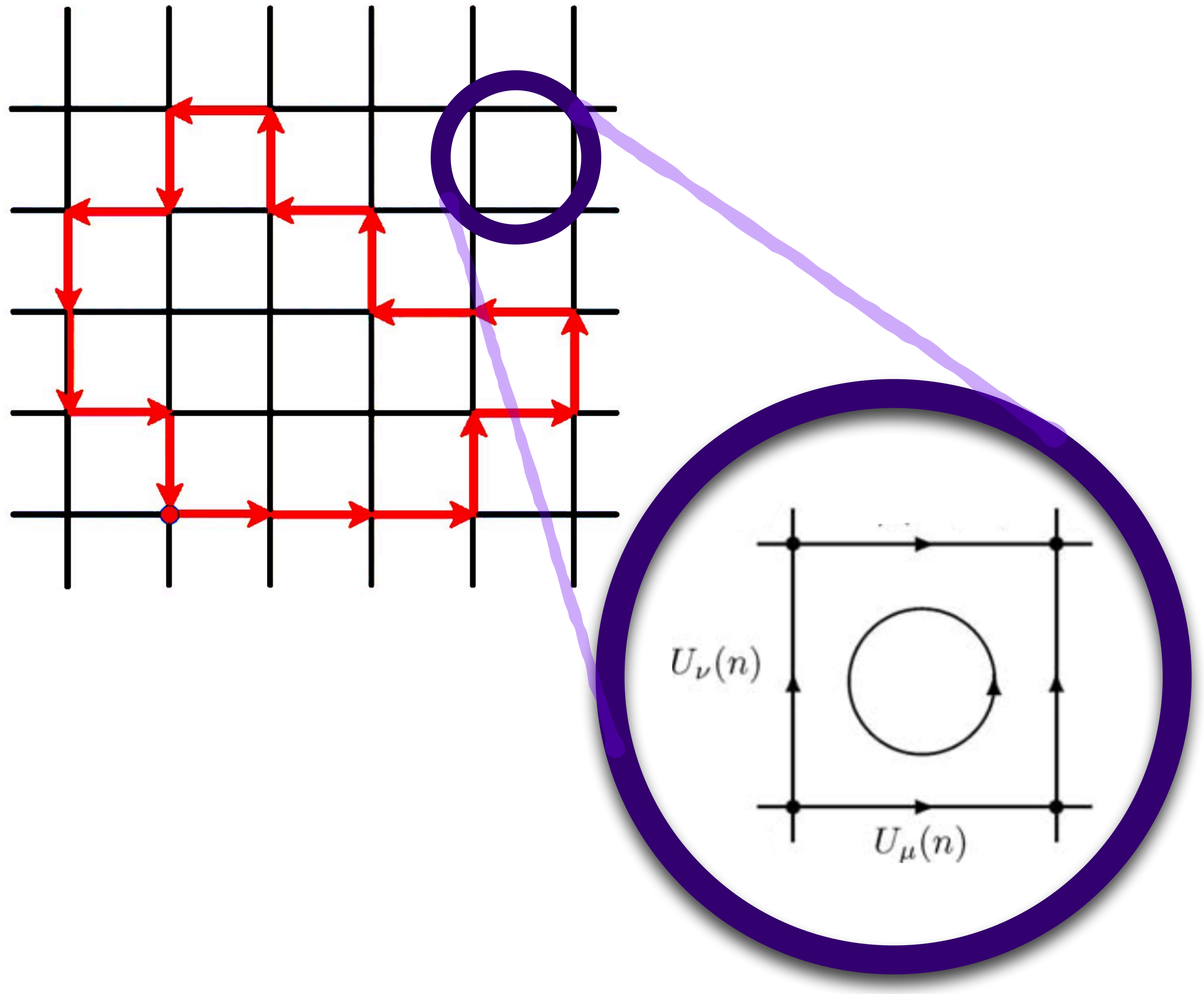


Amplitude



Resonances

Lattice QCD



Field theory as a statistical system

$$Z = \int \mathcal{D}\left[\psi, \overline{\psi}\right] \; \mathcal{D}[U] \; \mathrm{e}^{-S_F[\psi,\overline{\psi},U]-S_G[U]}$$

Monte-Carlo simulation produces **correlation functions**

used to extract **finite-volume energy levels**

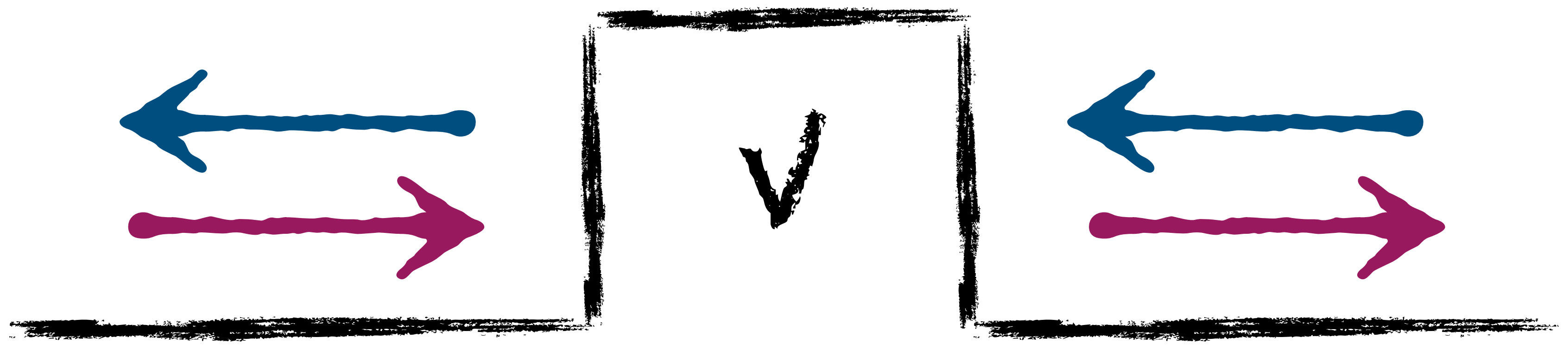
NO CLEAR DEFINITION OF SCATTERING!

$$C(\tau) = \int_L \underbrace{e^{-i\boldsymbol{P}\cdot\boldsymbol{x}}}_{\text{Total momentum}} \langle\Omega| \underbrace{\mathcal{O}_h(\tau,\boldsymbol{x})\,\mathcal{O}_h^\dagger(0,\boldsymbol{0})}_{\text{Creation/ annihilation of hadron(s) "h"}} |\Omega\rangle \propto \sum_n |\langle\Omega|\mathcal{O}(0,\boldsymbol{0})|\underbrace{h,\boldsymbol{P},n}_{\text{Finite-volume states of h's quantum numbers}}\rangle|^2 \; \underbrace{e^{-E_n\tau}}_{\text{Time dependence} \to \text{energy levels}}$$

Finite–volume scattering formalism

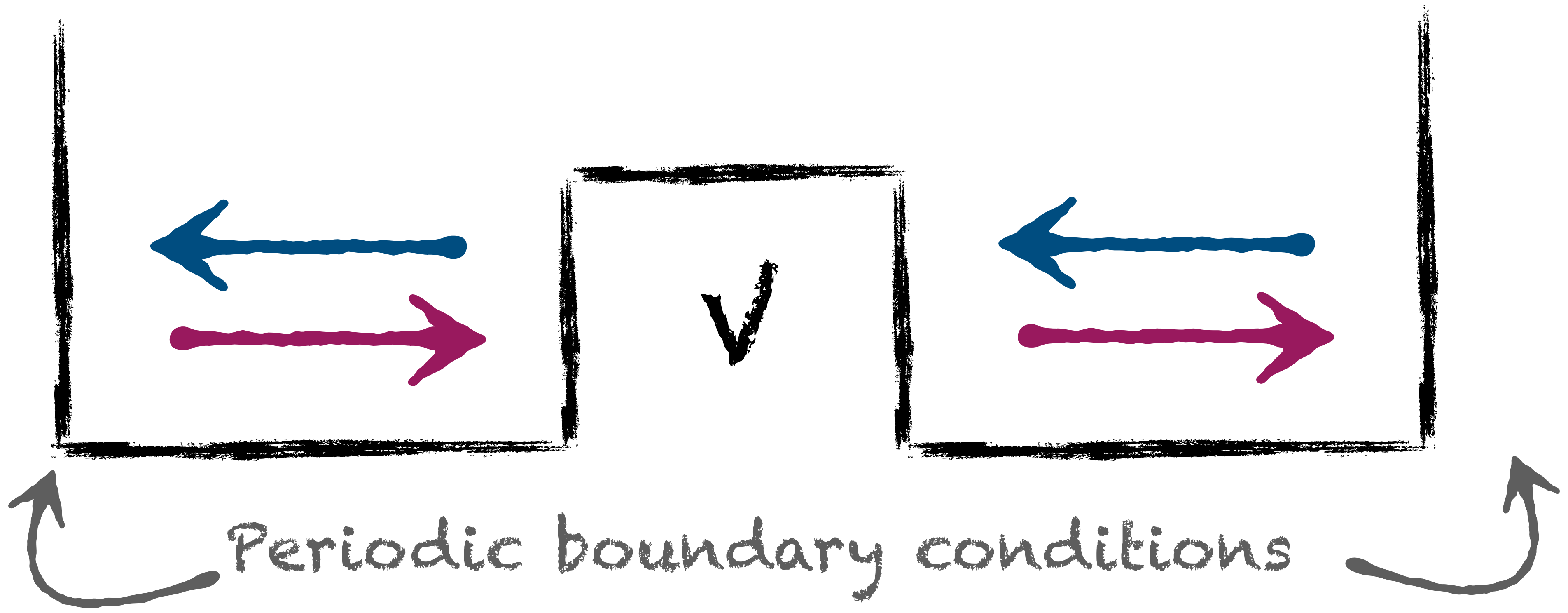
Lüscher, *Volume dependence of the energy spectrum in massive QFTs*
Rummukainen & Gottlieb, *Resonance scattering phase shifts on a non-rest frame lattice*
Kim, Sachrajda, Sharpe, *Finite-volume effects for two hadron states in moving frames*

One-dimensional scattering in the infinite volume



$$\psi(x) \propto \cos(k|x| + \delta(k))$$

One-dimensional "scattering" problem in the finite volume



Quantization condition

$$\frac{kL}{2} + \delta(k) = n\pi$$

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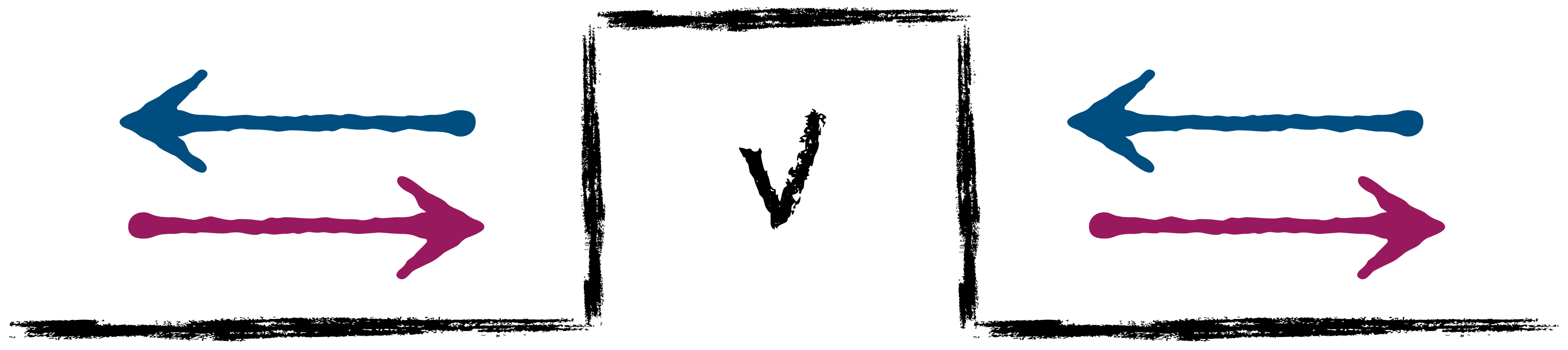

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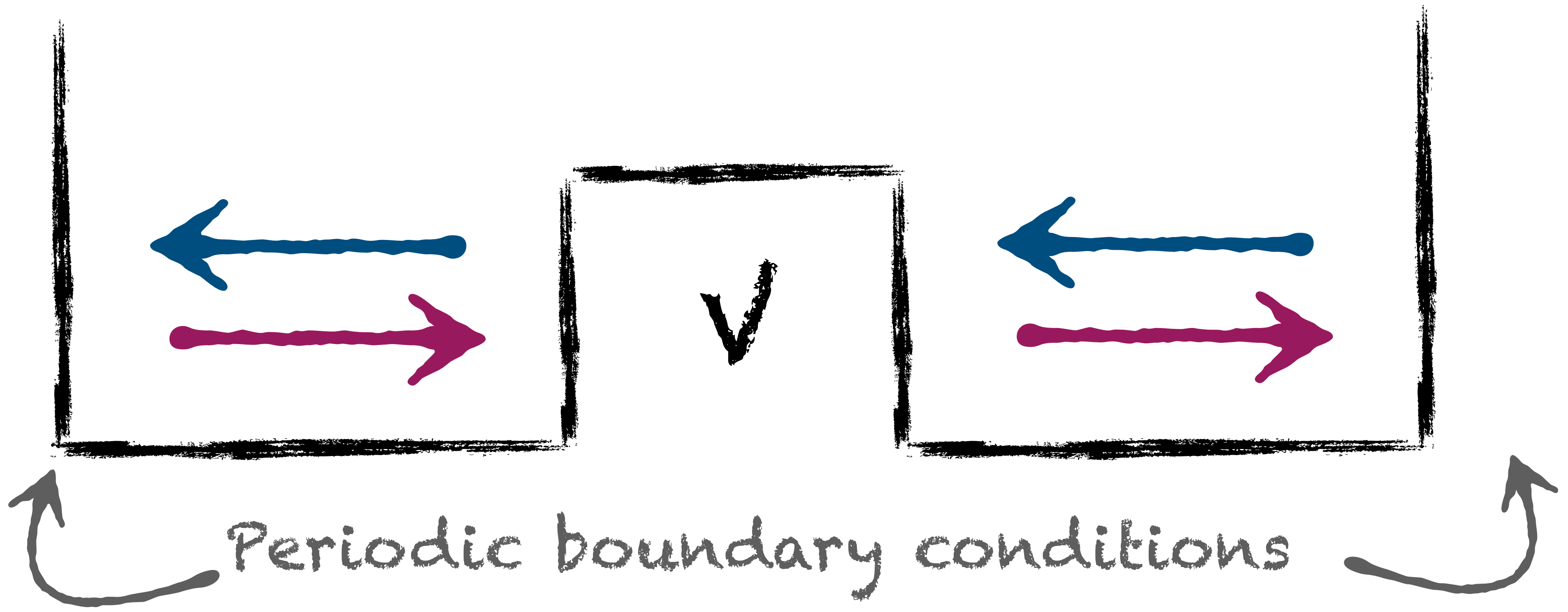

Quantization condition

$$\det_{\ell m_\ell}\left[\mathcal{M}_2(E)^{-1} + F(E, \boldsymbol{P}, L)\right] = 0$$

Matrices in the angular momentum space

Known special function

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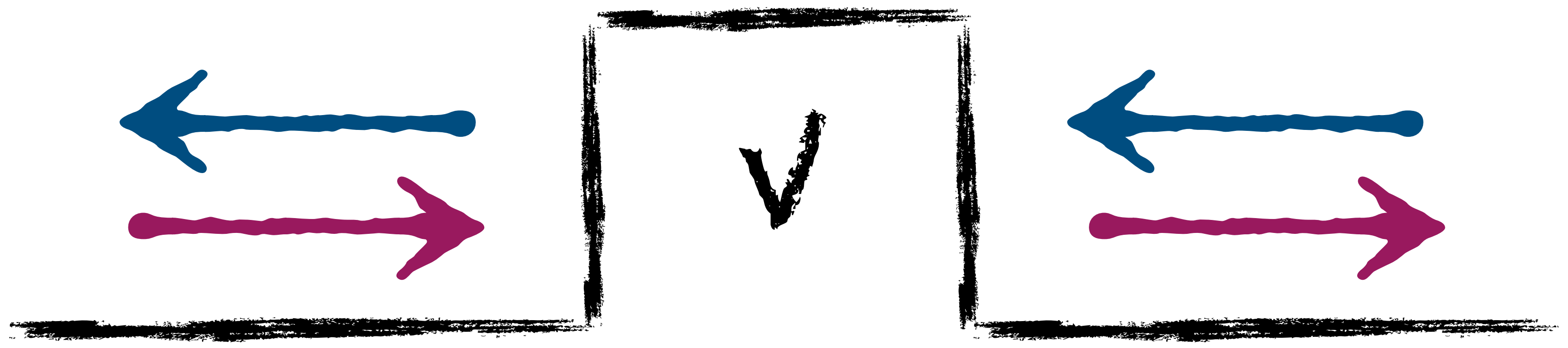

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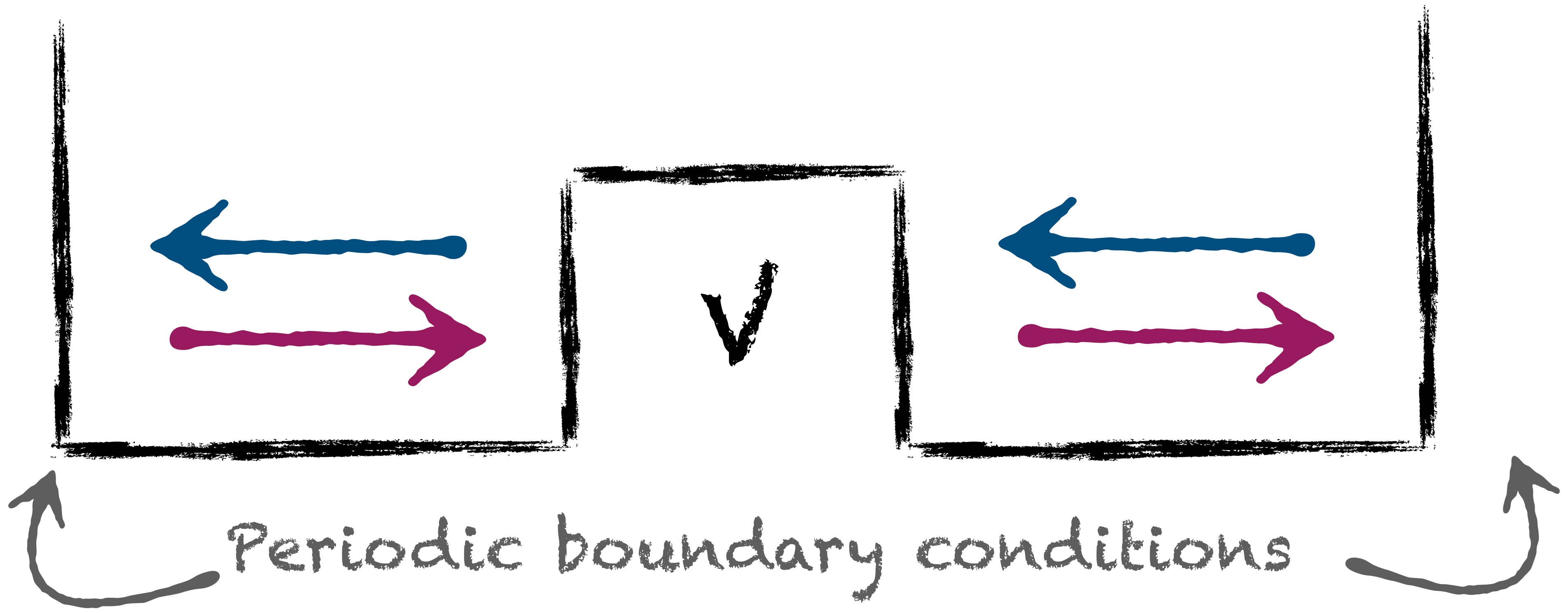

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Known special function

Power-law volume dependence related to the unitarity cut: on-shell particles travel large distances

S-matrix theory

$$S_{ab}(E) = \langle \text{out}|\hat{\boldsymbol{S}}|\text{in}\rangle$$

The transition operator T is

$$S = \underbrace{\mathbb{1}}_{\text{Disconnected}} + \underbrace{iT}_{\text{Connected}}$$

Unitarity of the S operator

$$T - T^\dagger = iTT^\dagger$$

Properties of the S matrix

Analyticity (causality)

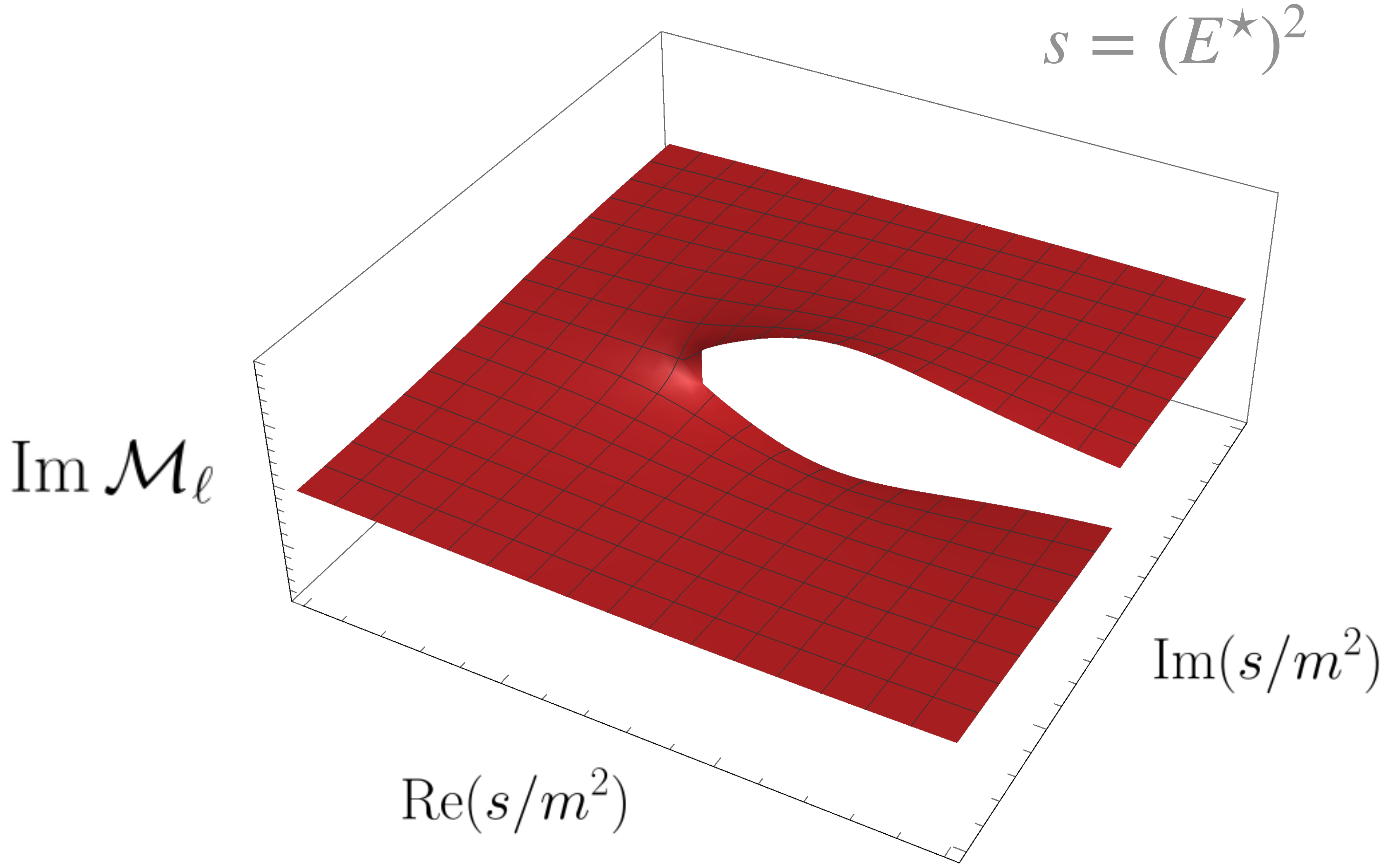
Unitarity (probability conservation)

Poincaré symmetry (frame independence)

Crossing symmetry (particles–antiparticles)

Internal symmetries (charge, isospin, G-parity)

$s = (E^\star)^2$



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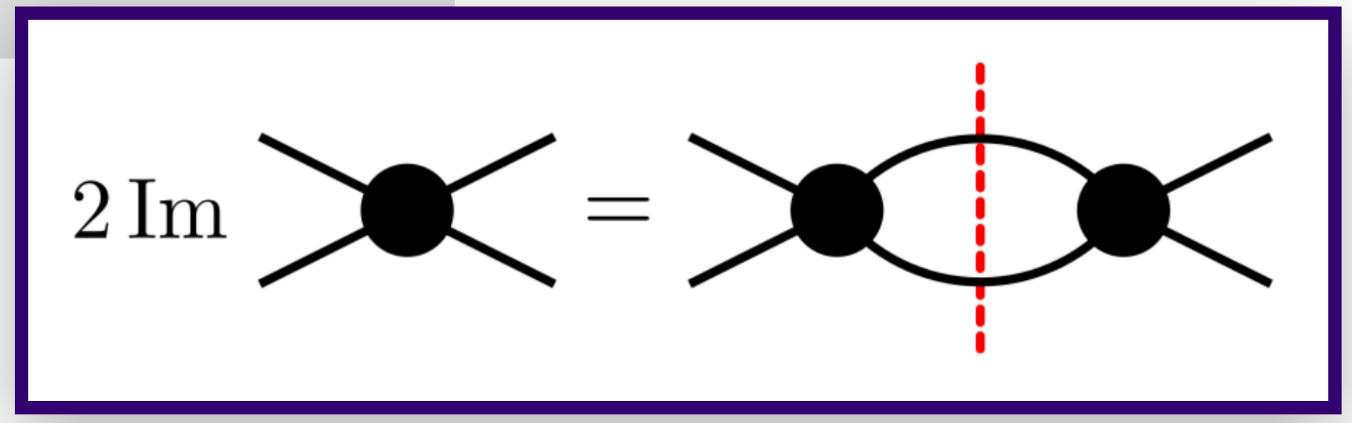

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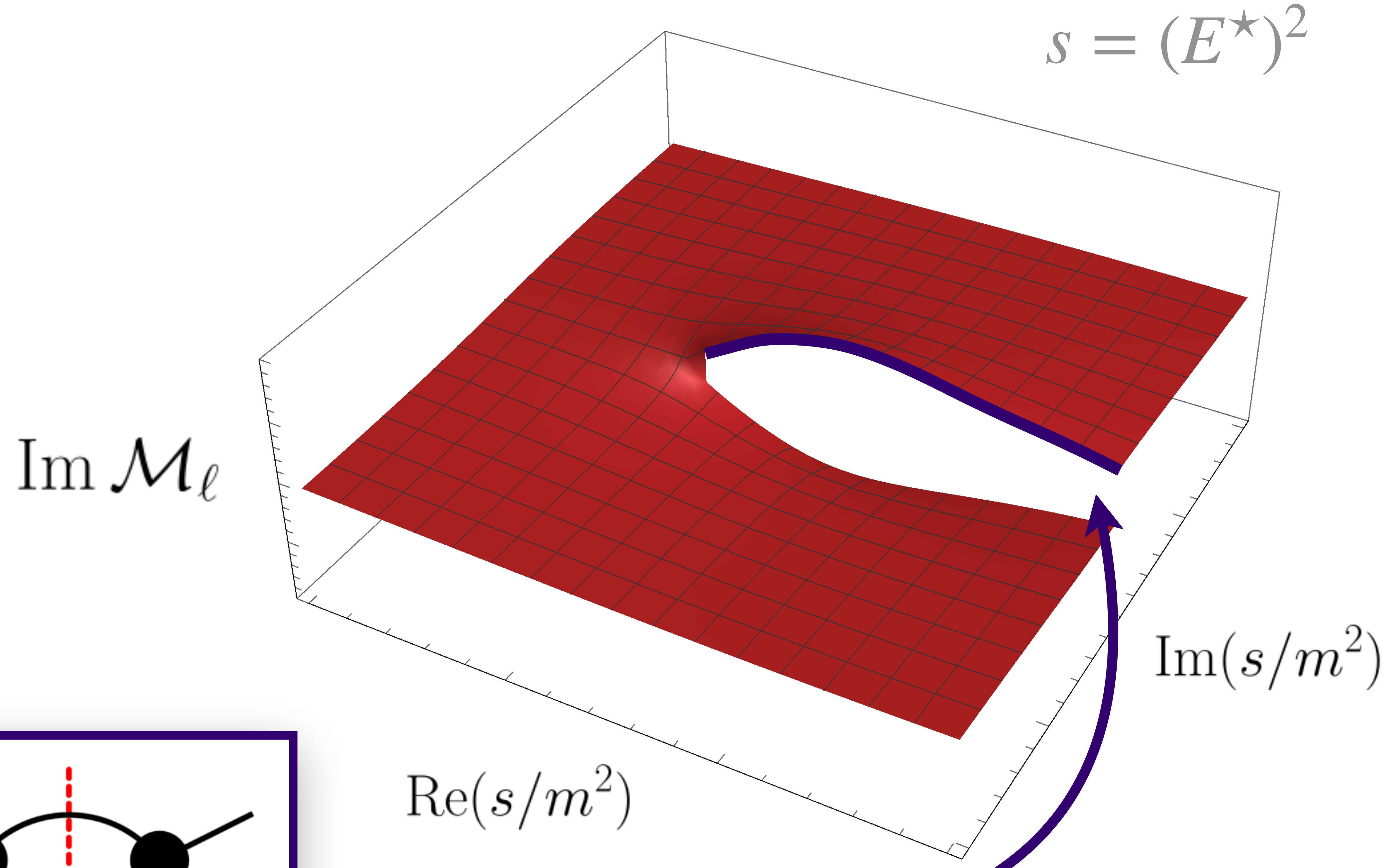

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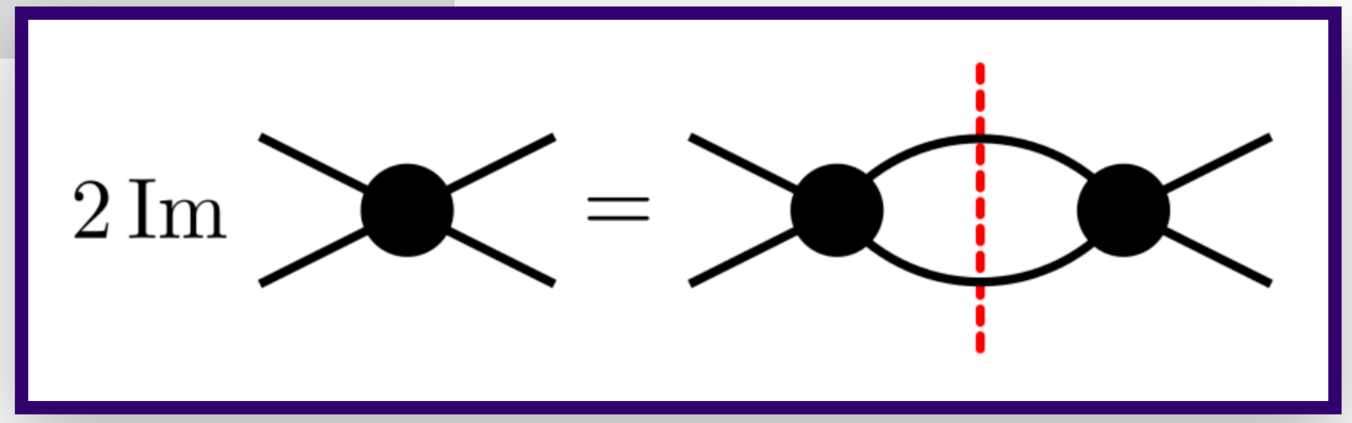
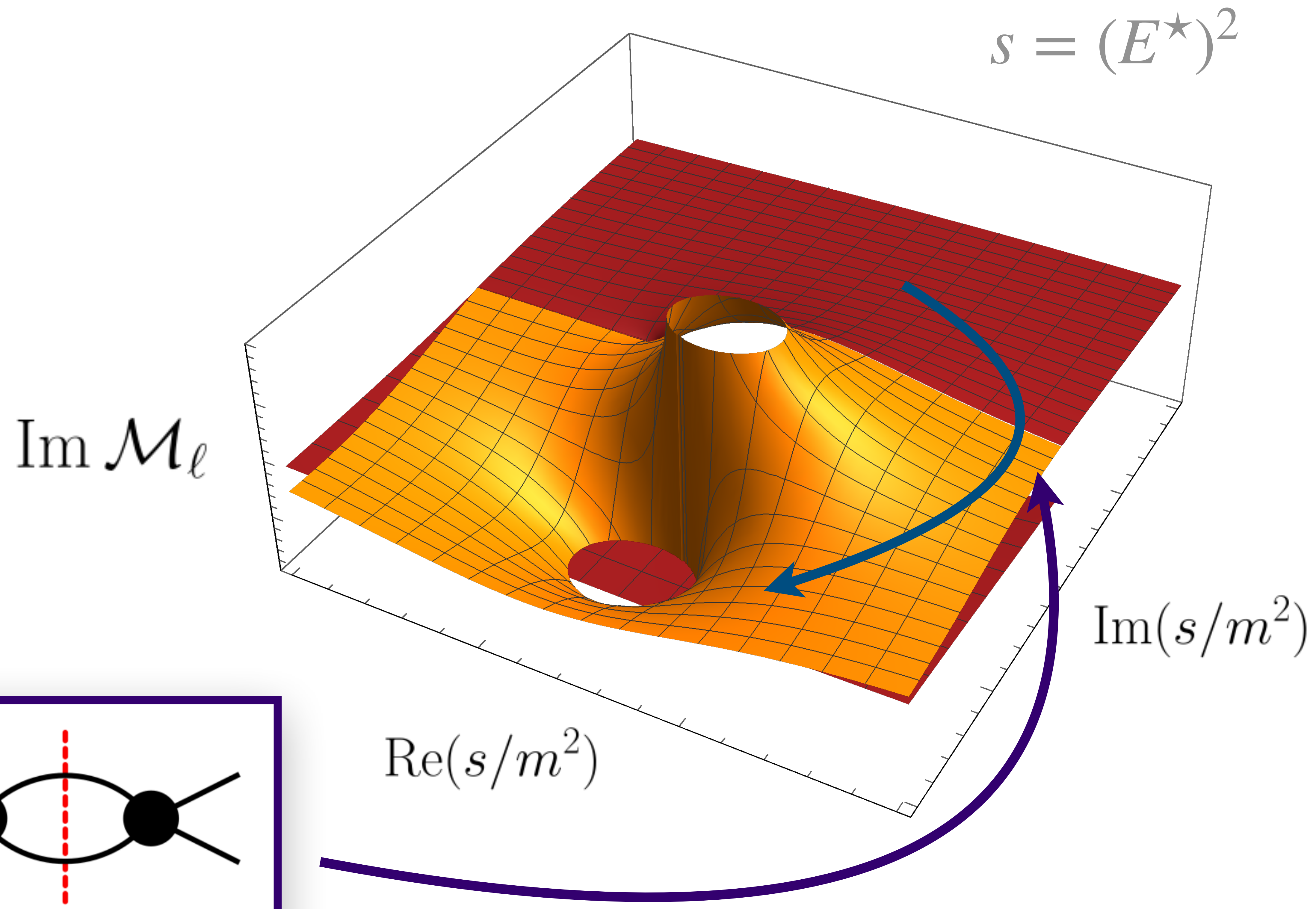




S-matrix theory

$$S_{ab}(E) = \langle \text{out}|\hat{\boldsymbol{S}}|\text{in}\rangle$$

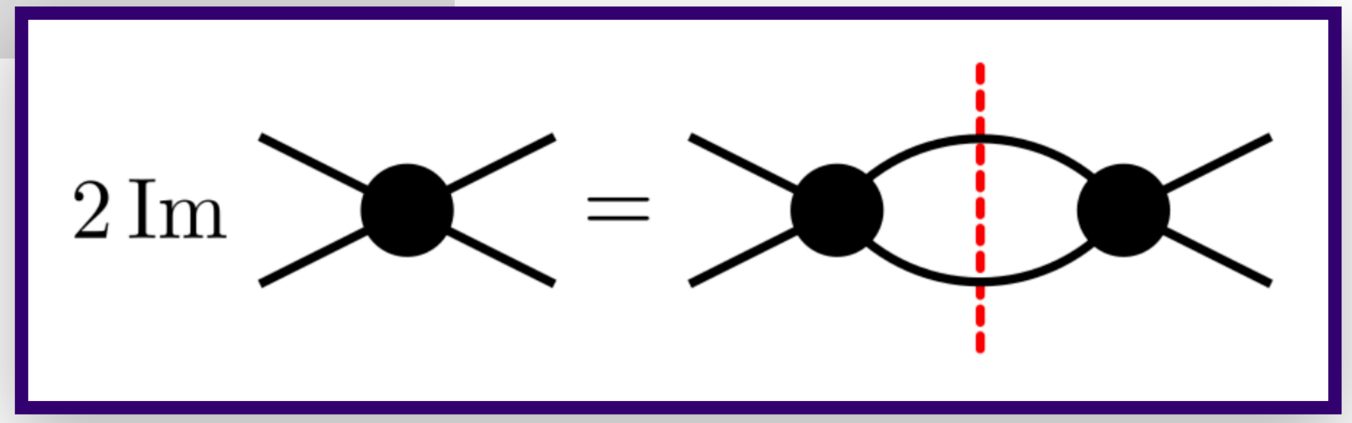
The transition operator T is

$$S = \mathbb{1} + iT$$

Disconnected **Connected**

Unitarity of the S operator

$$T - T^\dagger = iTT^\dagger$$



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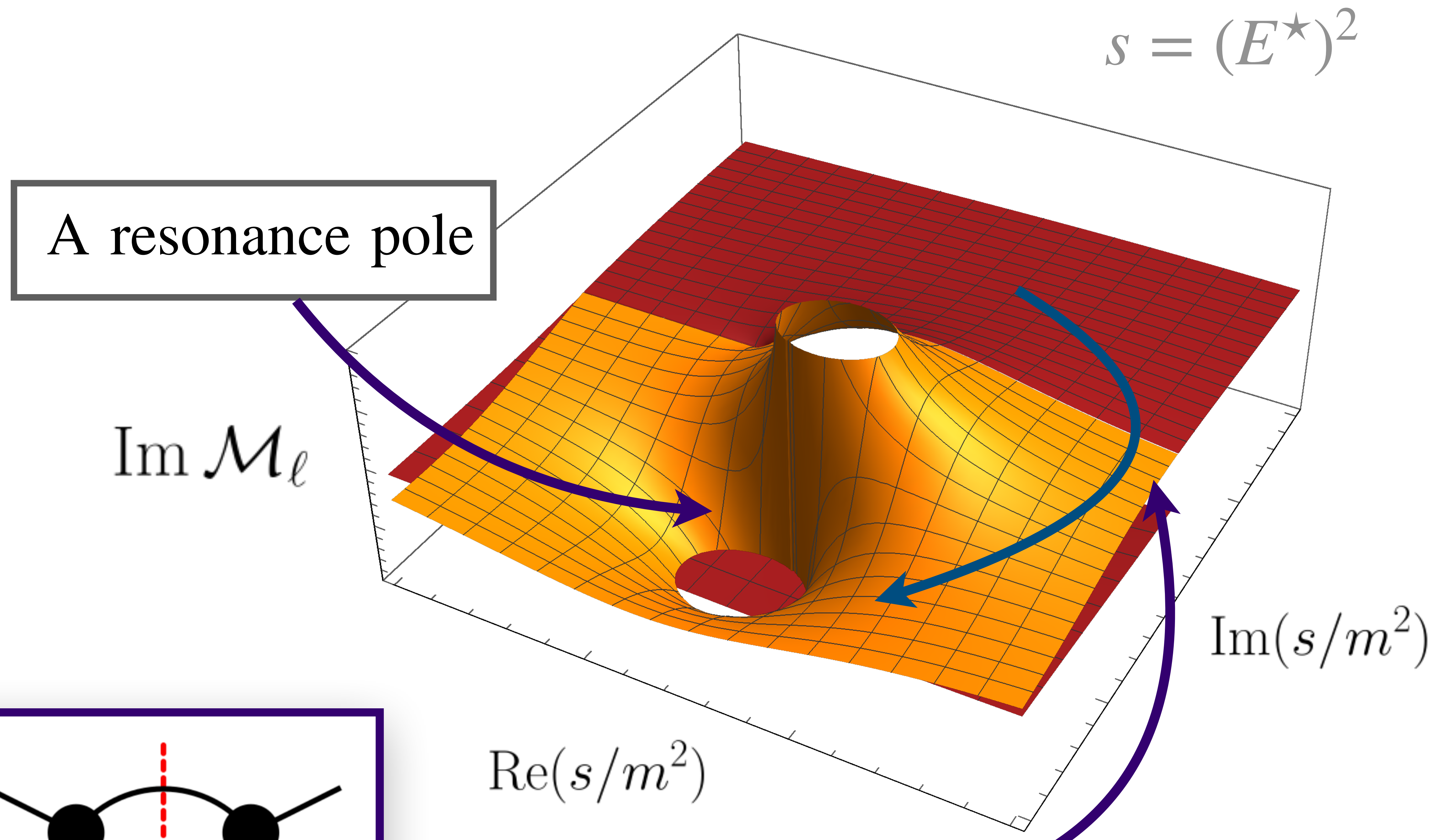

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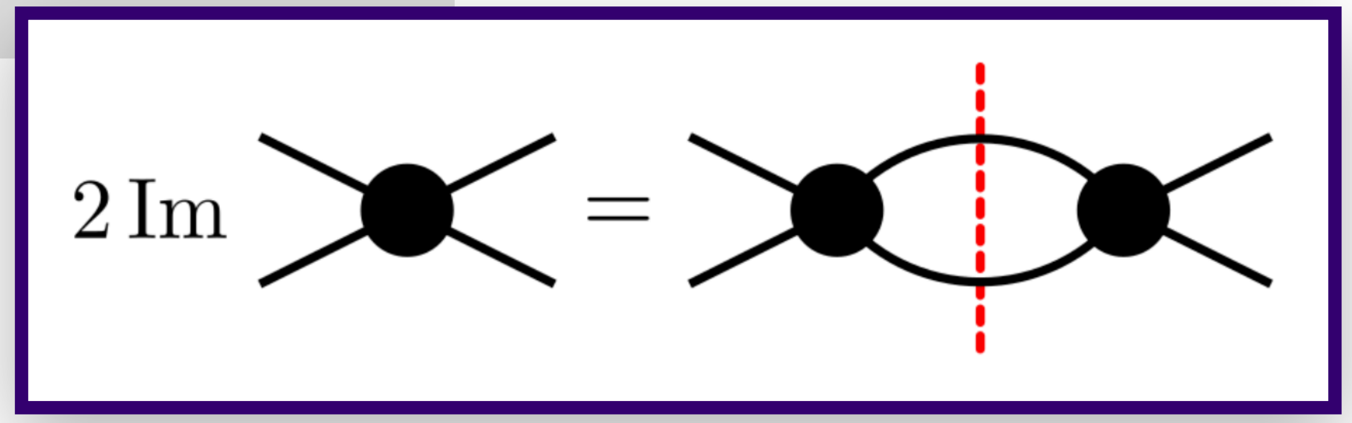
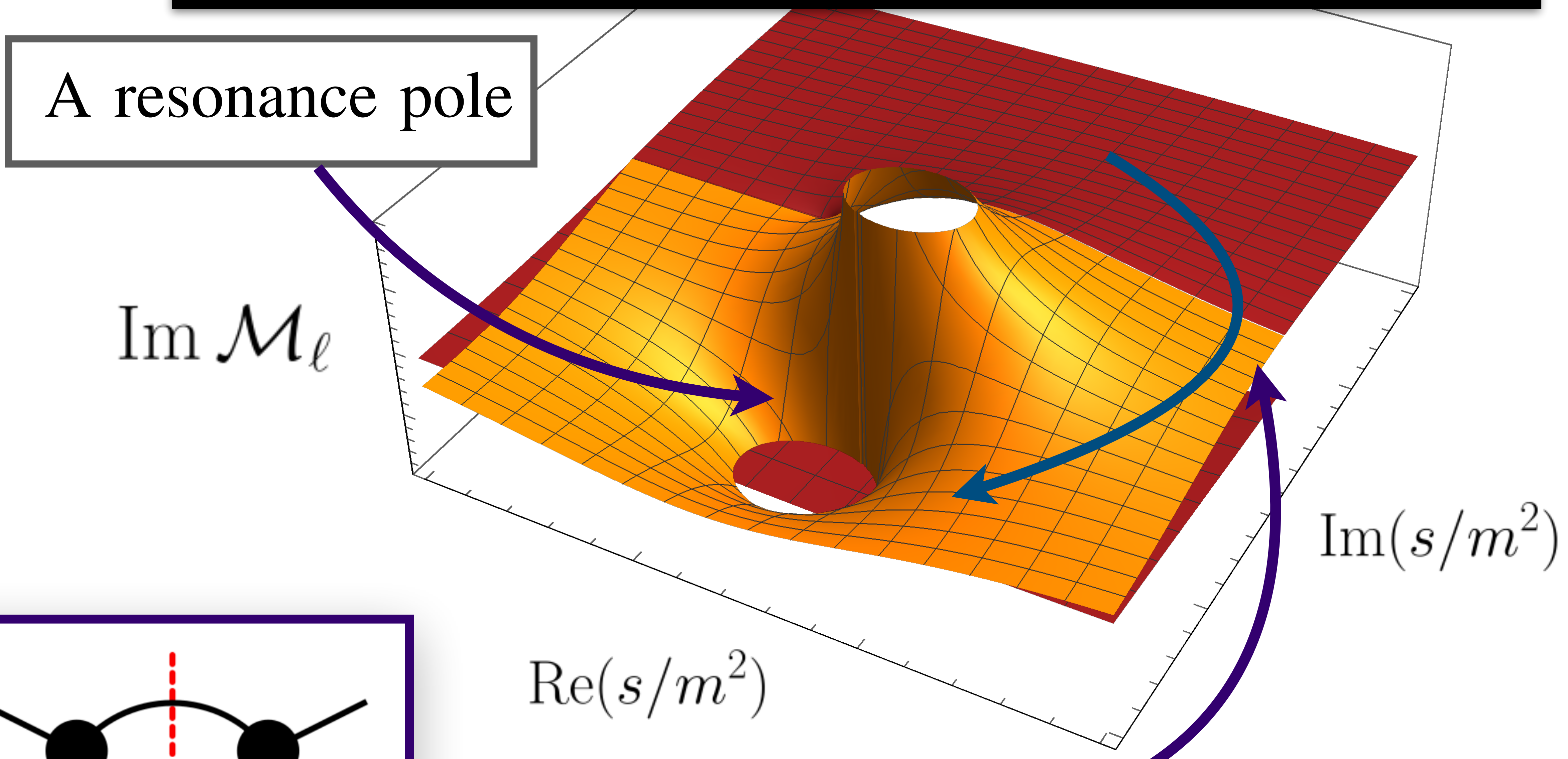


K–matrix parametrization

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell^{-1}(s) - i\rho(s)}$$

Phase–shift

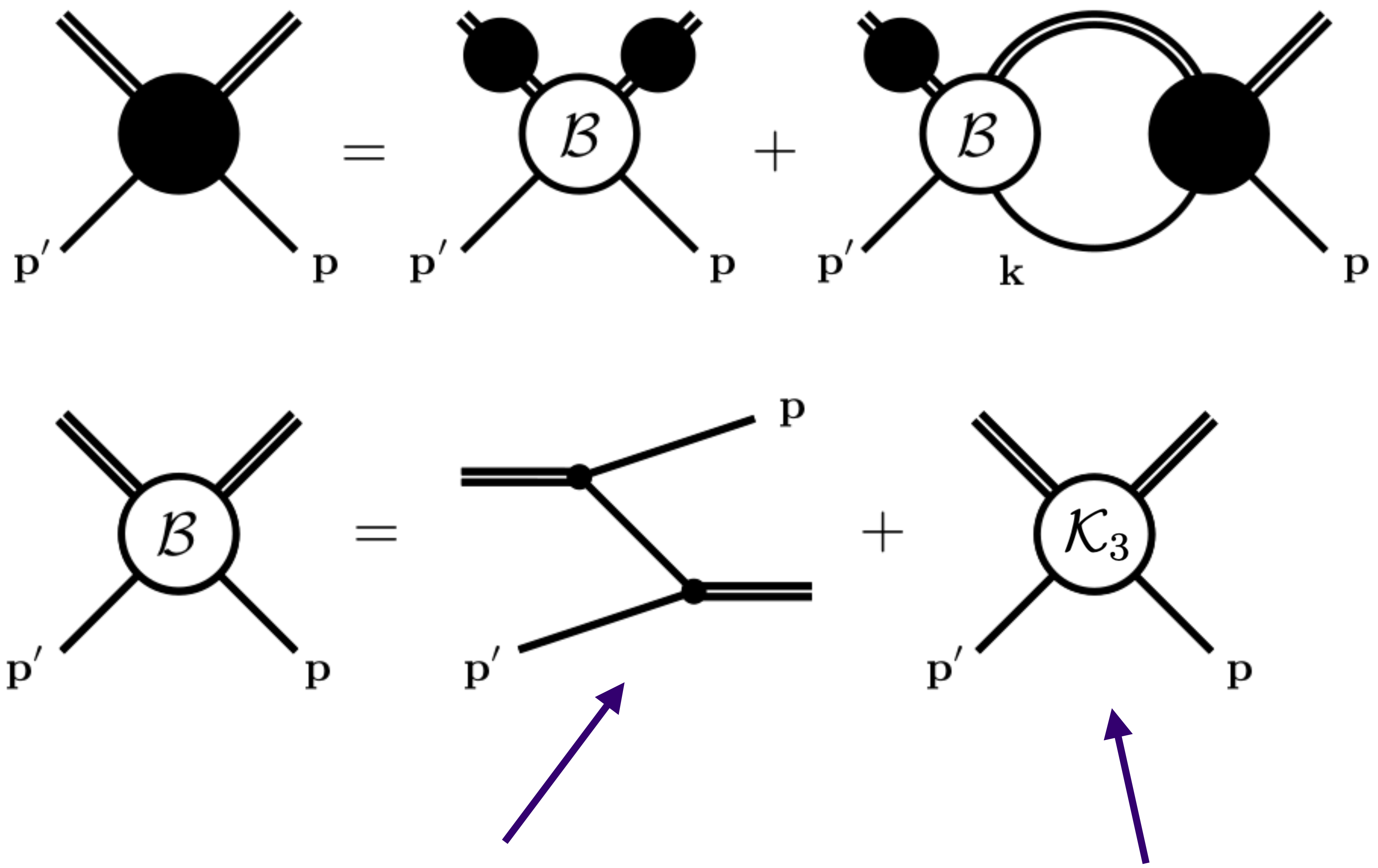
$$\mathcal{K}_\ell^{-1}(s) = \frac{q^\star}{8\pi\sqrt{s}} \cot(\delta_\ell(s))$$



The three-body amplitude

Jackura et al. (JPAC), Phenomenology of Relativistic 3-to-3 Reaction Amplitudes within the Isobar Approximation

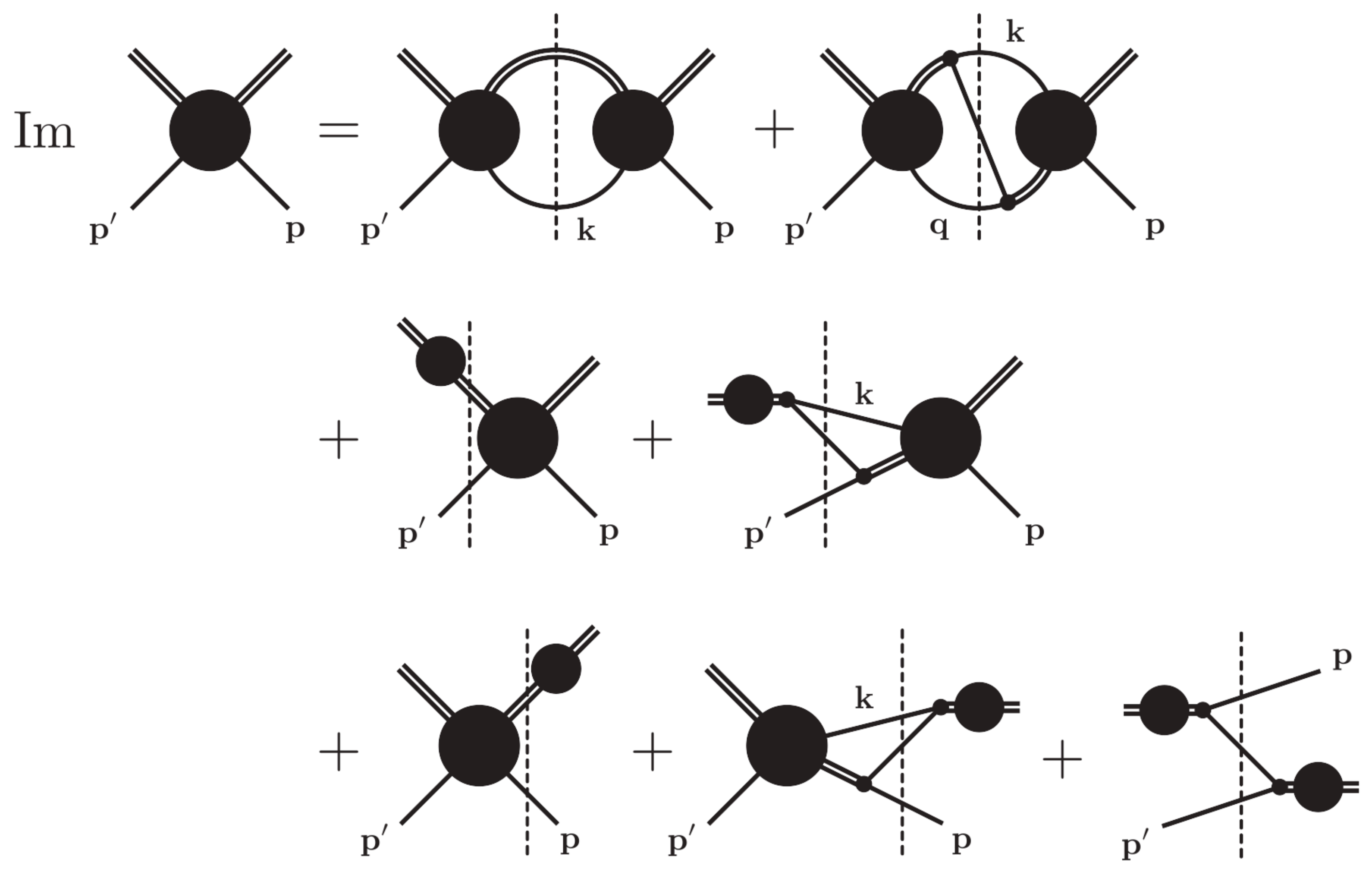
Dawid & Szczepaniak, *Bound states in the B-matrix formalism for the three-body scattering*



One Particle Exchange

Short Range Interactions

Three-particle unitarity



$$\mathcal{M}_3 = \mathcal{M}_2 \mathcal{B} \mathcal{M}_2 + \mathcal{M}_2 \int \mathcal{B}\, \rho_3\, \mathcal{M}_3$$

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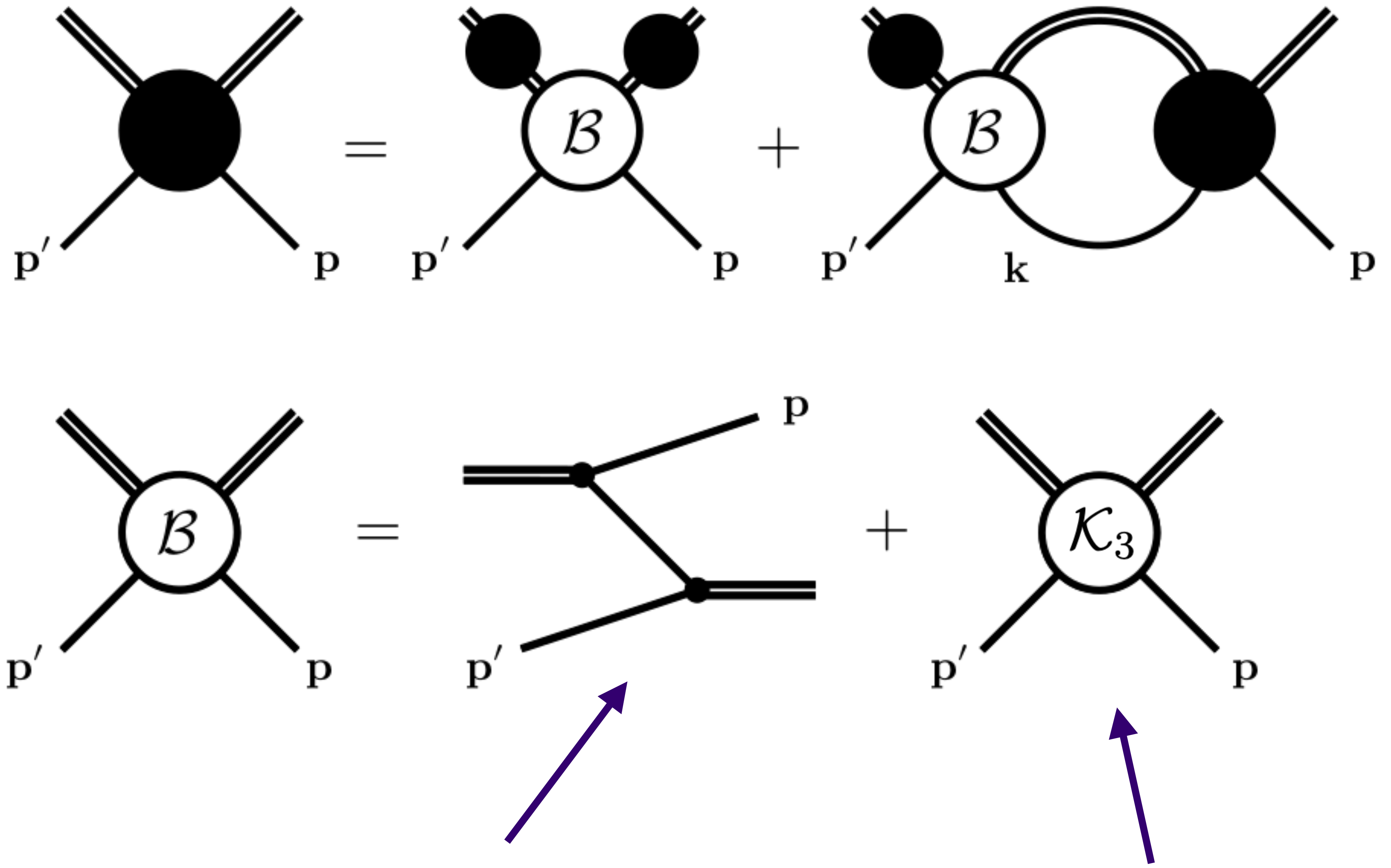

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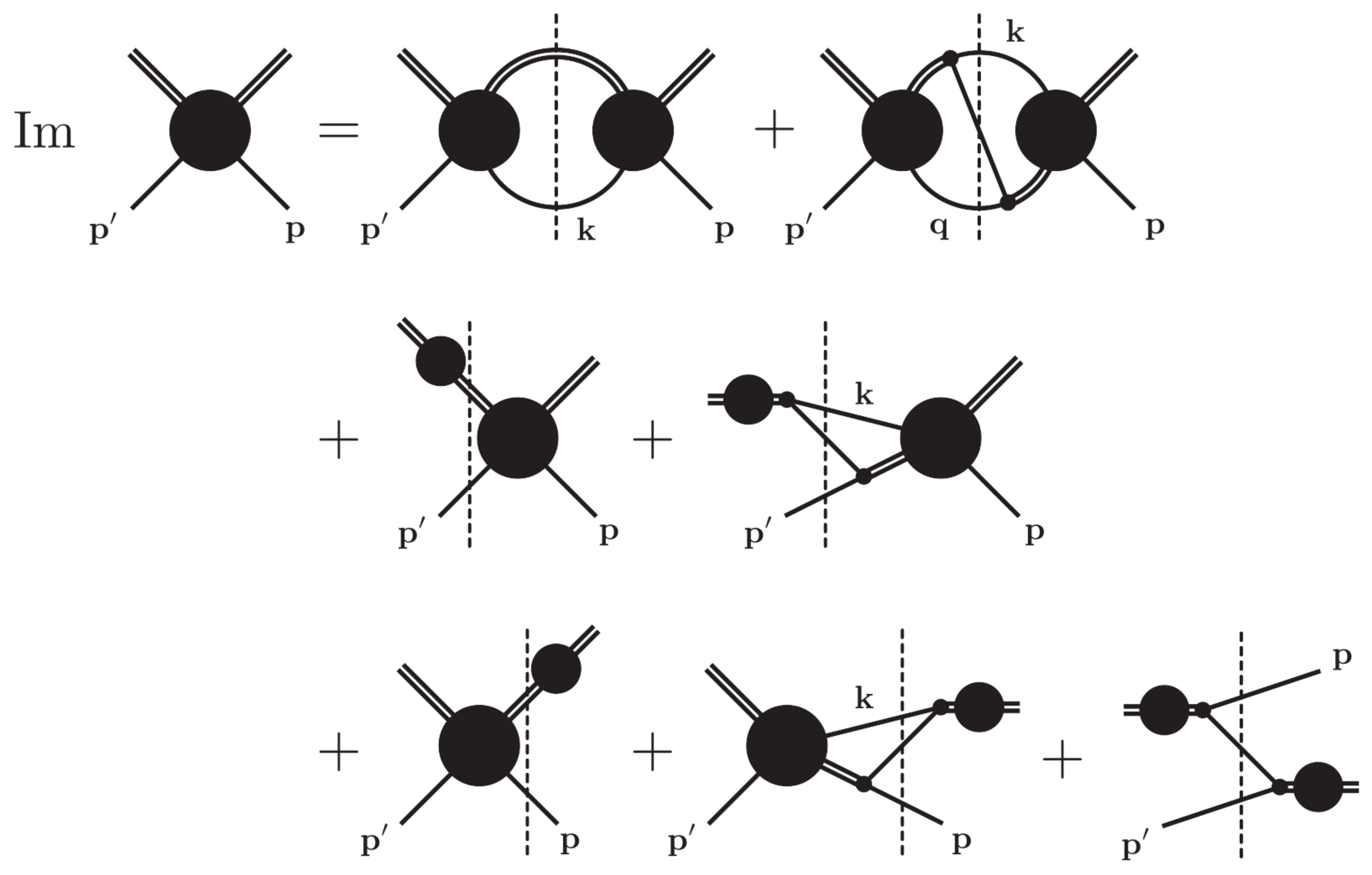
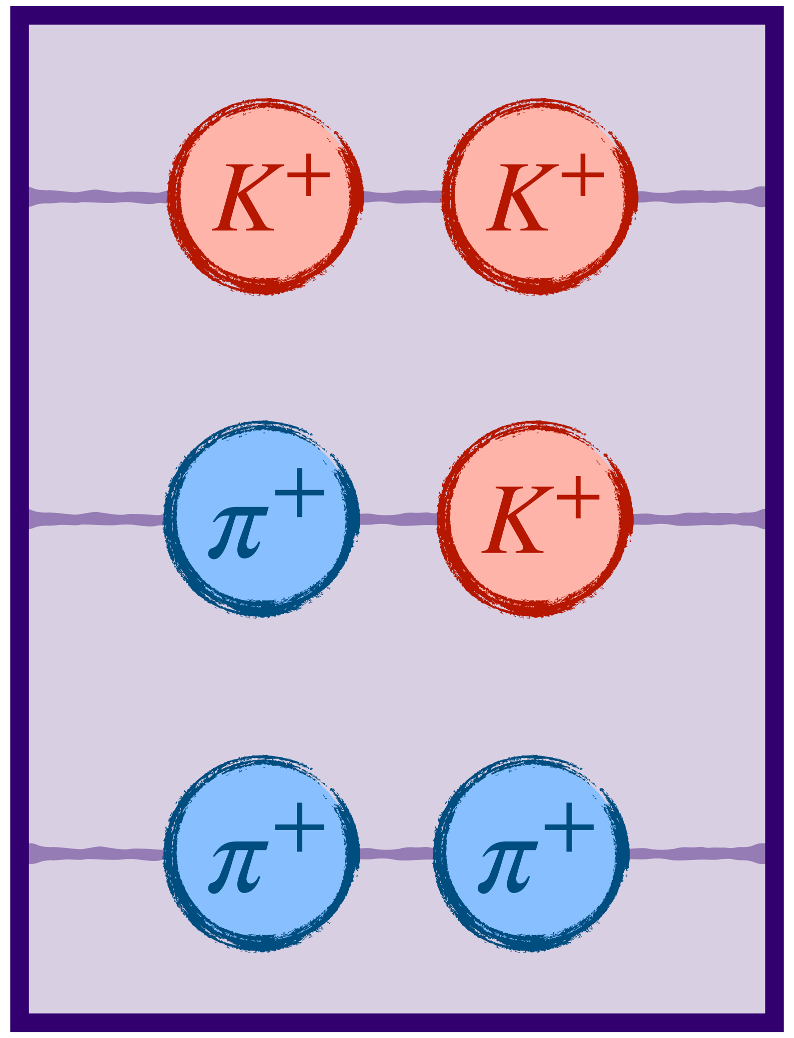


$$\widetilde{\mathcal{M}}_3 = \mathcal{B} + \int \mathcal{B}\, \mathcal{M}_2\, \rho_3\, \widetilde{\mathcal{M}}_3$$

$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i \rho_2 \mathcal{M}_2$$

Two-body reactions: pion-kaon

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner
"Two- and three-meson scattering amplitudes at physical quark masses from lattice QCD"



Light mesons at maximal isospin

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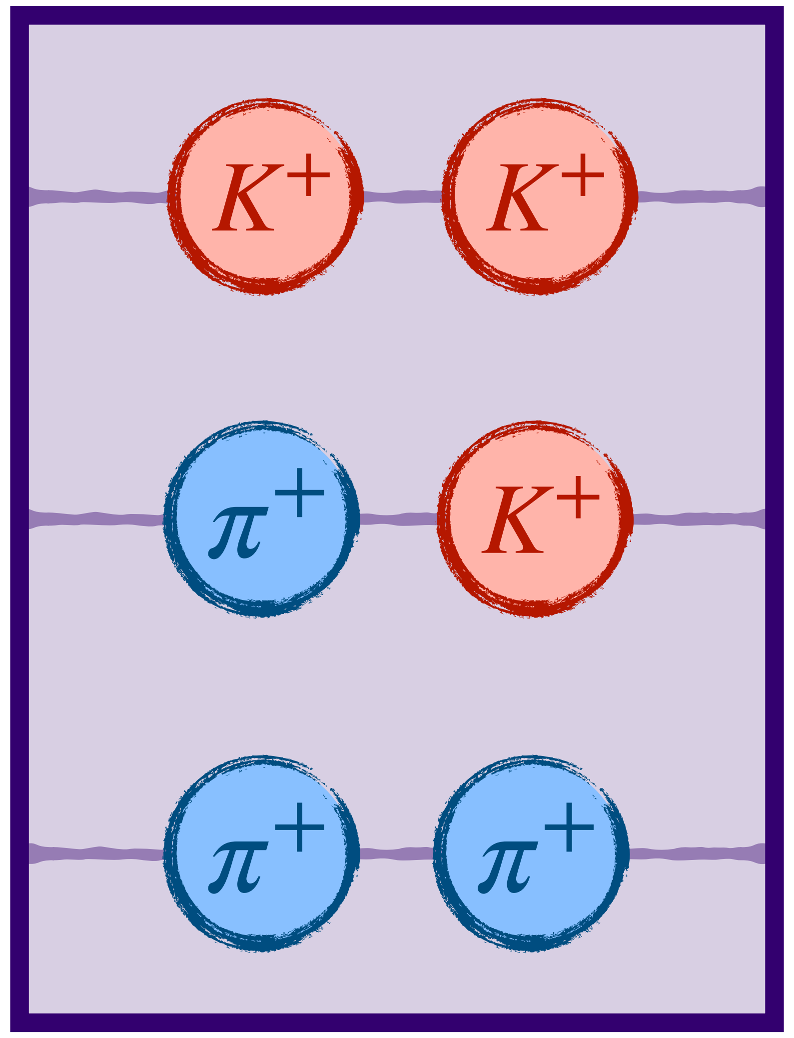

Light mesons at maximal isospin

CLS $N_f = 2+1$ "E250" ensemble

volume: $(L/a)^3 \times (T/a) = 96^3 \times 192$

masses: $M_\pi = 130$ MeV and $M_K = 500$ MeV

spacing: $a = 0.063$ fm

number of configurations: 505

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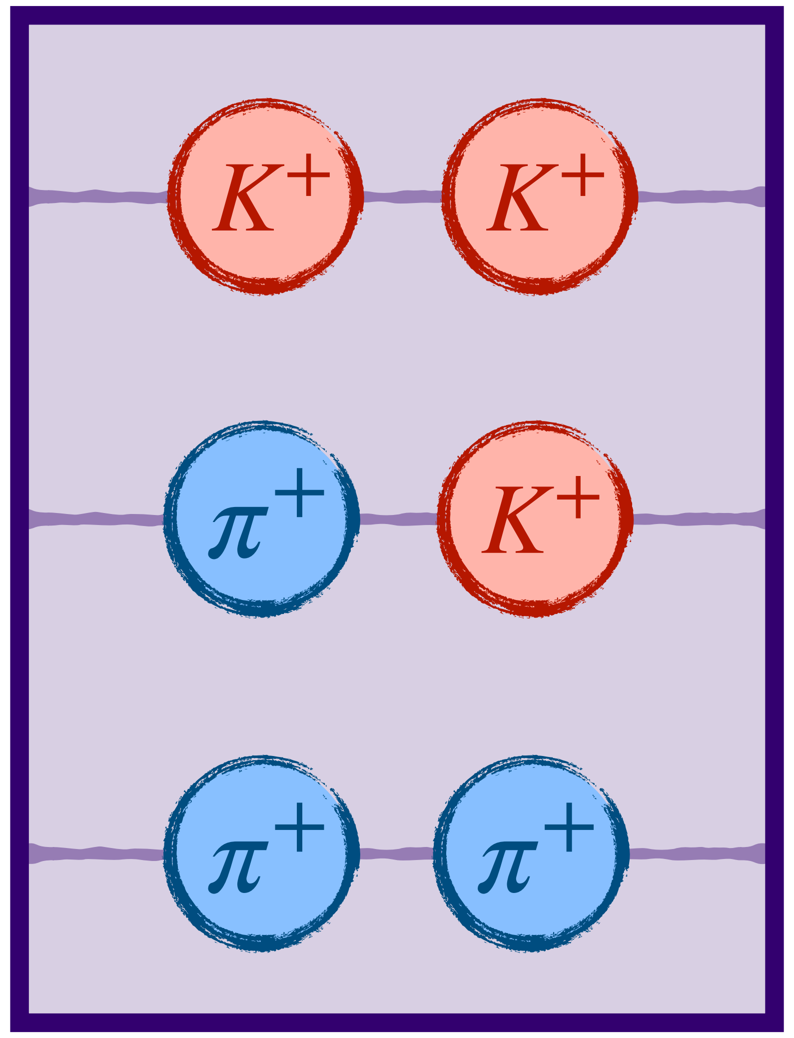

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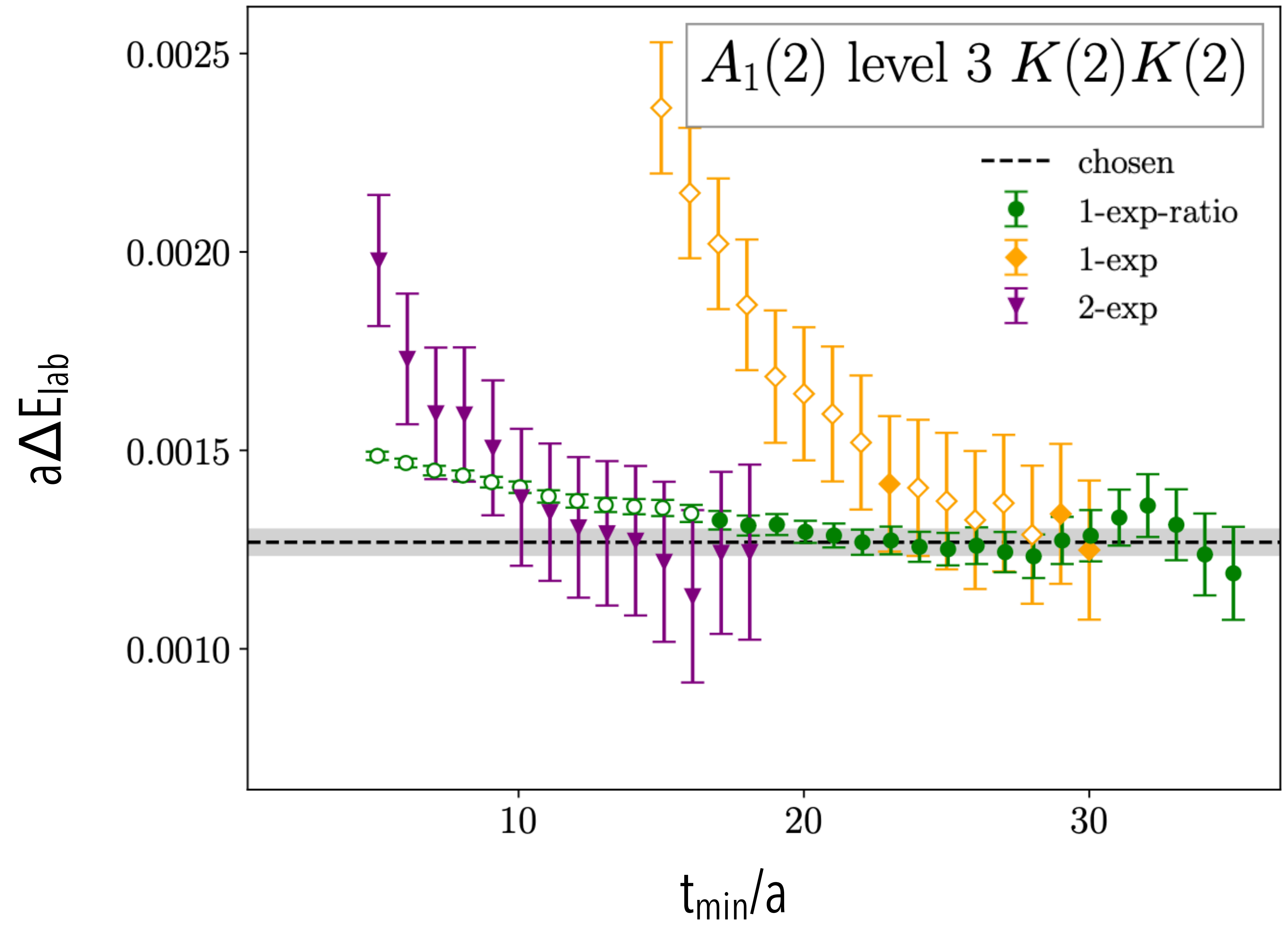
Time dependence of correlators



$$\widehat{C}(t)\, v_n(t,t_0) = \lambda_n(t,t_0)\, \widehat{C}(t_0)\, v_n(t,t_0)$$

Two-kaon spectrum

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner
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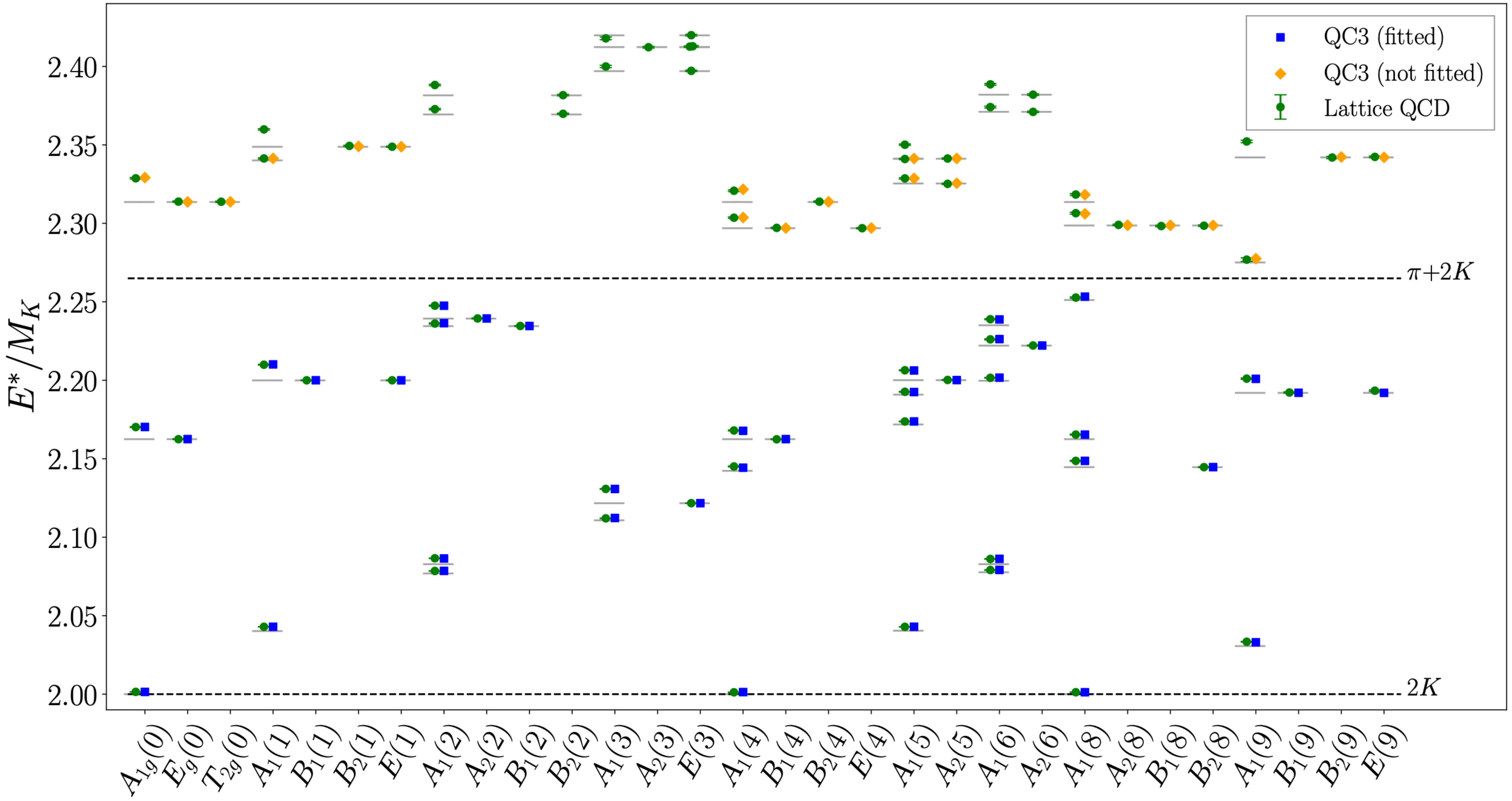
Energies constrain the amplitude via the QC

$$\frac{q}{M}\cot\delta_0(q) = \sum_{n=0}^{n<n_{\max}} b_n \left(\frac{q^2}{M^2}\right)^n$$



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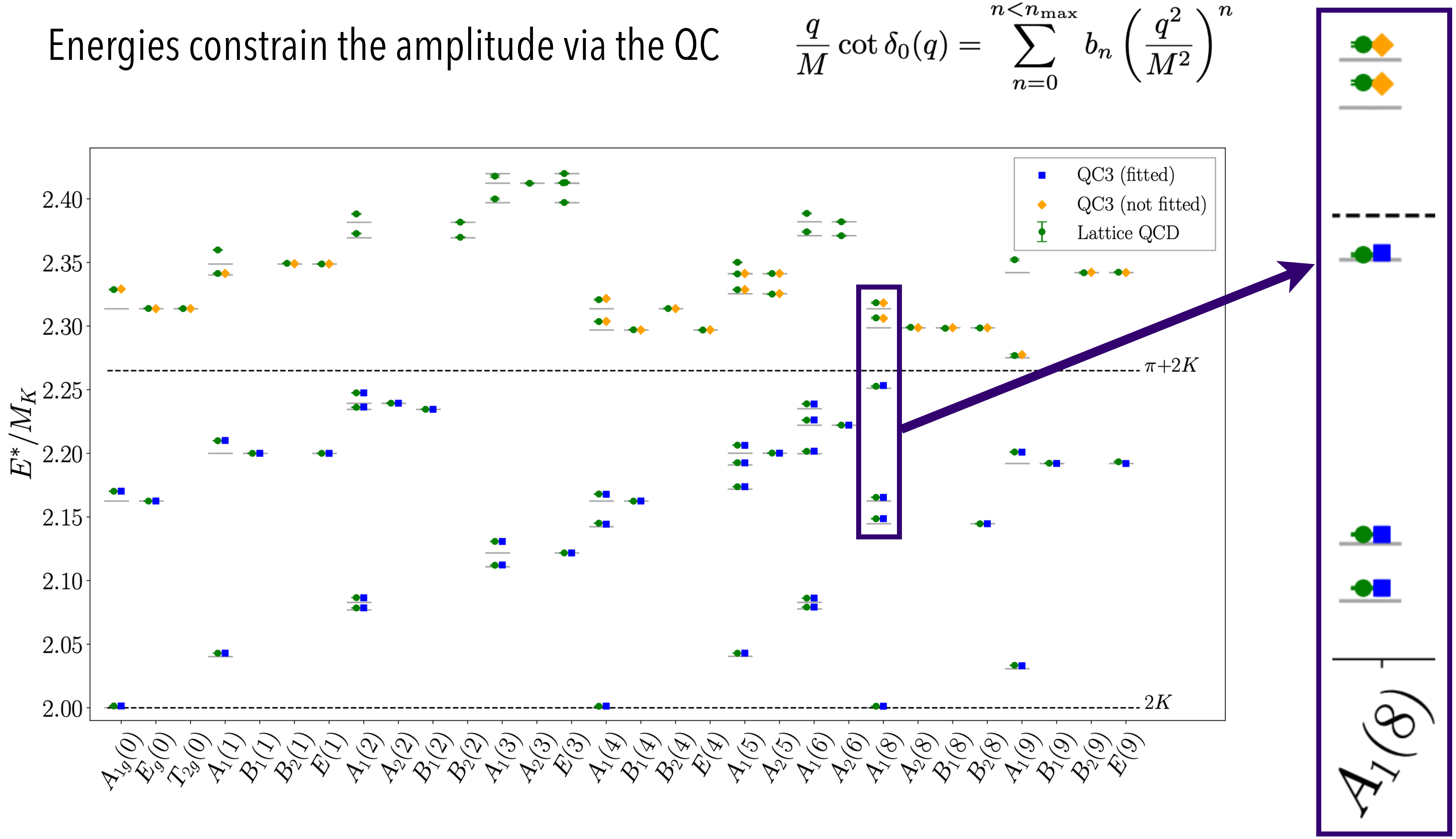

Cutoff	$2.27 M_K$
Description	s, d waves
χ^2	56.7
DOF	40-3=37
p	0.020
b_0^{KK}	$-2.621(46)$
b_1^{KK}	$0.59(20)$
b_2^{KK}	0 (fixed)
D_0^{KK}	$-0.059(30)$

Two-body amplitudes

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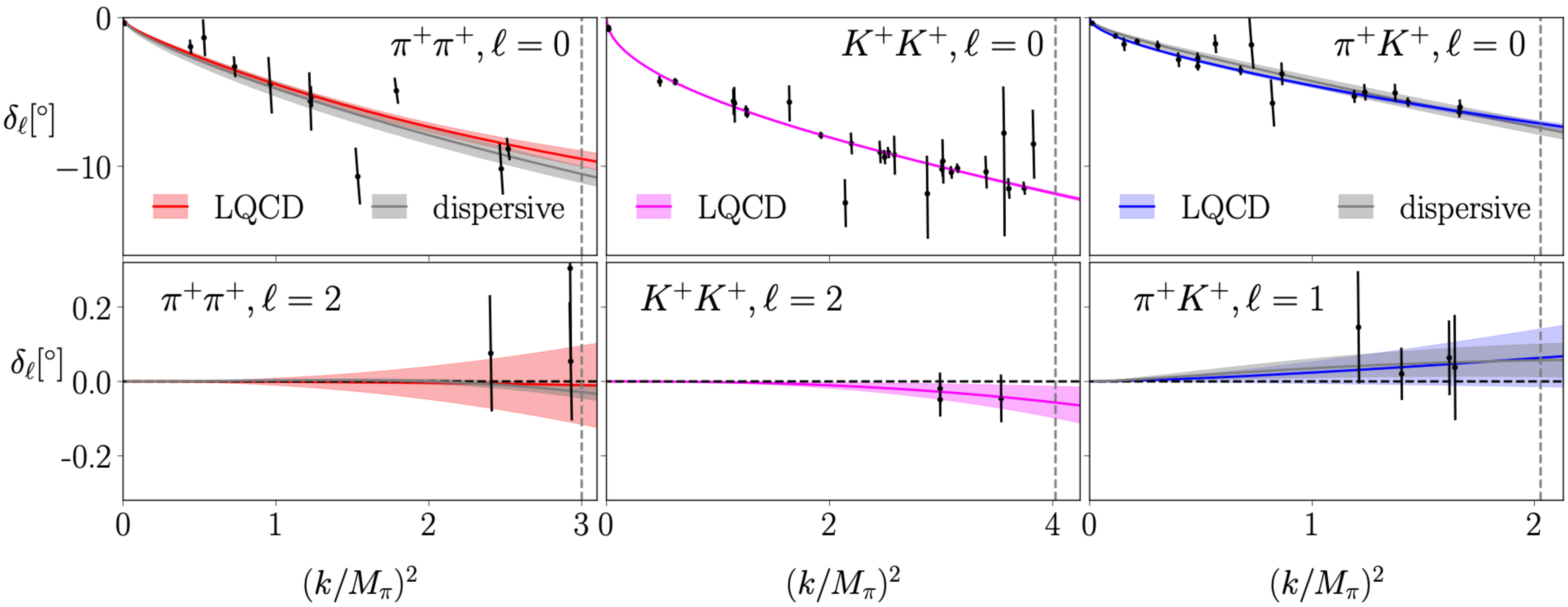

Garcia-Martin et al., *The pion–pion scattering amplitude*

Pealez & Rodas, *Dispersive πK–πK and ππ–KK amplitudes from scattering data*

Two-body scattering amplitudes at "physical" quark masses!

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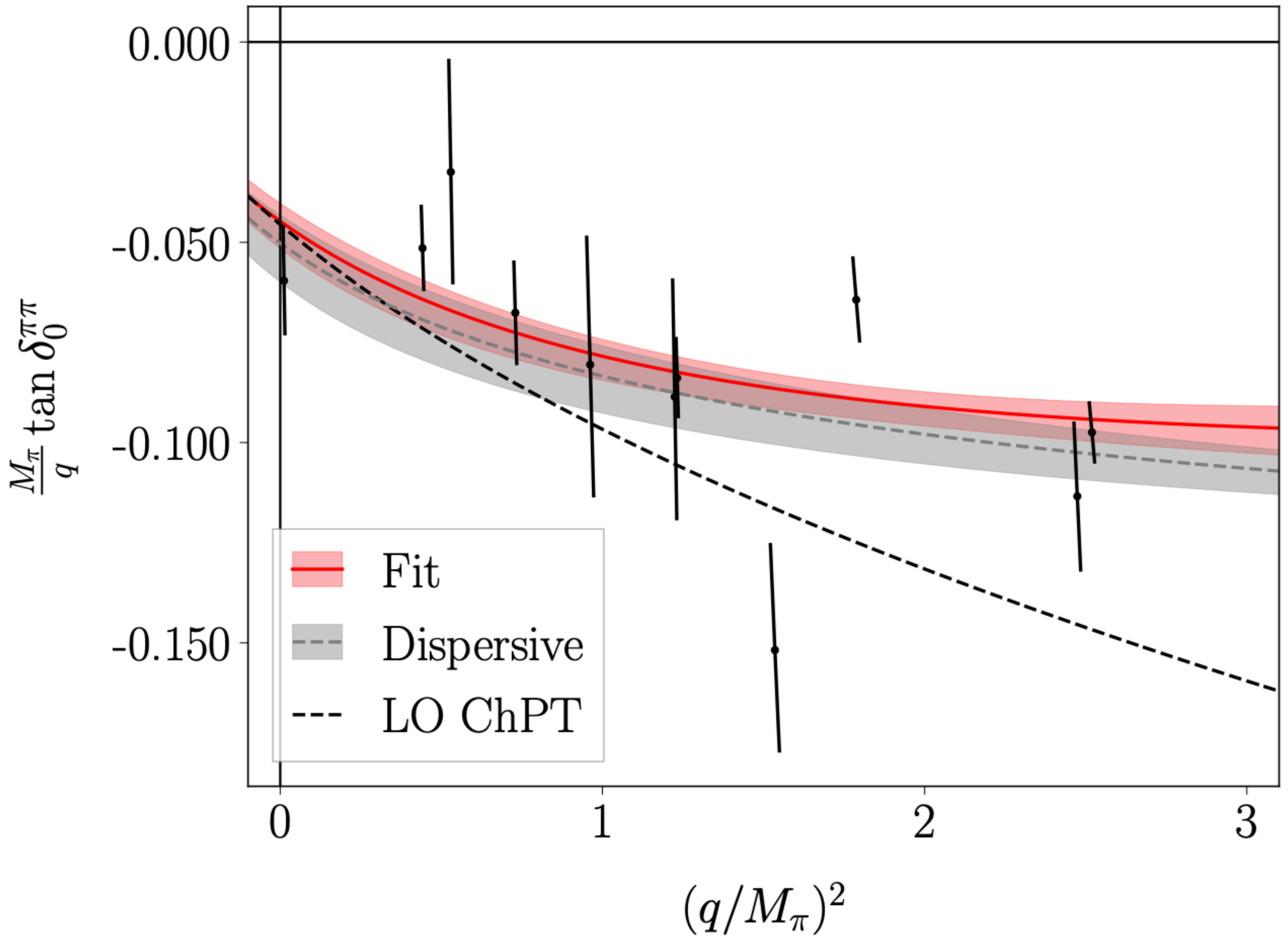
$\pi^+\pi^+$ phase shift



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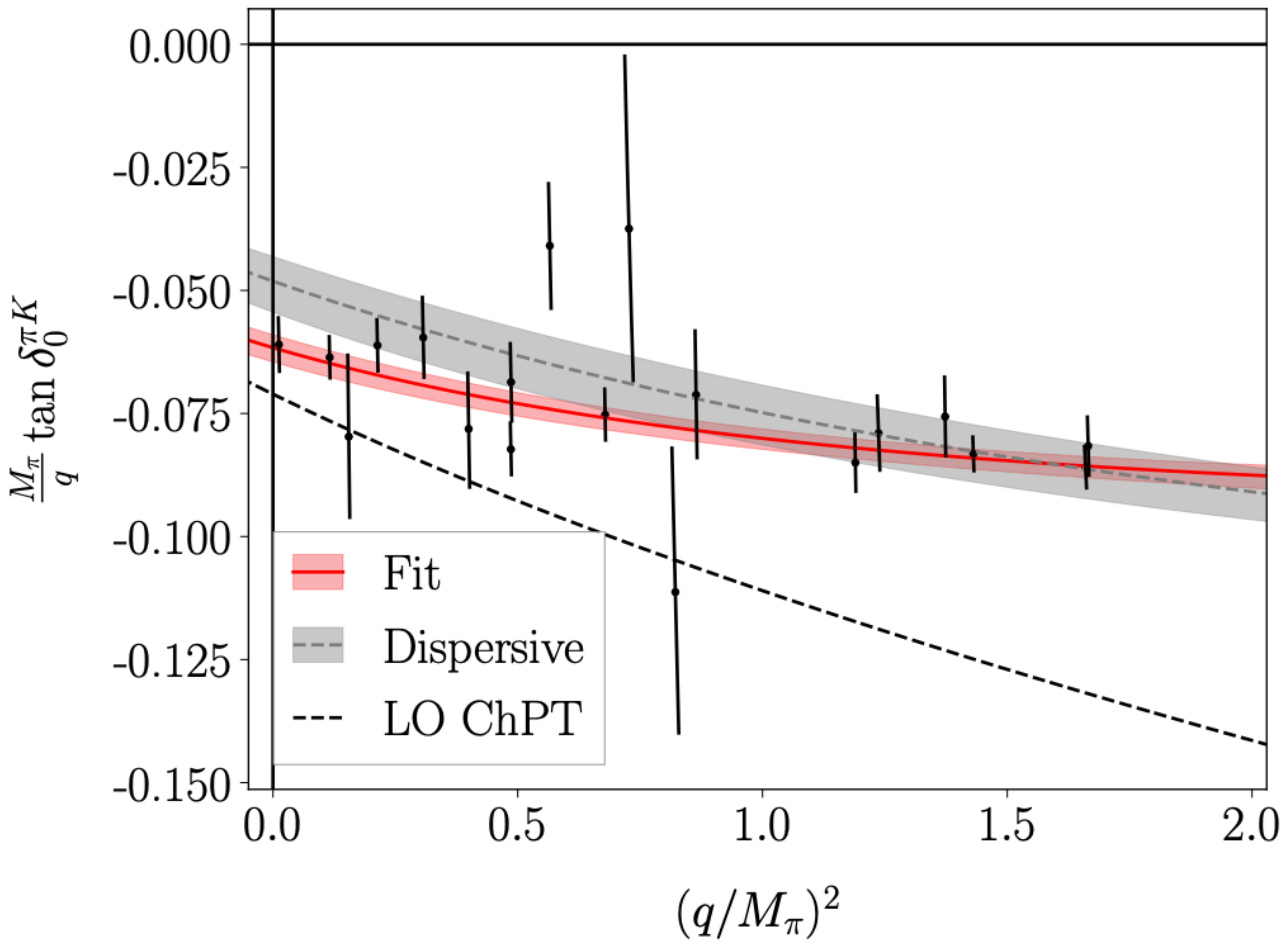

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Chiral extrapolation

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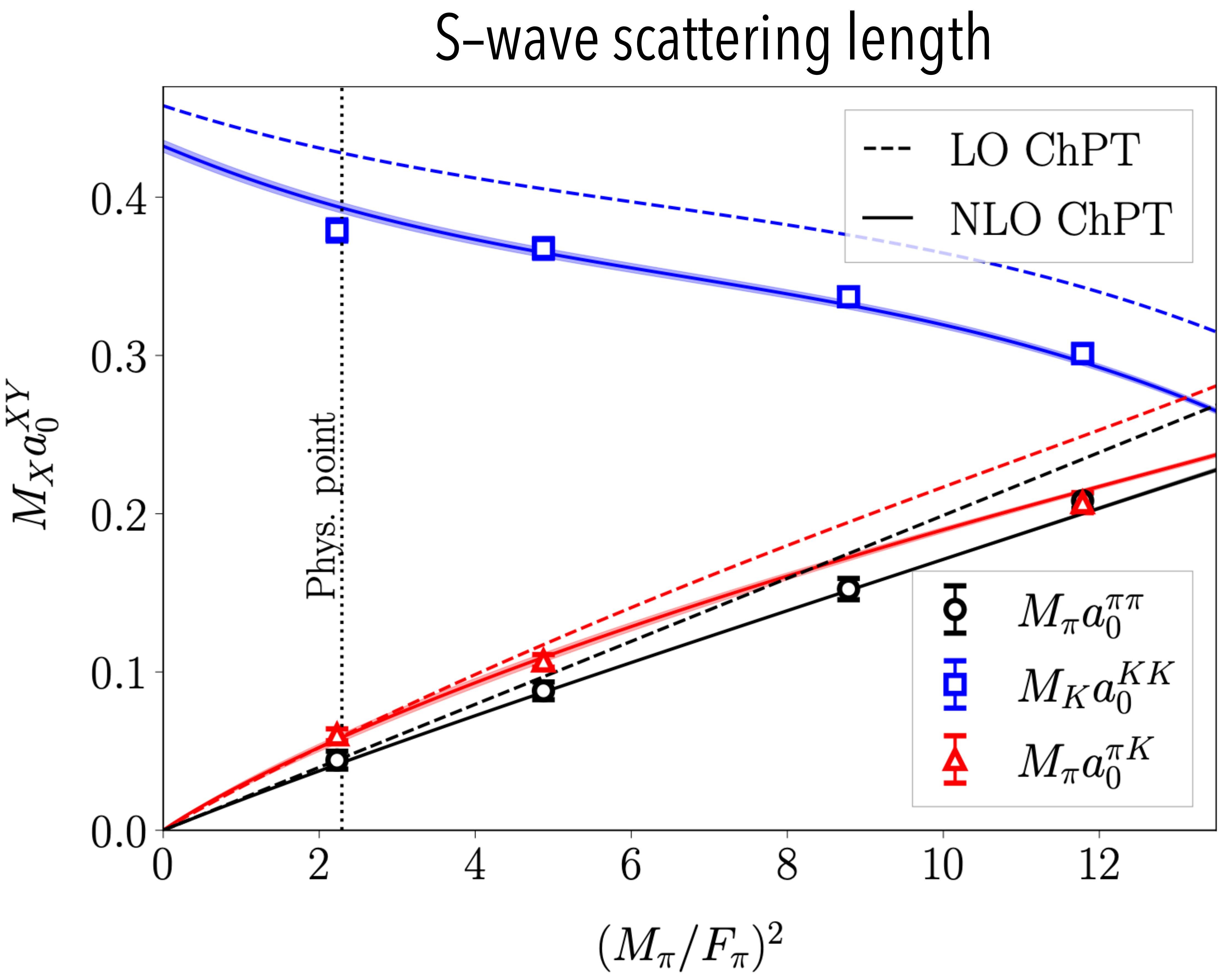
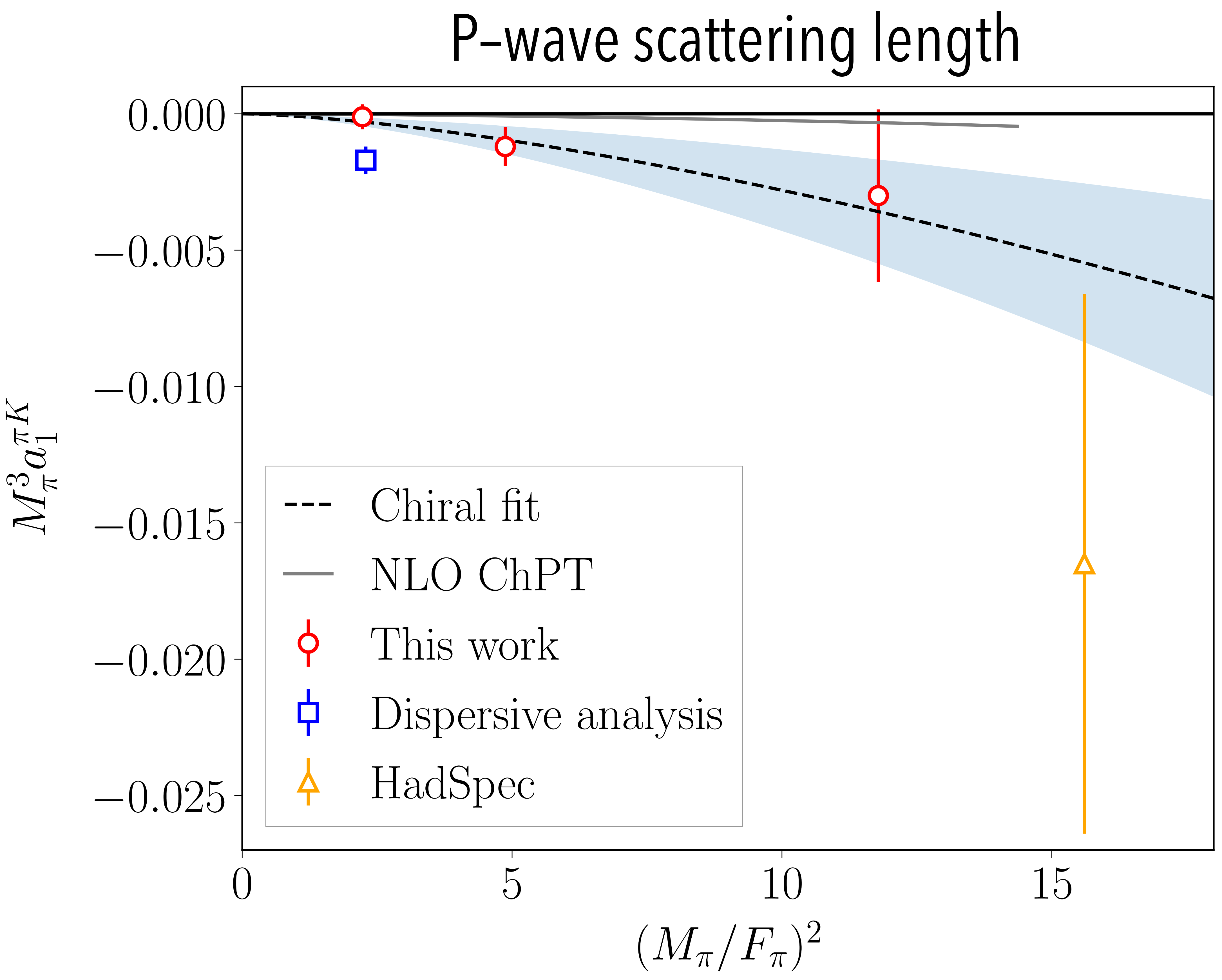
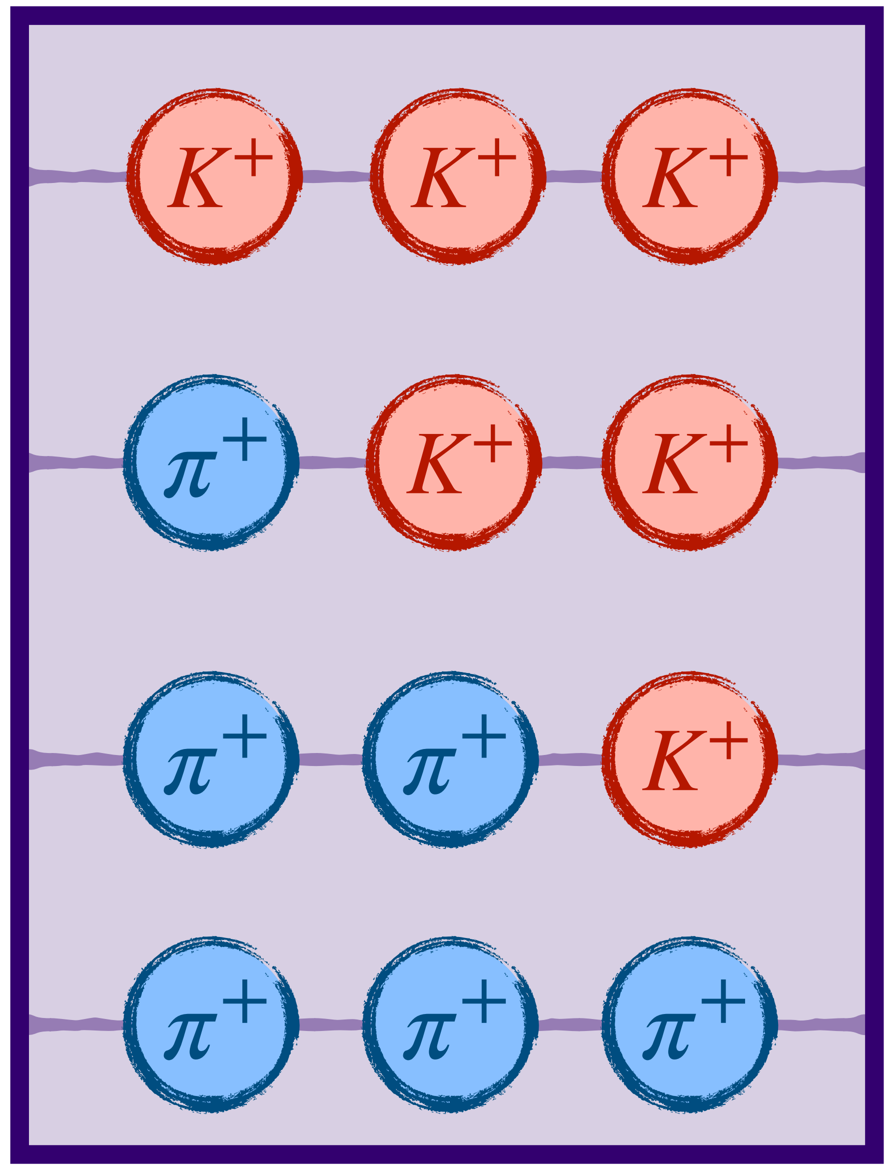




Low-energy scattering parameters as functions of quark masses

Reactions of three light mesons

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner
"Two- and three-meson scattering amplitudes at physical quark masses from lattice QCD"



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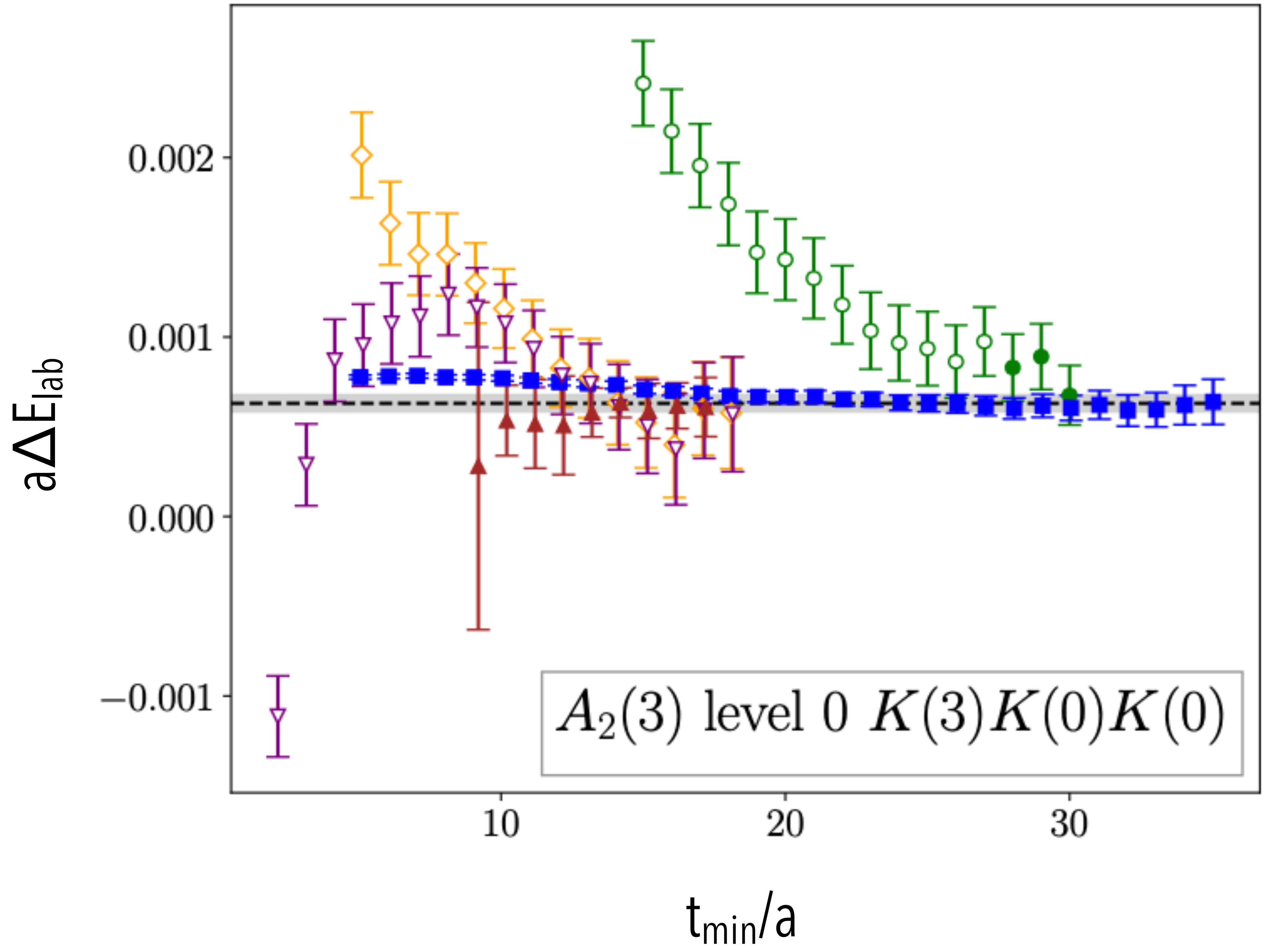

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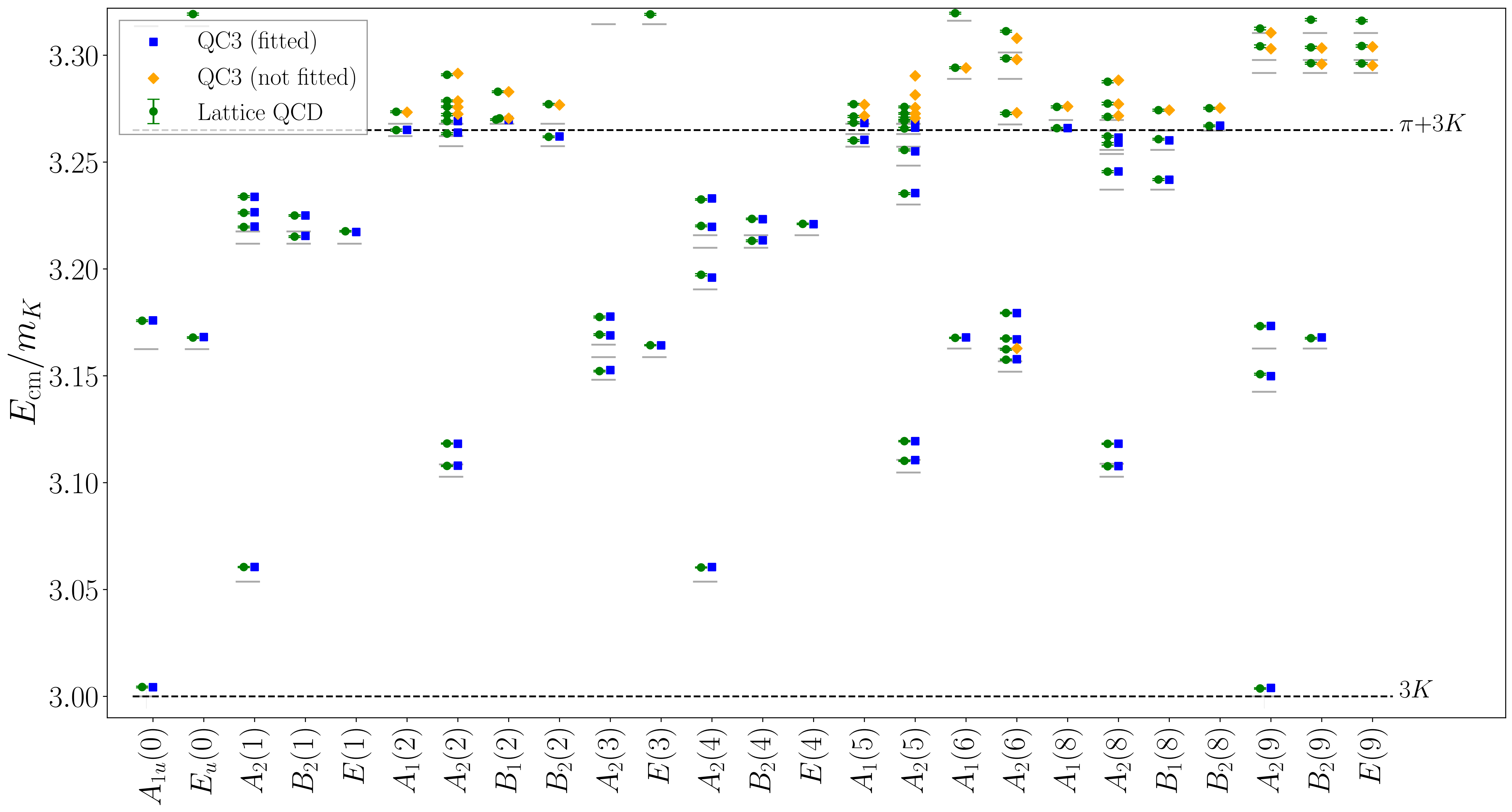
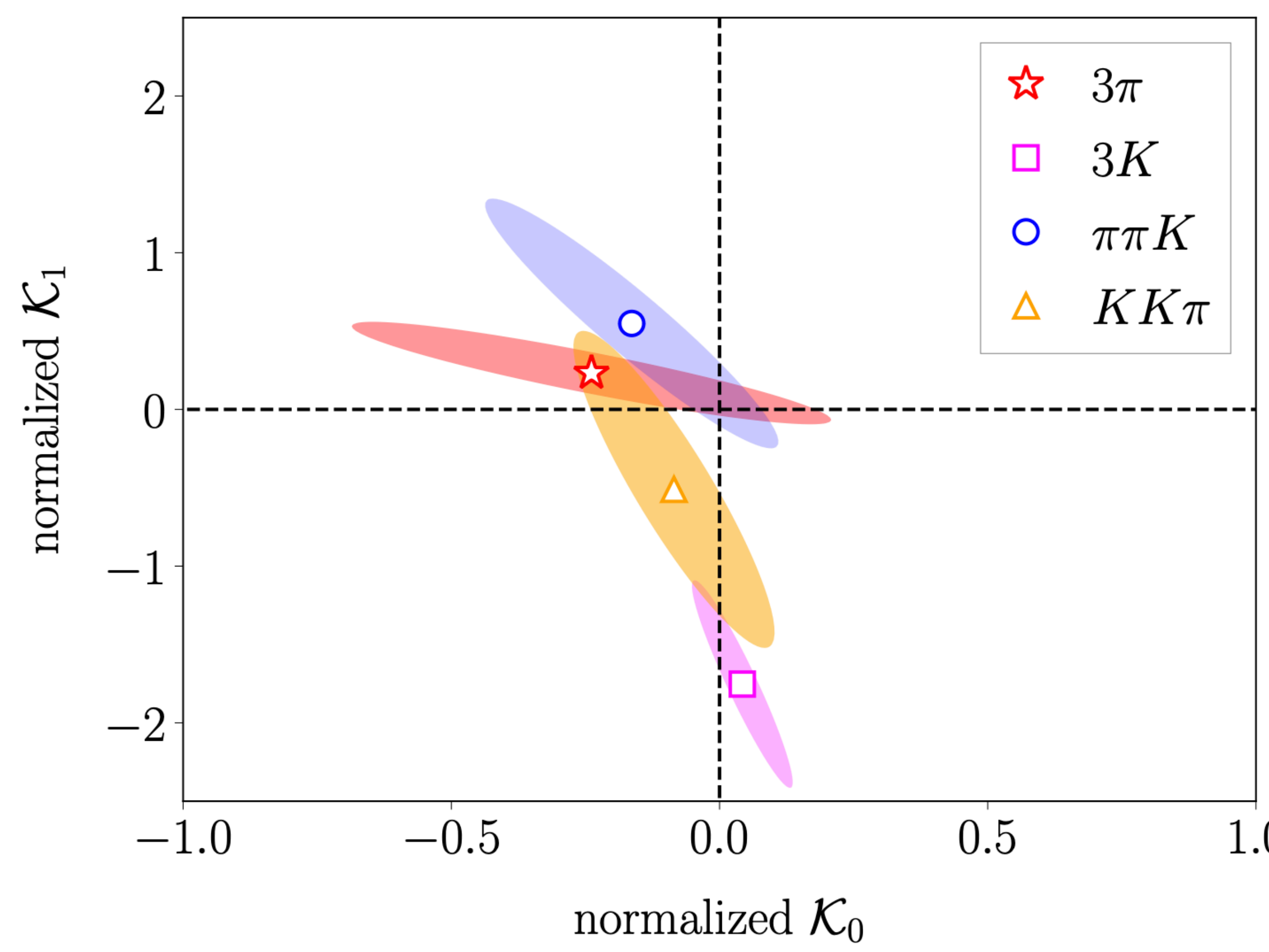
Energies constrain the three-body K matrix via the QC

$$\det_{k,\ell,m_\ell}\left[\mathcal{K}_3^{-1}(E) + F_3(E,\boldsymbol{P},L)\right] = 0$$



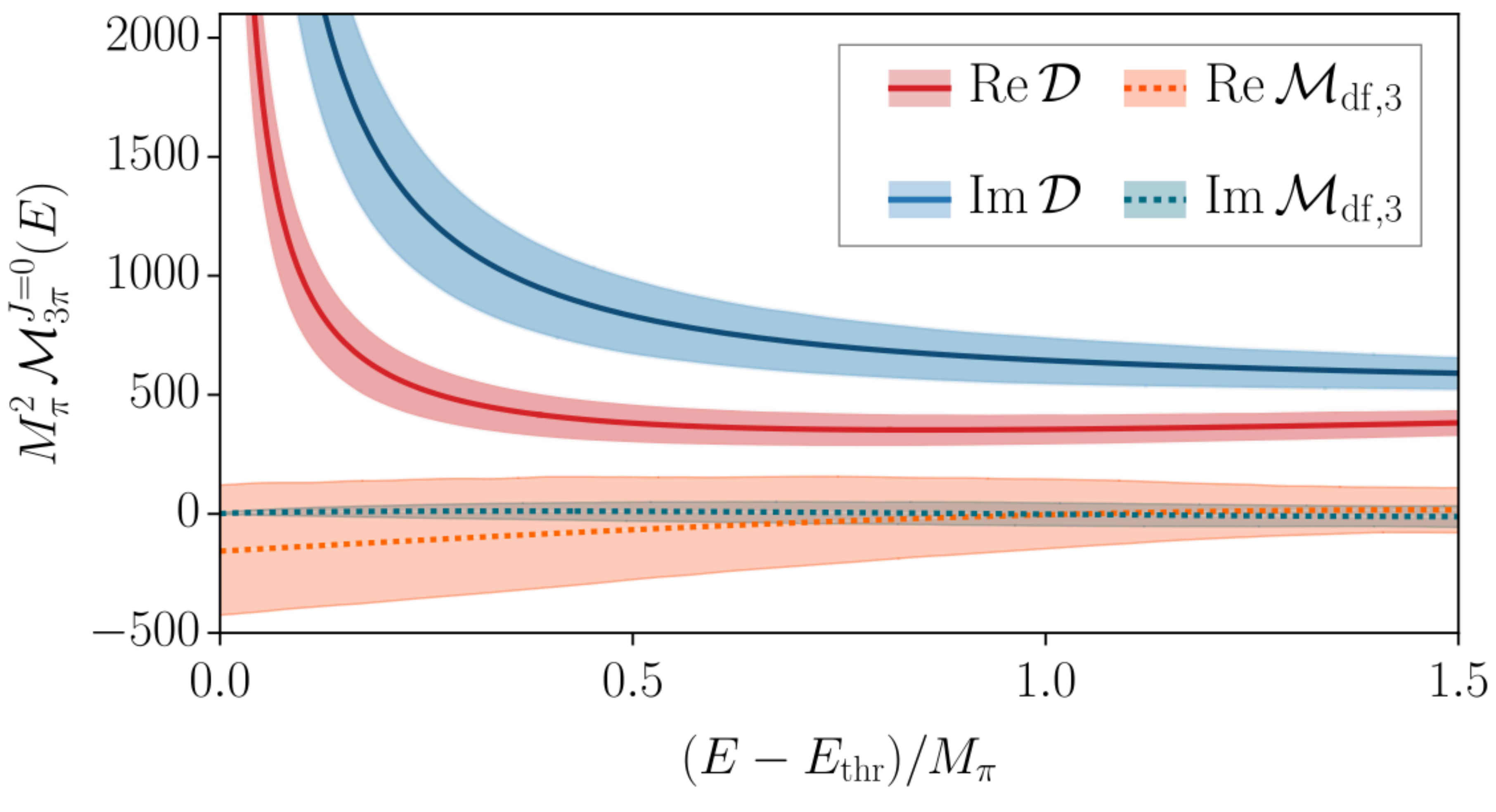
Threshold expansion of the three-body forces

$$\mathcal{K}_3 = \mathcal{K}_0 + \mathcal{K}_1\Delta + \ldots \qquad \Delta = \frac{s - E_{\mathrm{th}}^2}{E_{\mathrm{th}}^2}$$

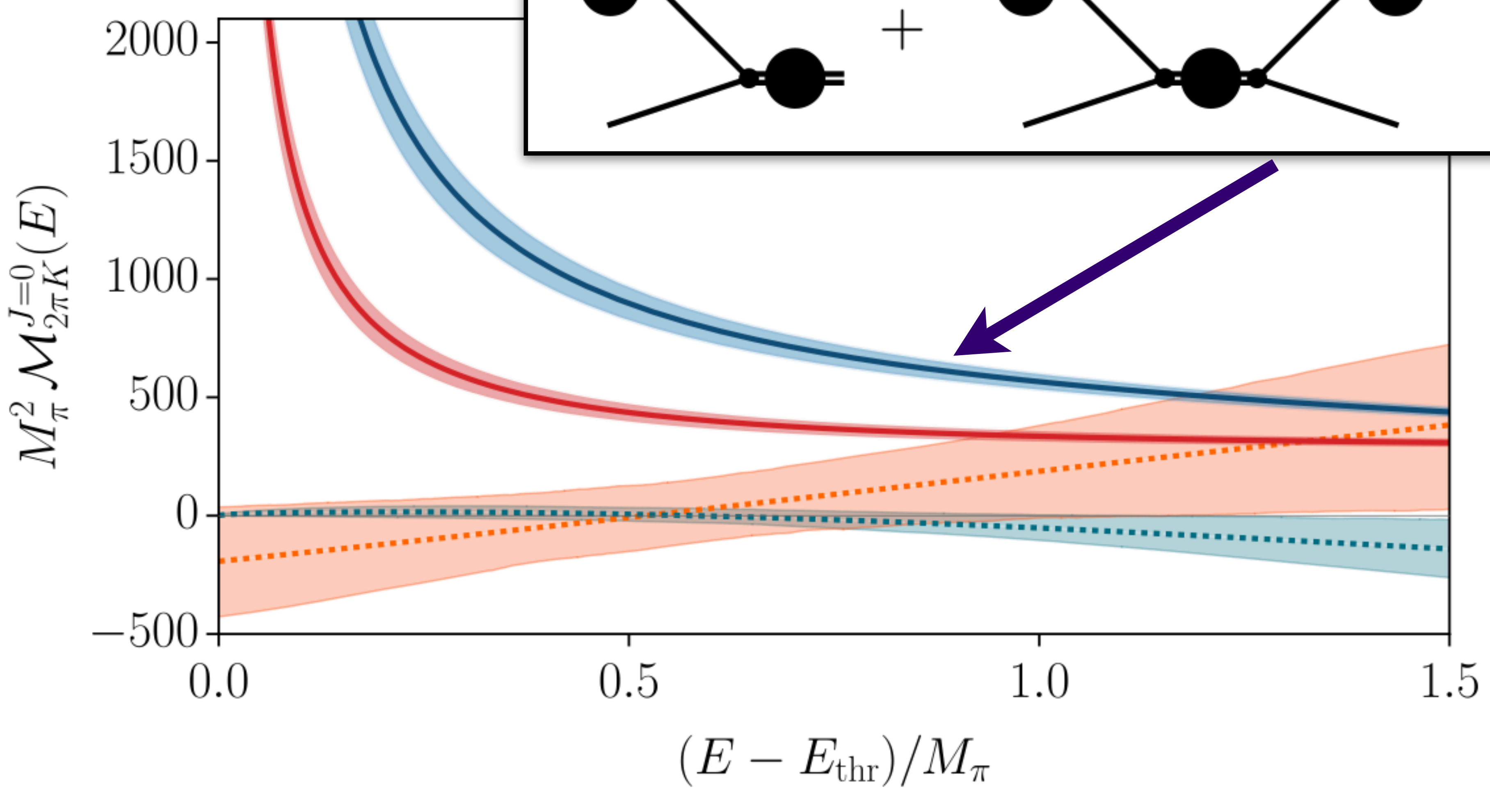


Three-meson amplitudes

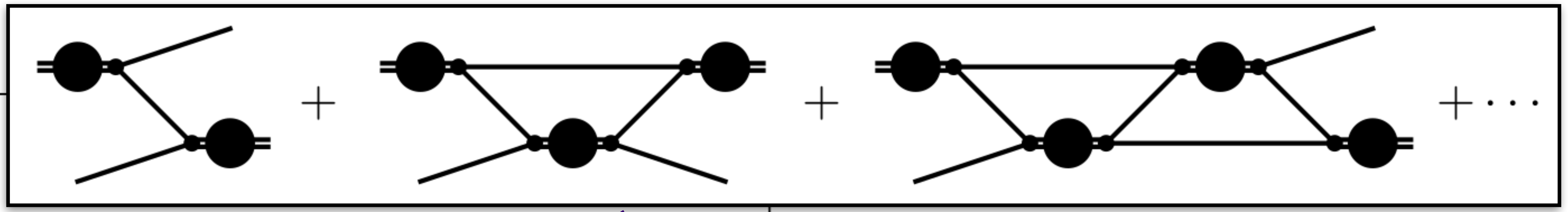
Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner
"Two- and three-meson scattering amplitudes at physical quark masses from lattice QCD"



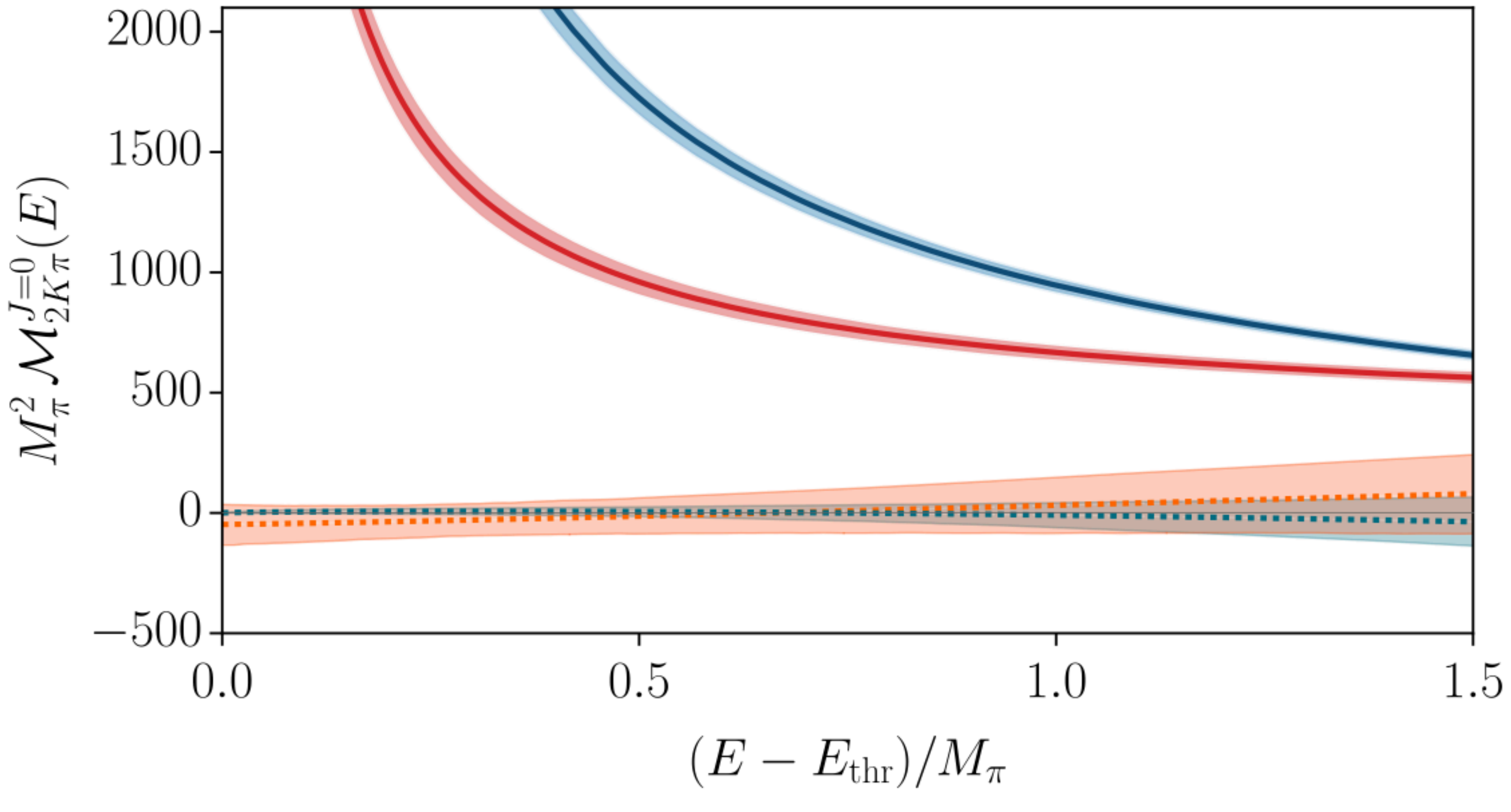
(a) 3π amplitude



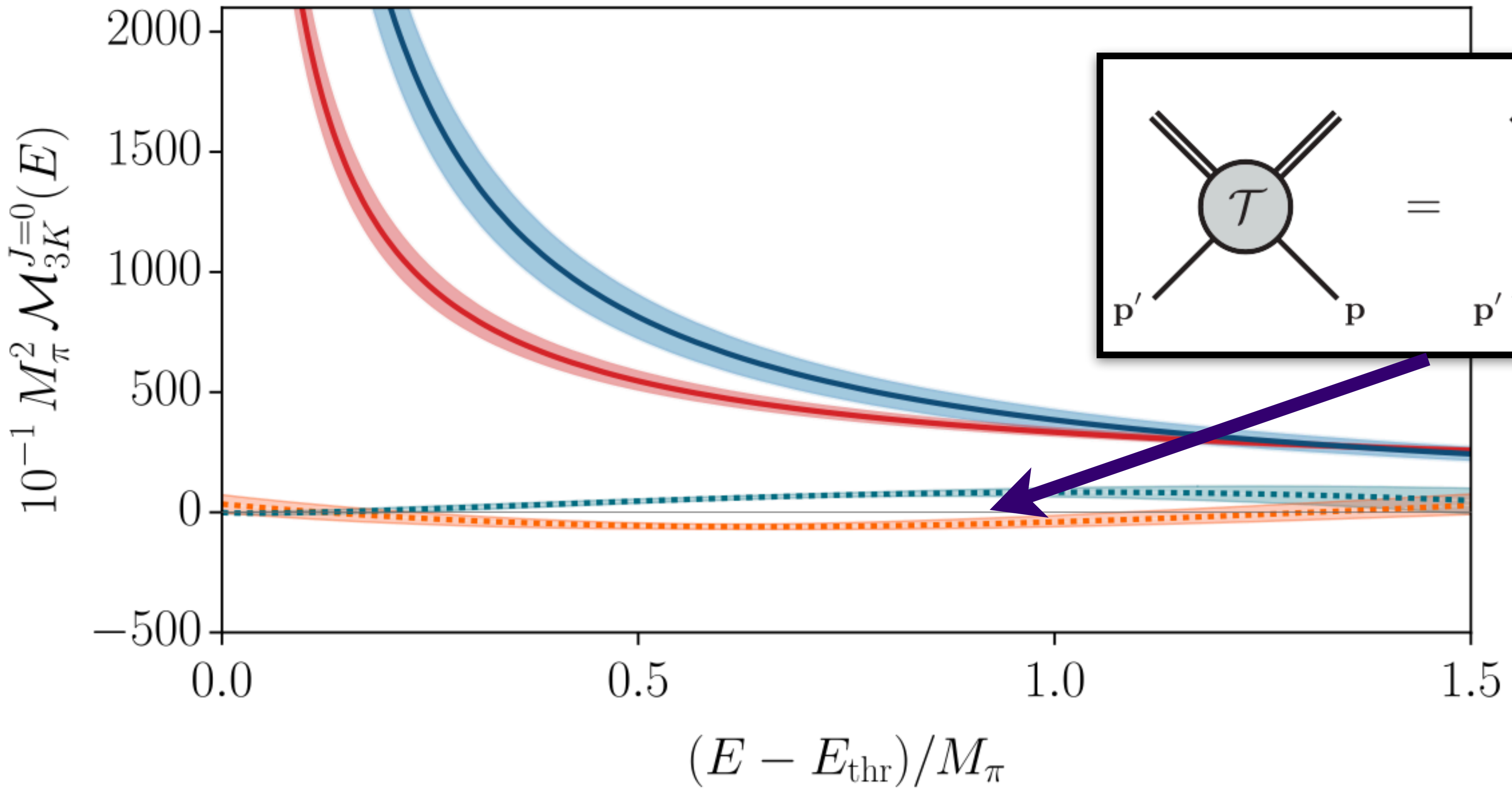
(b) $\pi\pi K$ amplitude



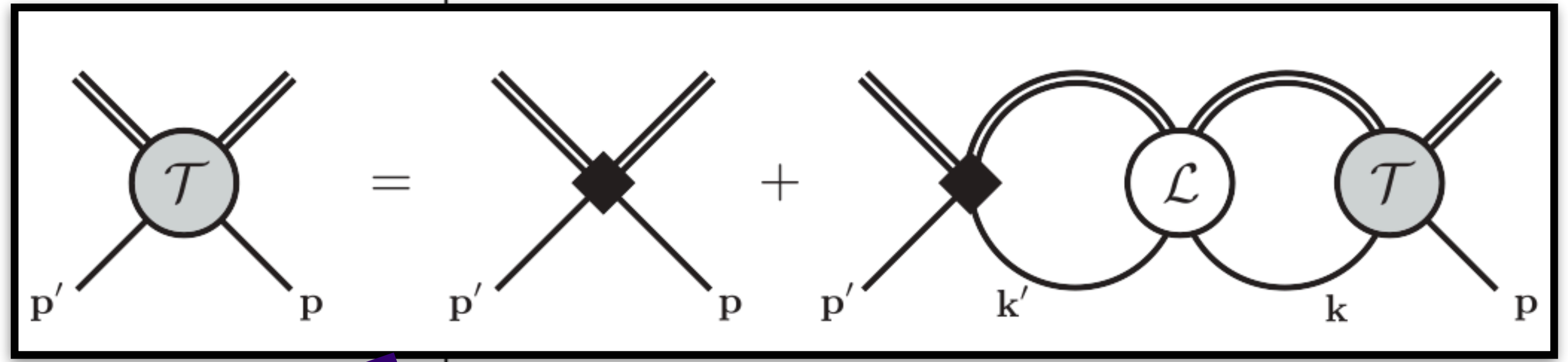
Ladder amplitude
(pairwise interactions)



(c) $KK\pi$ amplitude



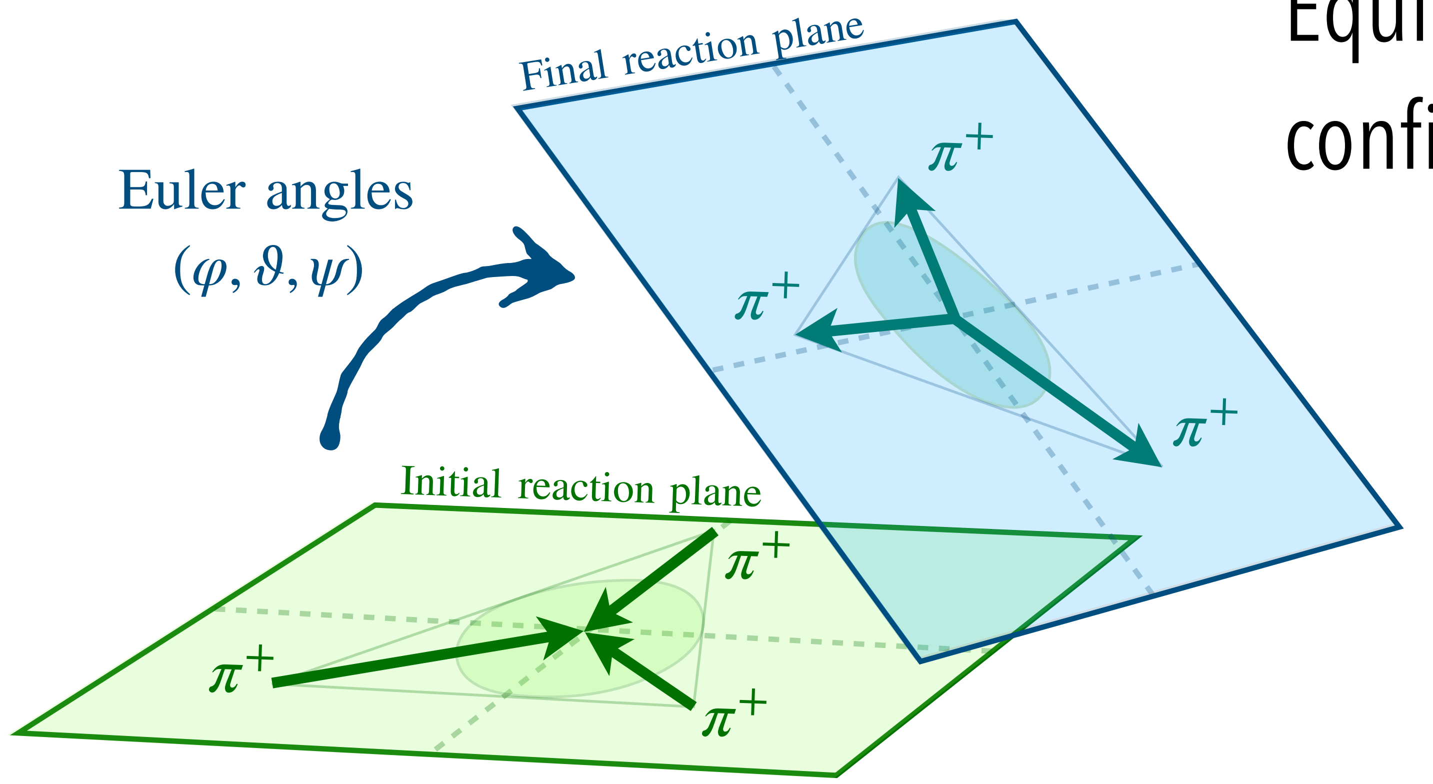
(d) $3K$ amplitude



Short−range amplitude
(contact interactions)

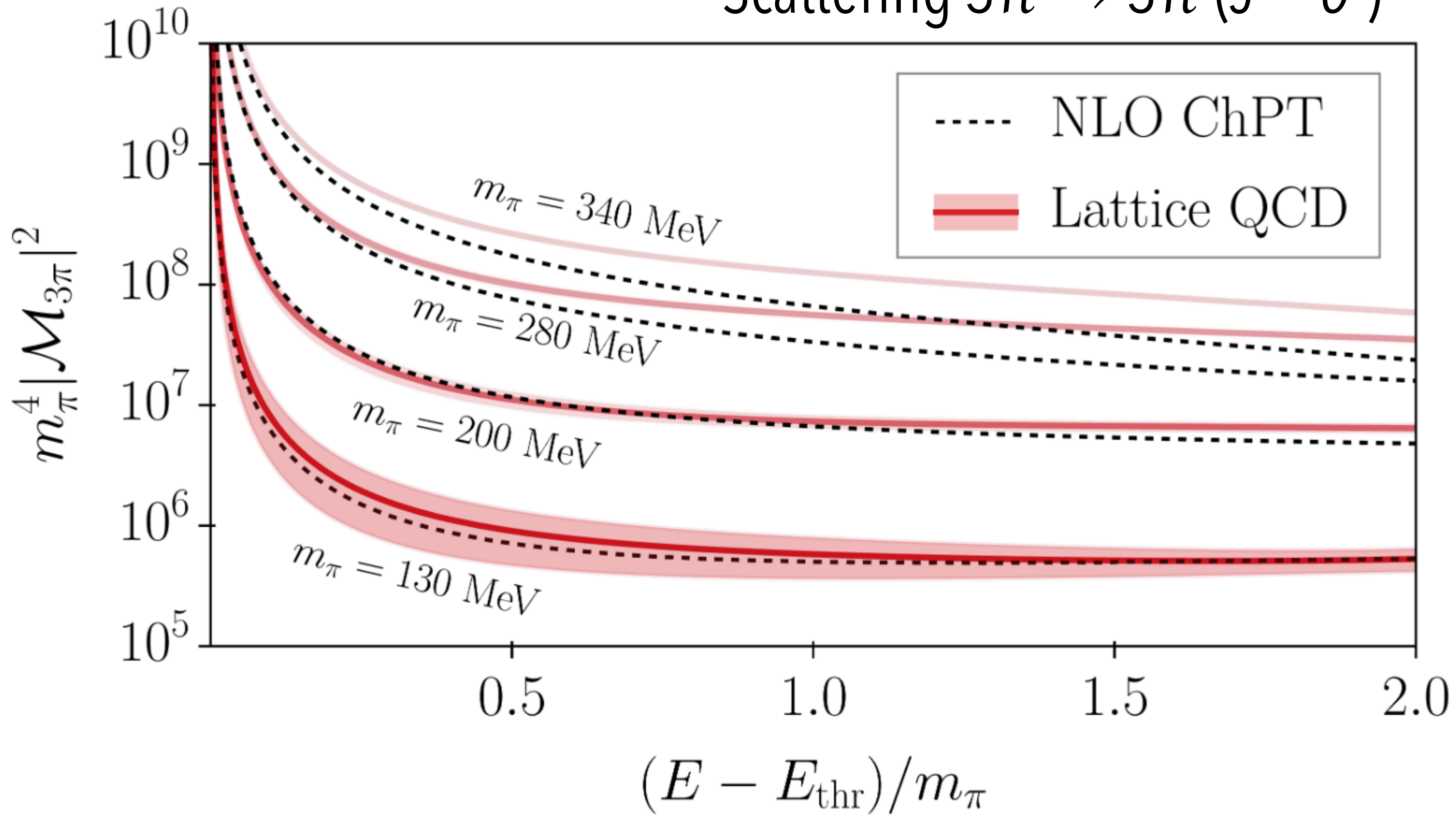
Three-meson amplitudes

Dawid, Draper, Hanlon, Hörz, Morningstar, Romero-López, Sharpe, Skinner
“Two- and three-meson scattering amplitudes at physical quark masses from lattice QCD”

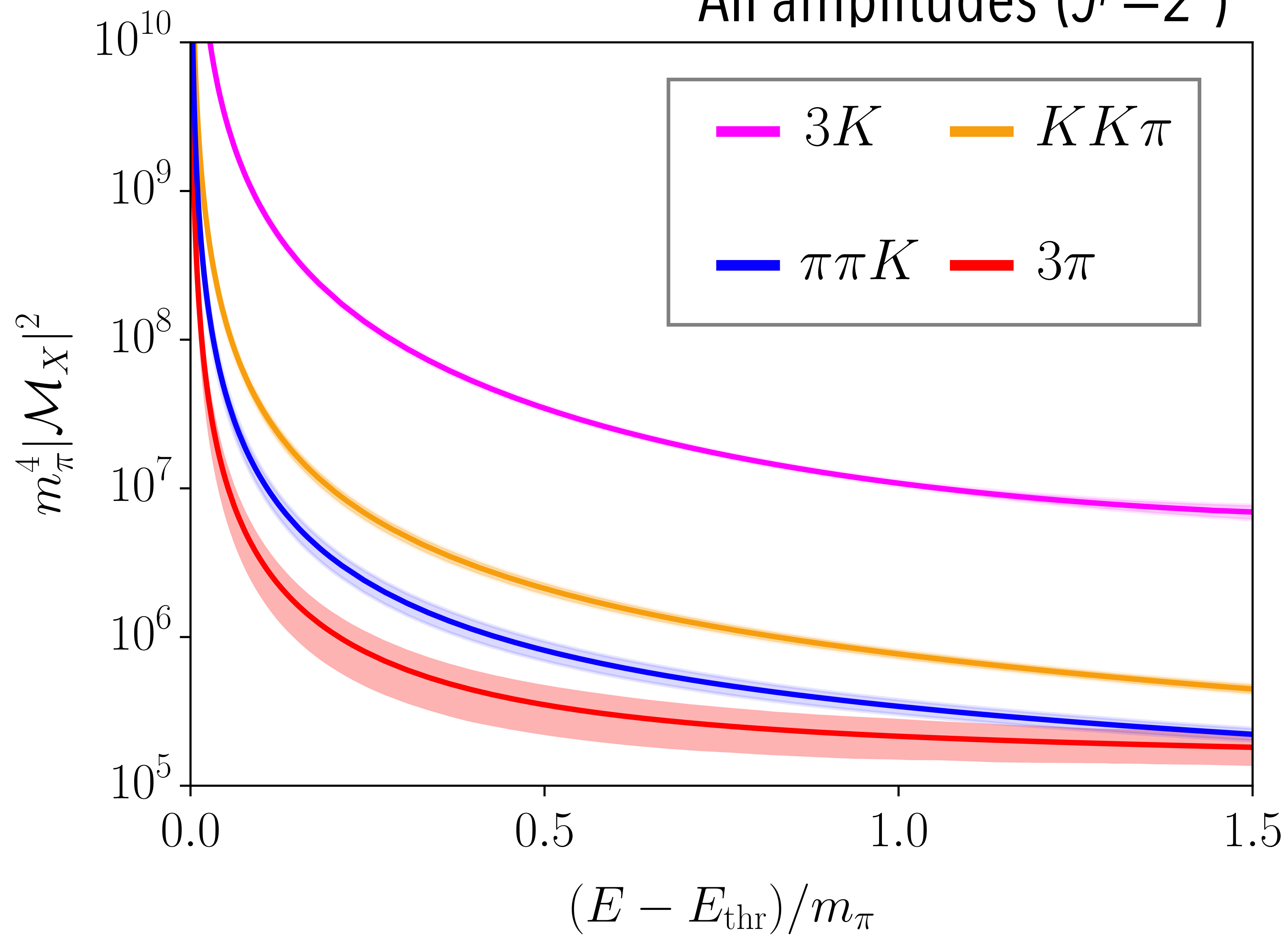


Equilateral triangle configuration of momenta

Scattering $3\pi \to 3\pi$ (J^P=0^-)

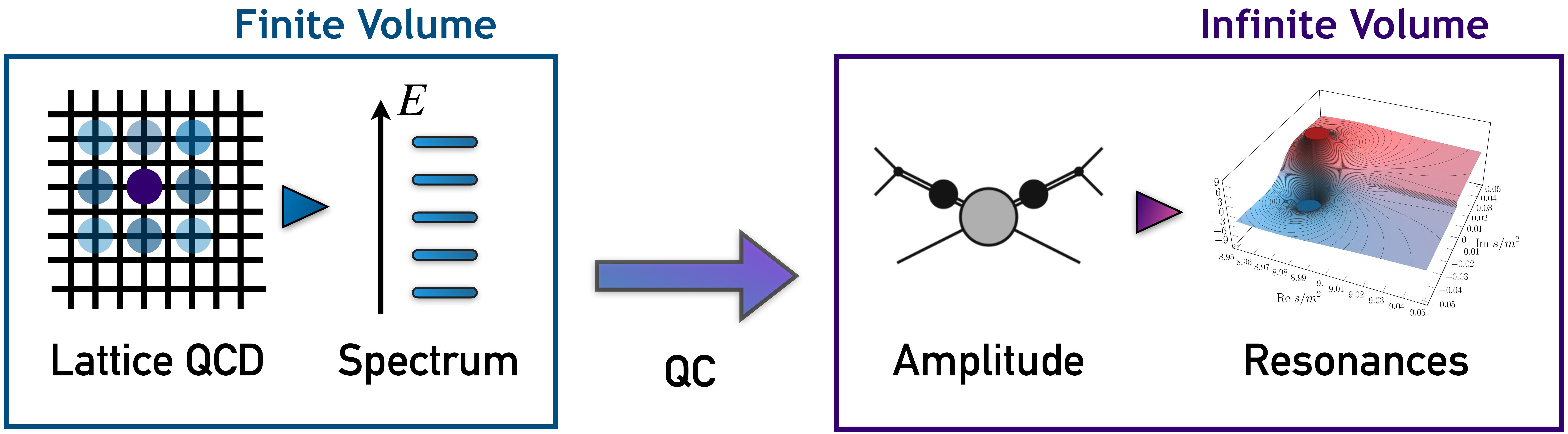


All amplitudes (J^P=2^-)



Solution of the **three-body integral equations**:
– non-zero partial waves in two-body sub channels
– non-zero total angular momentum
– non-trivial three-body forces
– physical quark masses & chiral extrapolation

Summary

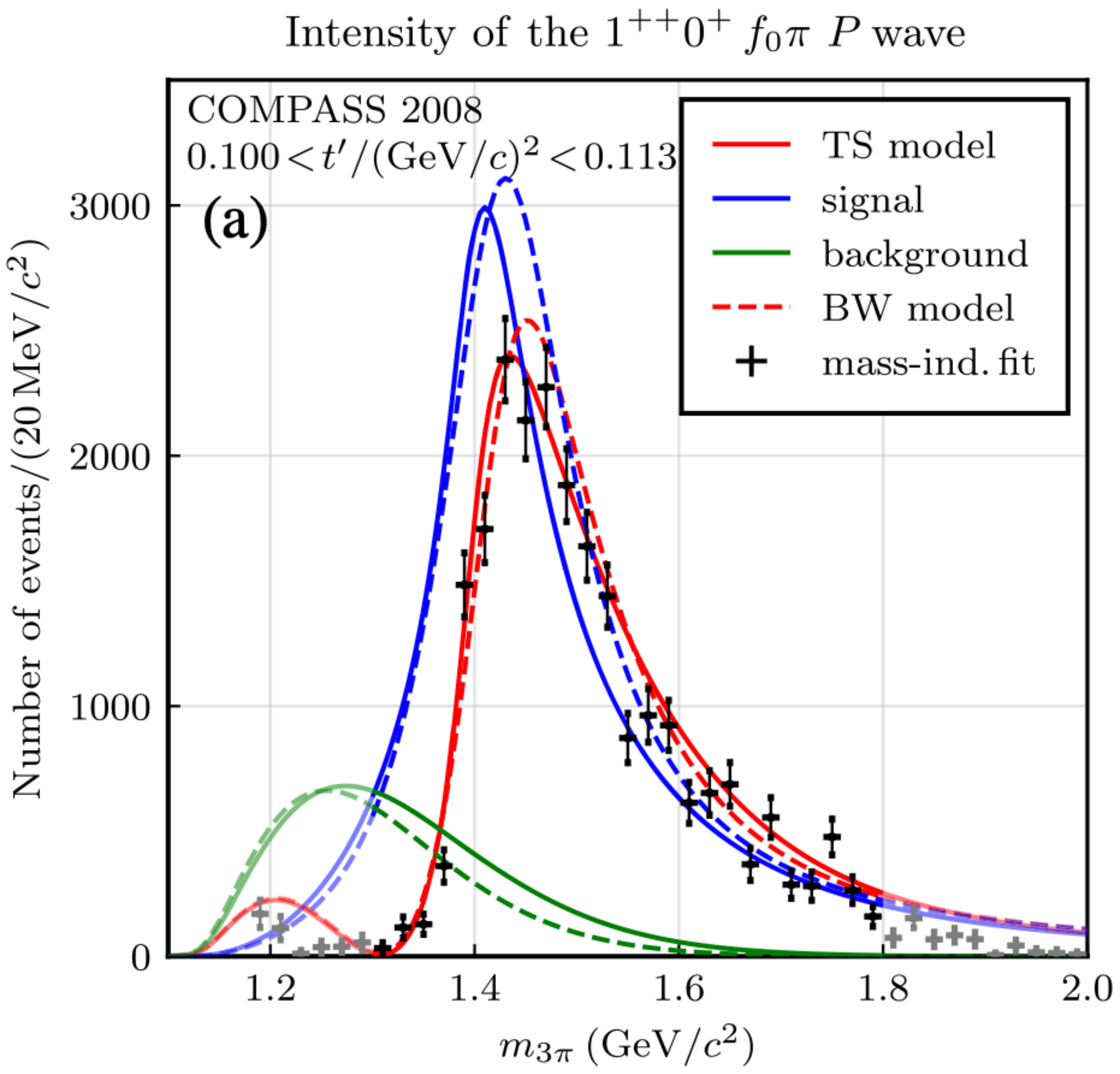


Toward determining hadronic resonances from LQCD

- ☑ Considerable progress in the three-body finite-volume formalisms
- ☑ Chiral dependence of the two-body scattering parameters
- ☑ Evidence for non-zero three-body force in 3K system
- ☑ Three-meson amplitudes at the physical pion mass

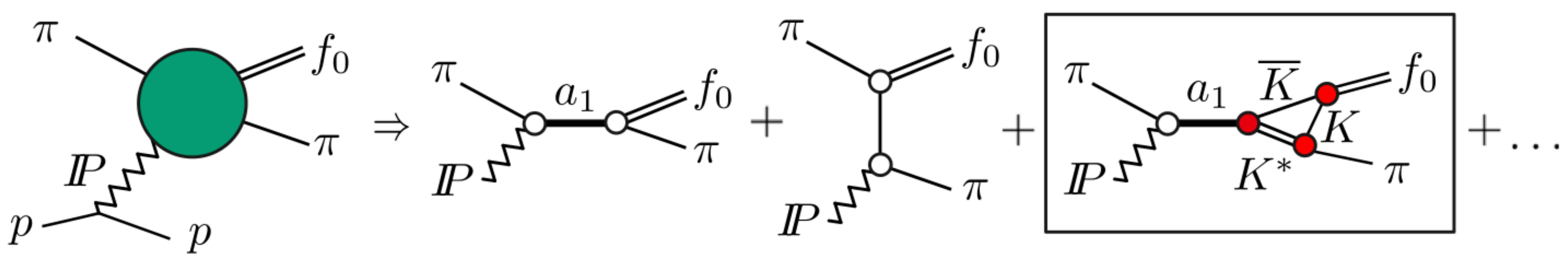
THANK YOU

Amplitude analysis



COMPASS, *Triangle singularity as the origin of a_1(1420)*

The a_1(1420)?



Most interesting states require inclusion three-body channels

χ_{c1}(3872), *N**(1440), a_1(1260), a_1(1420), π_1(1600),…

Bumps in complicated line-shapes can correspond to **kinematic enhancements** and not genuine resonances…

K matrix parametrization

Unitarity implies (Cayley transform)

$$S = \frac{\mathbb{1} + iK}{\mathbb{1} - iK}$$

where the K operator is Hermitian .

If S is symmetric, K is real .

$$T = K + iKT$$

This is the K matrix parametrization

Principle of "nearby singularities"

- Analyticity on the first Riemann sheet
- Bound-states & resonances correspond to poles
- Branch cuts correspond to open channels

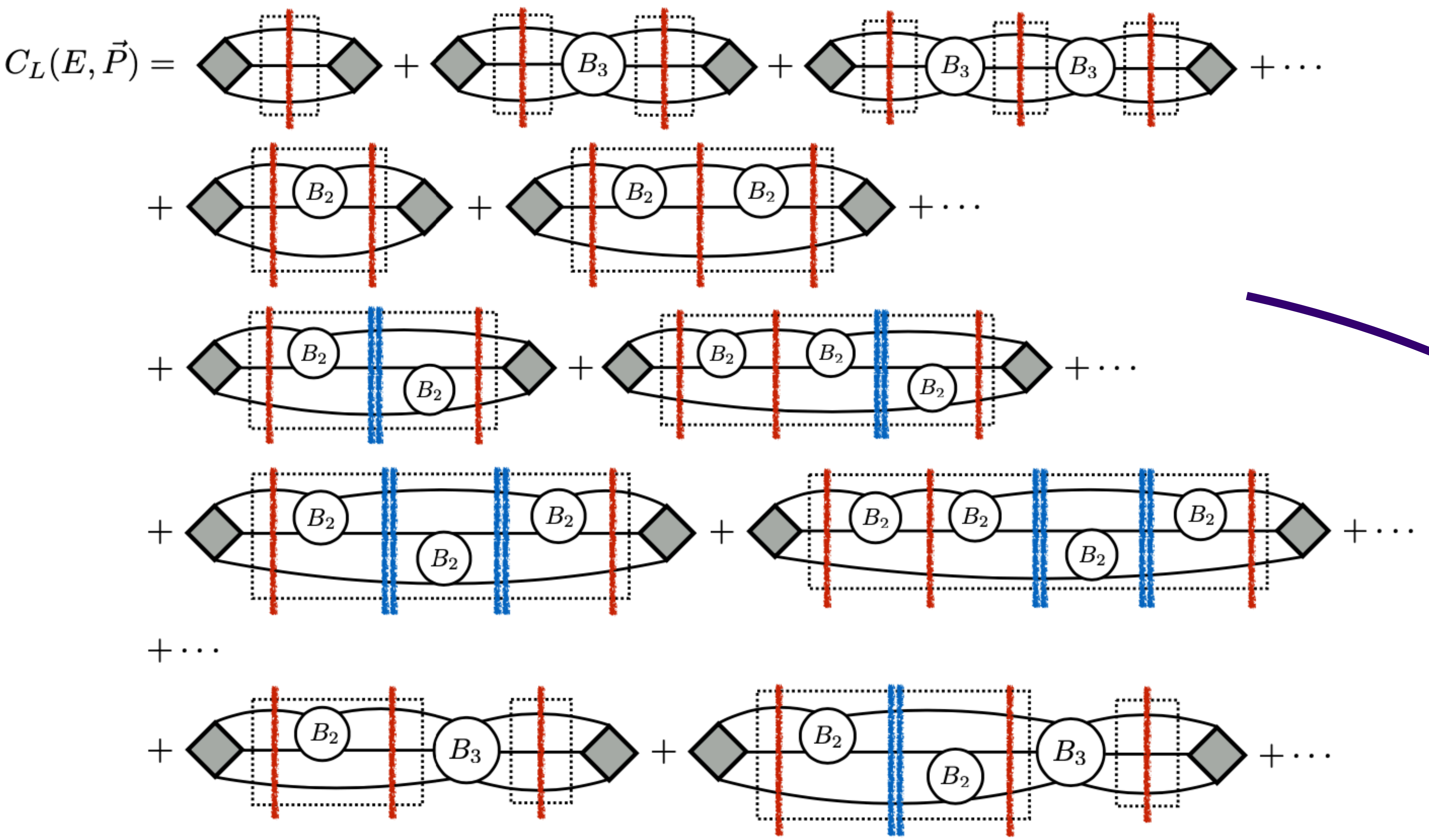
K−matrix parametrization

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell^{-1}(s) - i\rho(s)}$$

Phase−shift

$$\mathcal{K}_\ell^{-1}(s) = \frac{q^\star}{8\pi\sqrt{s}}\cot(\delta_\ell(s))$$

REFT formalism*



Poisson summation formula

$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d\boldsymbol{k}}{(2\pi)^3}\right] f(\boldsymbol{k}) = \frac{1}{L^3}\sum_{\boldsymbol{x}\neq 0} \tilde{f}(\boldsymbol{x})$$

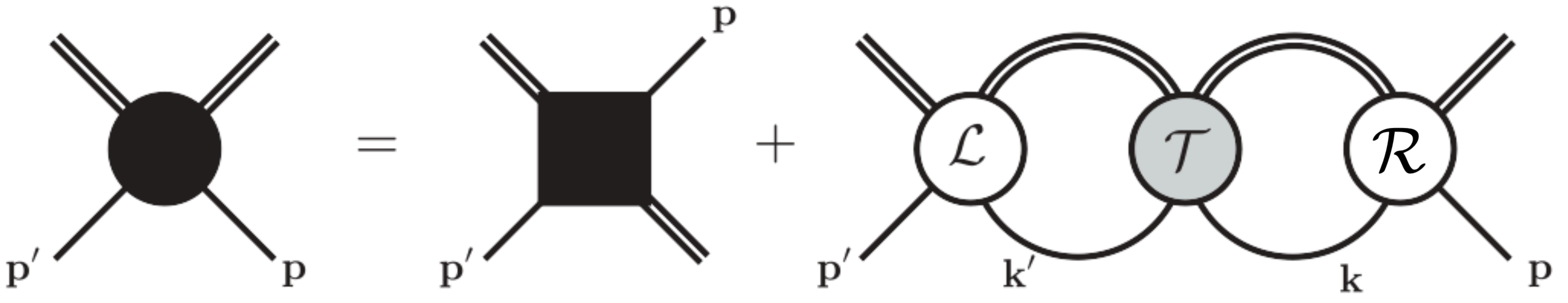
$$\det_{k,\ell,m}\left[\mathbb{1} - \boldsymbol{\mathcal{K}}_3(E^\star)\boldsymbol{F}_3(E,\boldsymbol{P},L)\right] = 0$$

Relativistic, model-independent, three-particle quantization condition
Hansen, Sharpe, PRD 90 (2014) 11, 116003

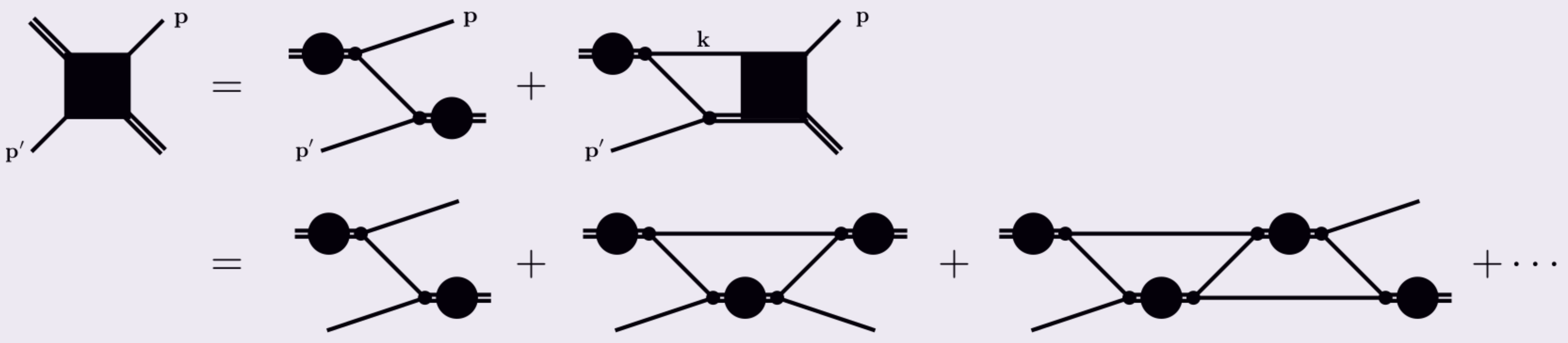
REFT three-body integral equations

Diagrams by Andrew Jackura

$$\mathcal{M}_3 = \mathcal{D} + \mathcal{M}_{3,\mathrm{df}}$$



One−particle exchanges

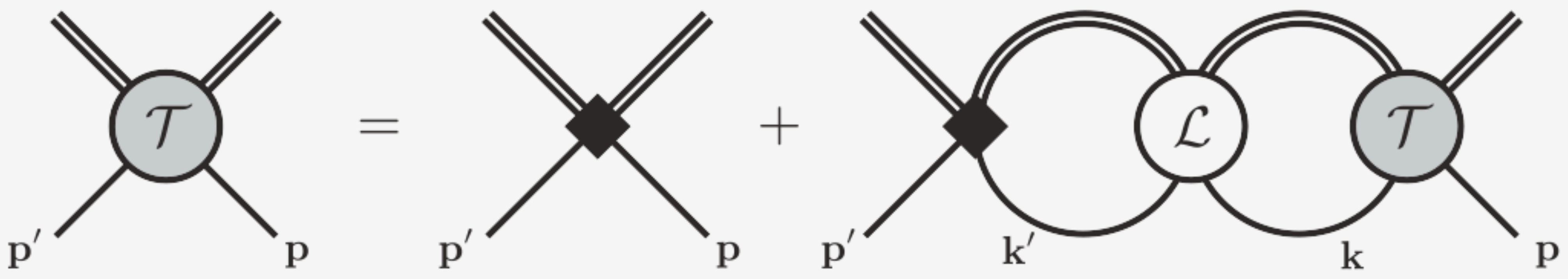


Three-body scattering: Ladders and Resonances
Mikhasenko, Wunderlich, Jackura, et al., *JHEP* 08 (2019) 080

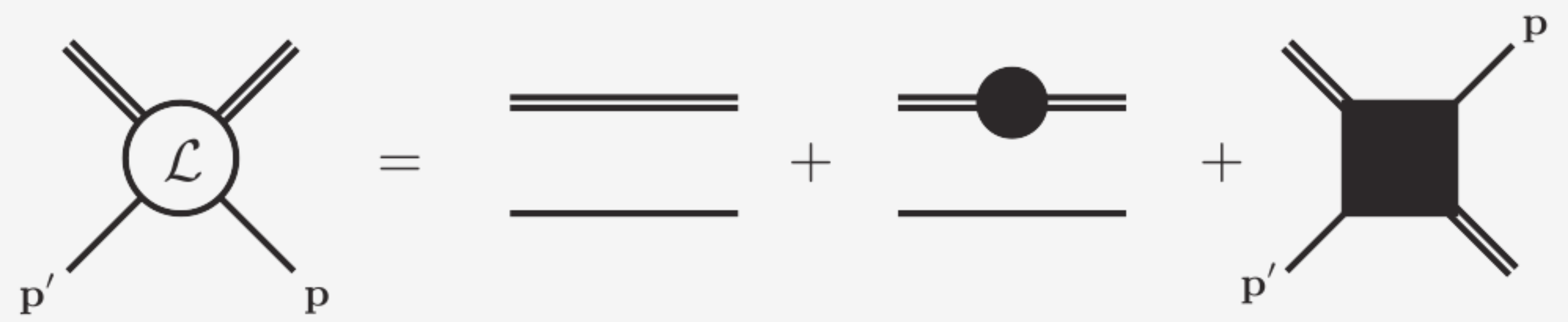
Equivalence of three-particle scattering formalisms
Jackura, Dawid, Fernandez-Ramirez, et al., PRD 100 (2019) 3, 034508

Equivalence of relativistic three-particle quantization conditions
Blanton, Sharpe, PRD 102 (2020) 5, 054515

Short−range amplitude

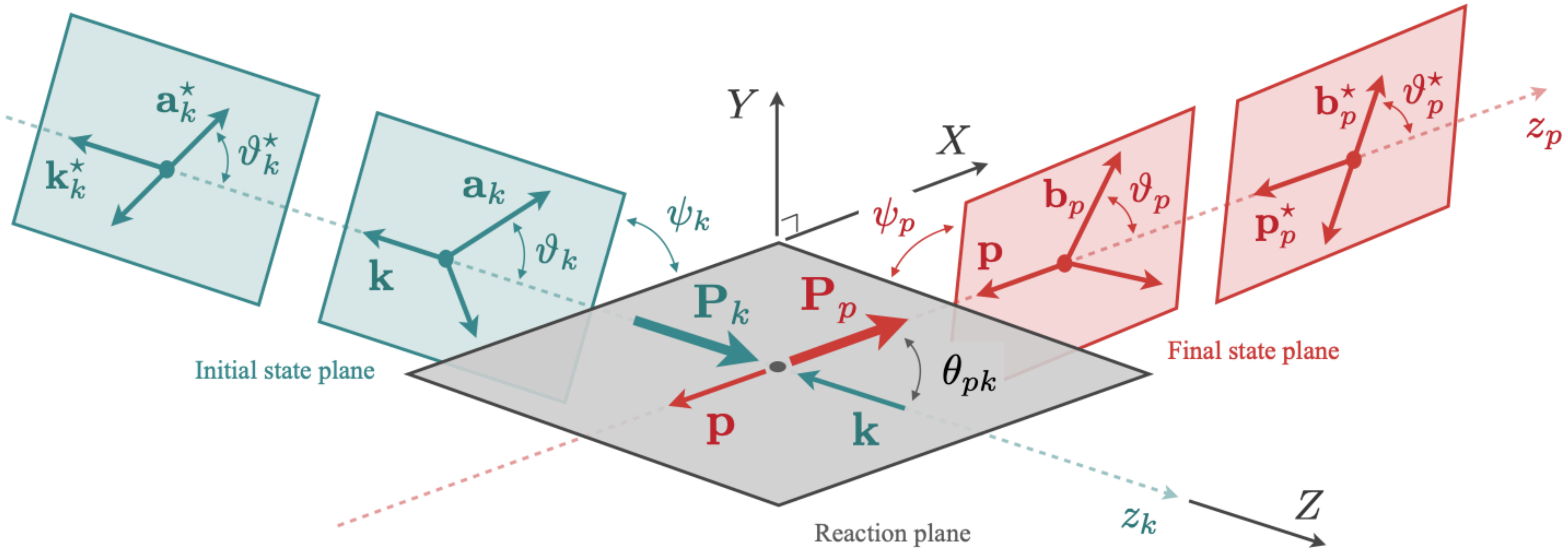


External−state rescatterings



Three-body amplitudes*

Pictures by A. Jackura (arXiv:2208.10587, arXiv:2312.00625)



On-shell three-body elastic amplitude depends on eight kinematical variables:
- two angles defining orientation of the initial-state pair
- two angles defining orientation of the final-state pair
- one angle defining orientation between spectators (pairs)
- invariant mass of the initial and final pair
- total energy

Three-body amplitudes*

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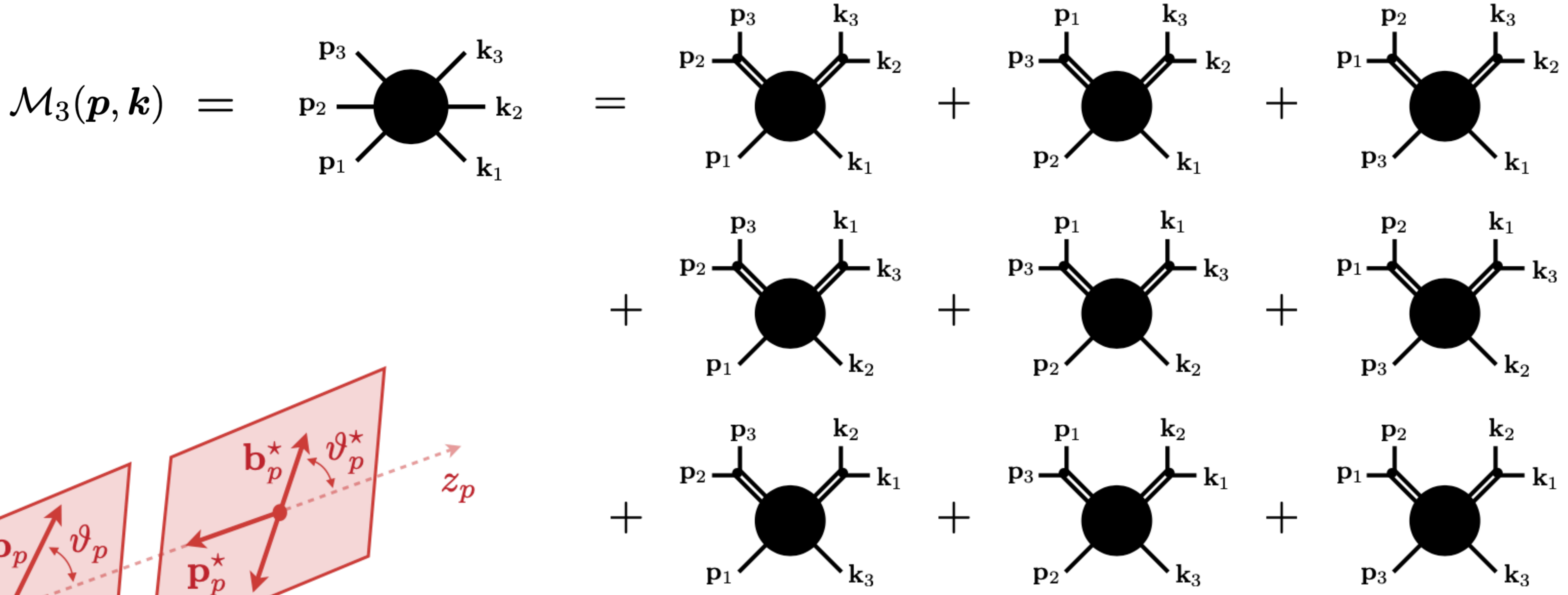
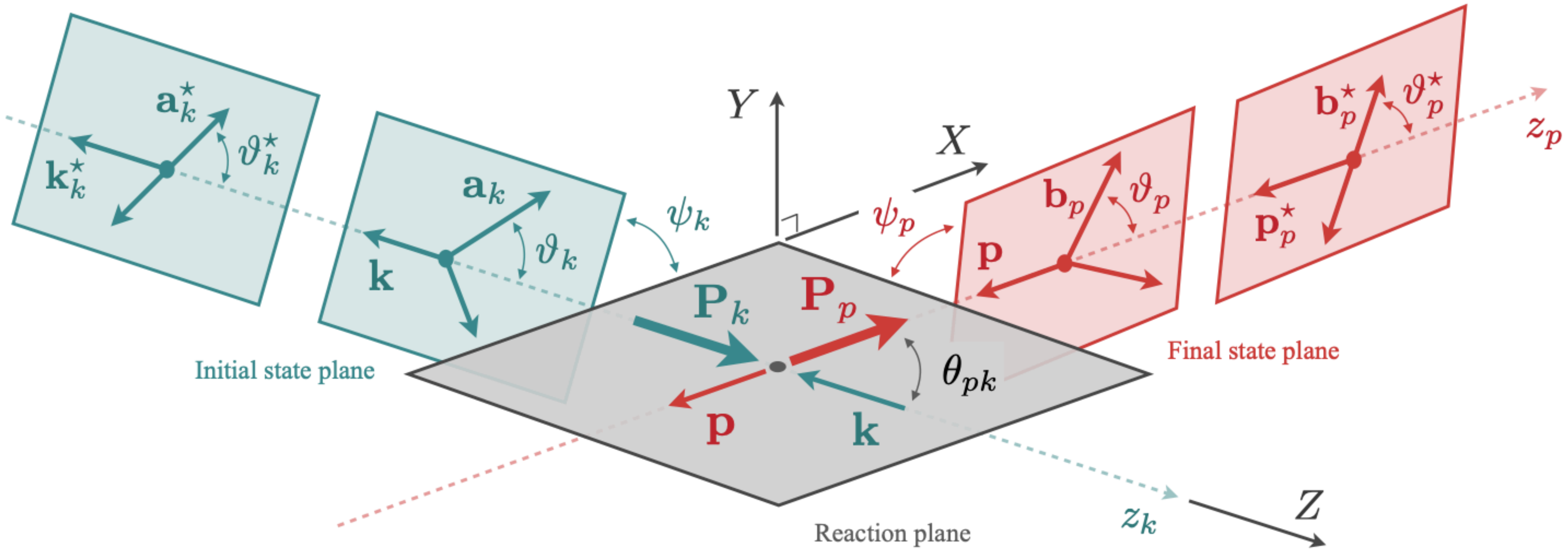
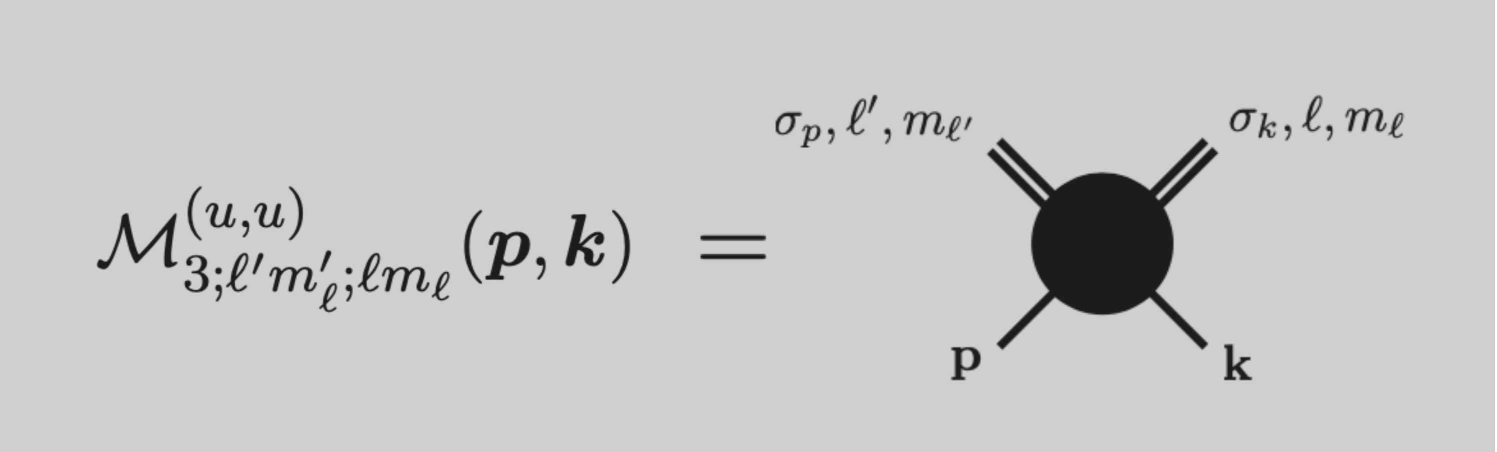




Pair$-$spectator amplitude



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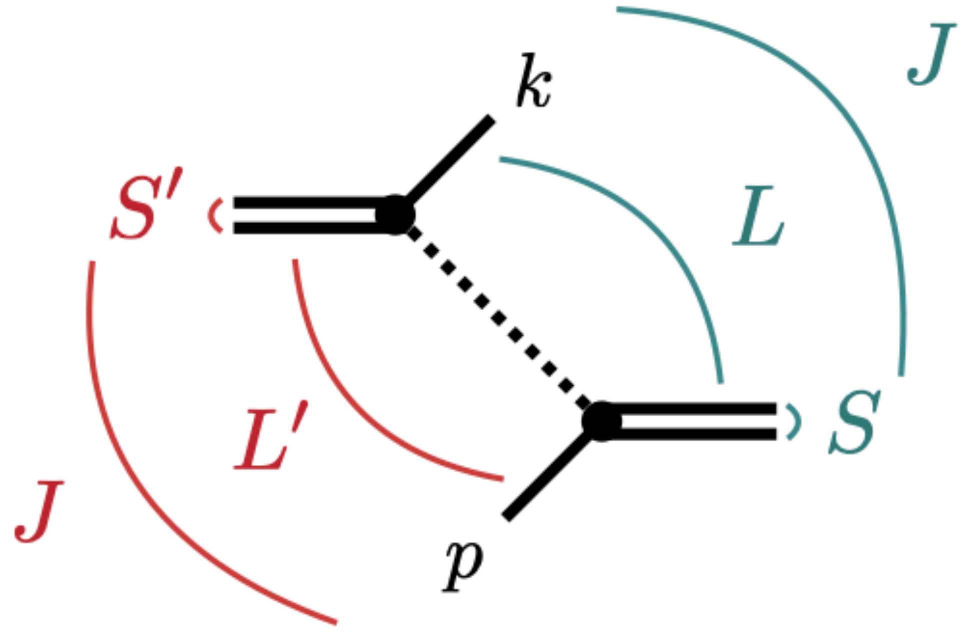

Status of the formalism and applications

From S. Sharpe, Multiparticle scattering from LQCD, Amplitudes24

3 identical spinless particles [Hansen & Sharpe; Hammer, Pang, Rusetsky; Mai & Döring]

(Applications: $3\pi^+$, $3K^+$, as well as φ^4 theory)

Mixing of two-and three-particle channels for identical spinless particles [Briceño, Hansen, Sharpe]

(Step on the way to $N^*(1440) \to N\pi$, $N\pi\pi$, etc.)

3 degenerate but distinguishable spinless particles, e.g., 3π with isospin 0,1,2,3 [Hansen, Romero-López, Sharpe]

(Potential applications: $\omega(782)$, $a_1(1260)$, $h_1(1170)$, $\pi(1300)$, ...)

3 nondegenerate spinless particles [Blanton & Sharpe]

(Potential applications: $D_s^+ D^0 \pi^-$)

2 identical+1 different spinless particles [Blanton & Sharpe]

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DDπ for all isospins (also BBπ, KKπ) [Draper, Hansen, Romero-López, Sharpe]

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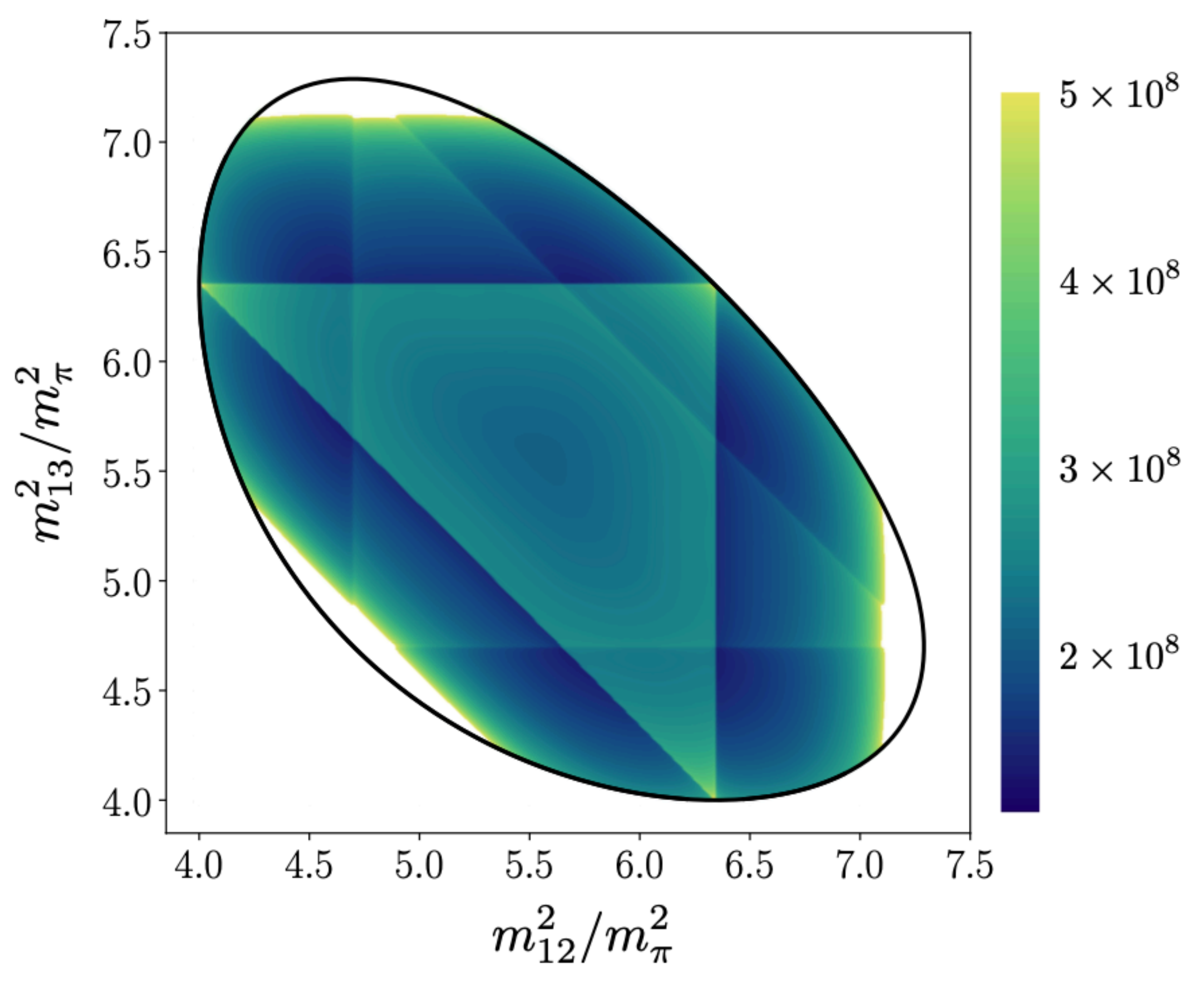

weakly interacting system in $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$

Hansen et al., Phys. Rev. Lett. 126 (2021), 012001

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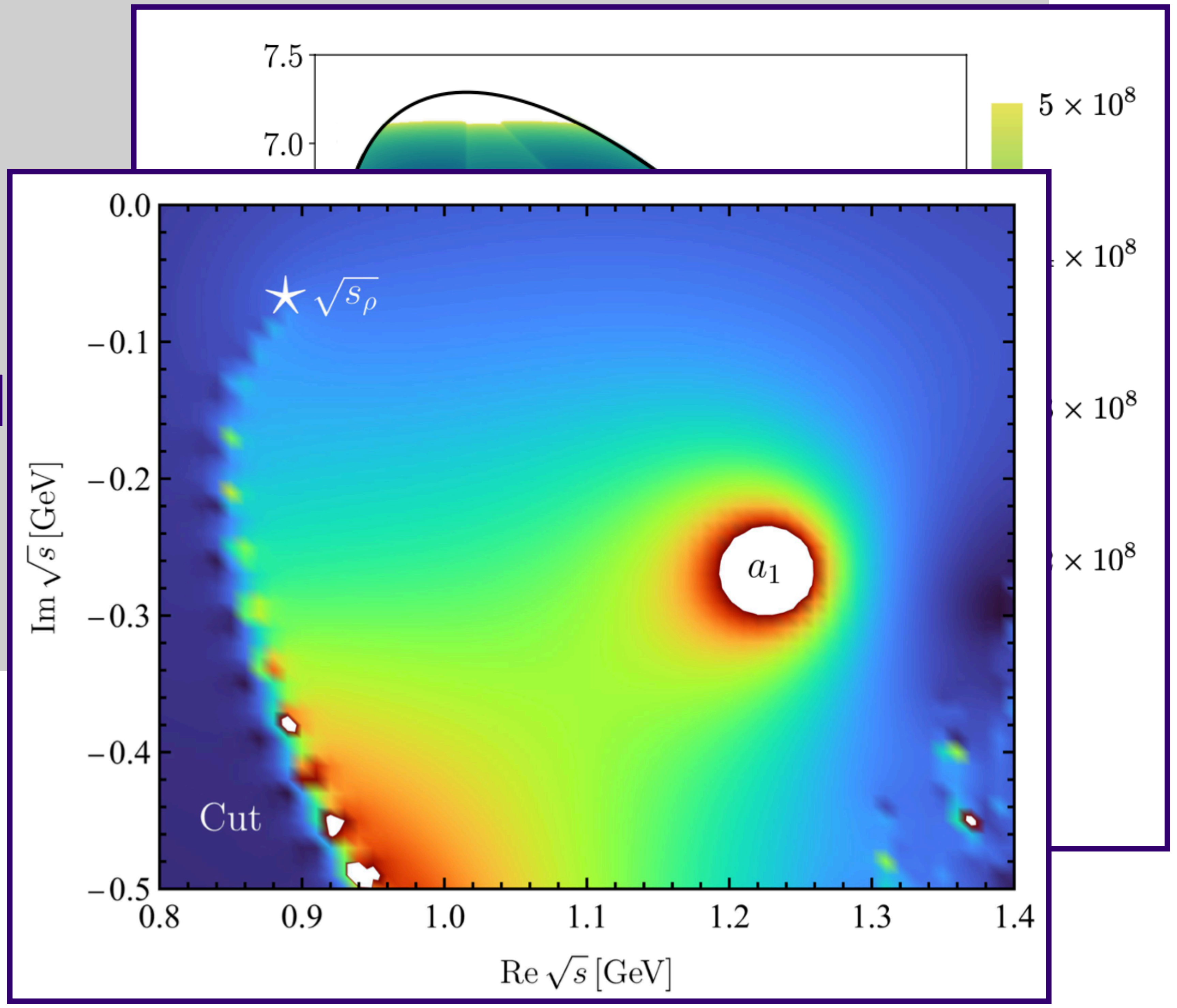
decay $a_1(1260) \to \rho^0\pi^- \to \pi^-\pi^+\pi^-$
Sadasivan et al., Phys. Rev. D 105 (2022) 5, 054020



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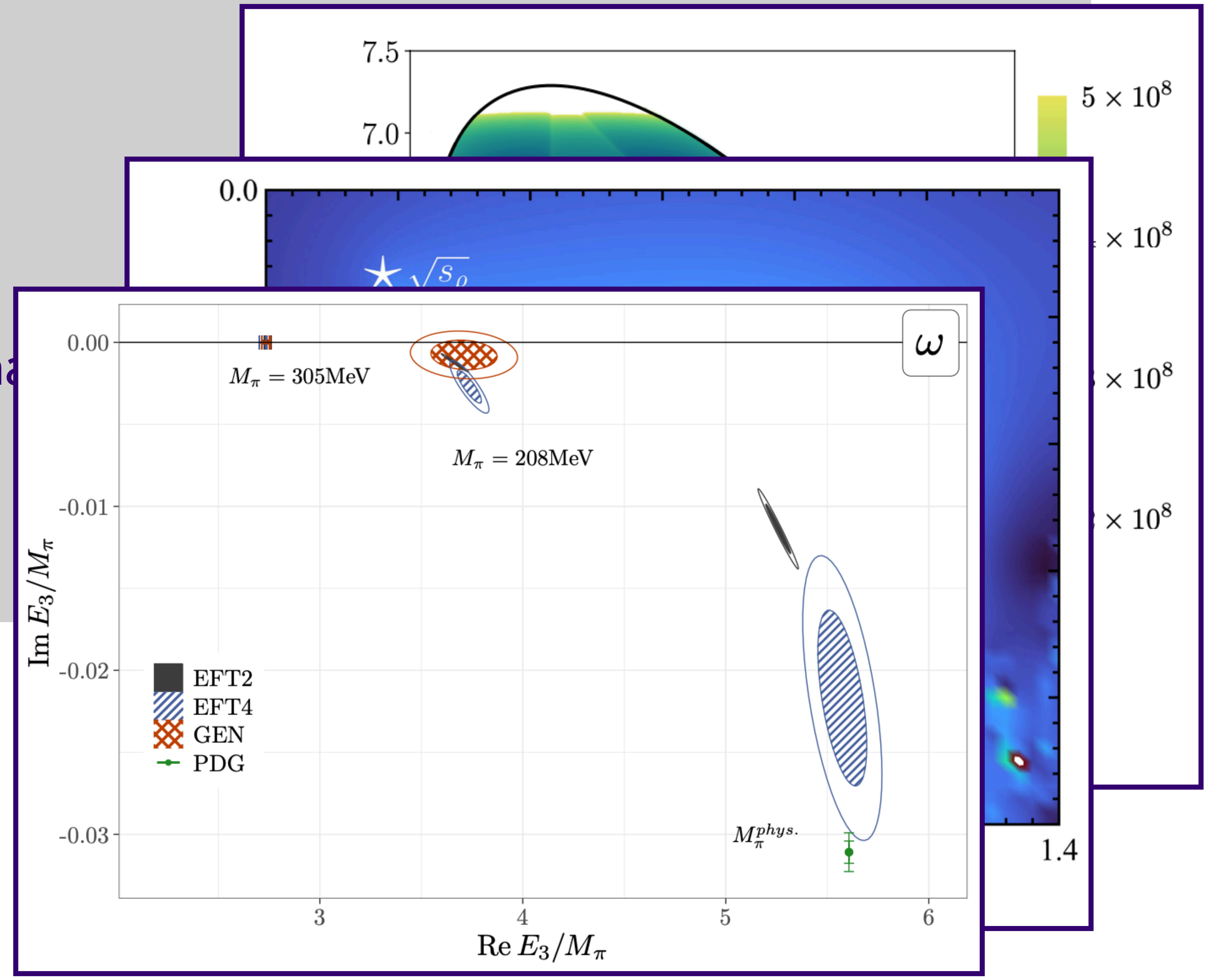

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Sadasivan et al., Phys. Rev. D 105 (2022) 5, 054020

ω meson

Yan et al., arXiv:2407.16659

Extracting correlators and energies*

$$\mathcal{O}_{3K^+}(\boldsymbol{P},t) = \sum_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3} c(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)\,\mathcal{O}_{K^+}(\boldsymbol{p}_1,t)\mathcal{O}_{K^+}(\boldsymbol{p}_2,t)\mathcal{O}_{K^+}(\boldsymbol{p}_3,t)$$

$$\mathcal{O}_{K^+}(\boldsymbol{p},t) = \sum_{\boldsymbol{x}} e^{-i\boldsymbol{p}\boldsymbol{x}}\,\bar{s}(x)\gamma_5 s(x)$$

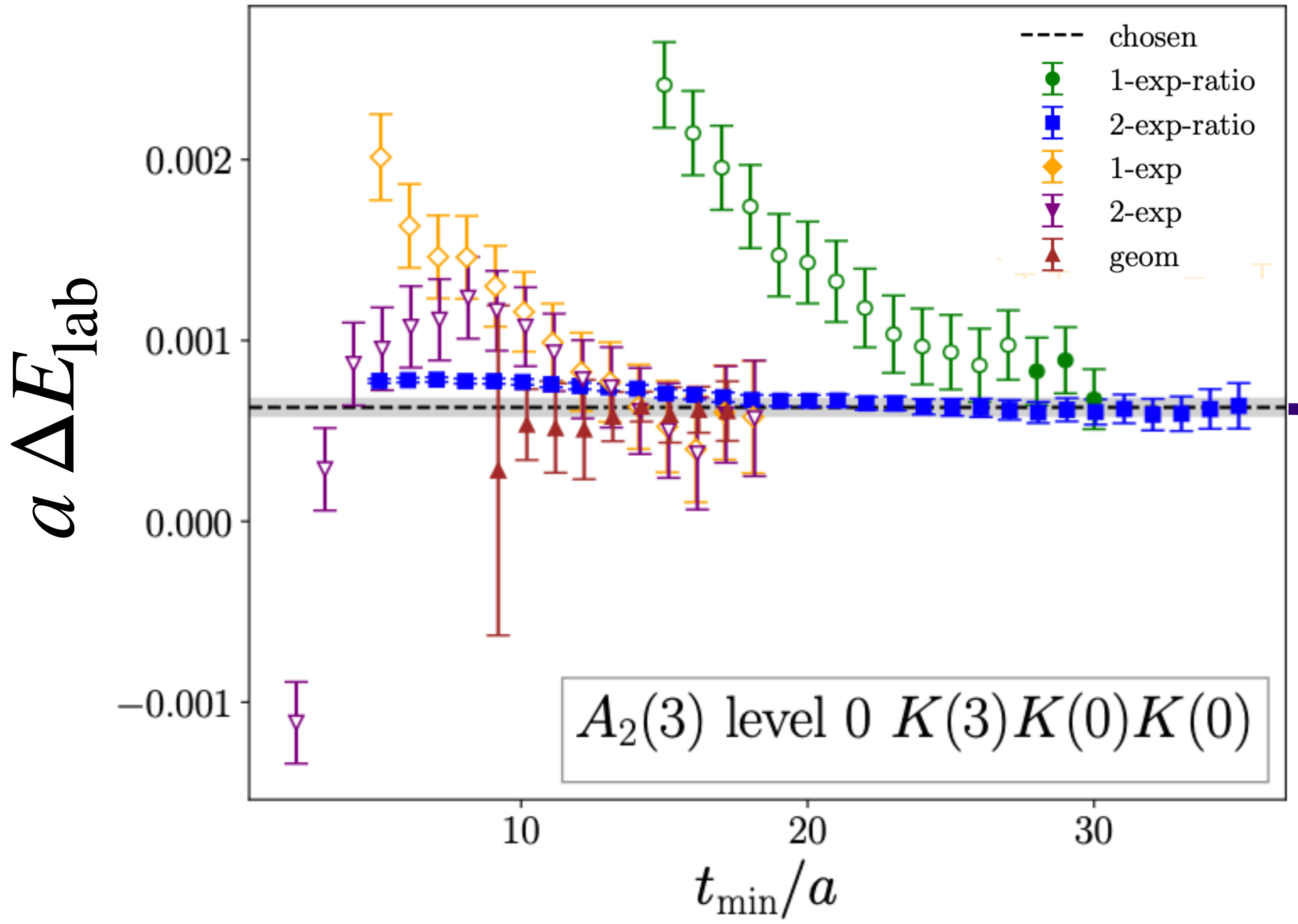
Multi−hadron operators

$$C_{ij}(|t-t_0|) = \langle\Omega|\mathcal{O}_i(t)\mathcal{O}_j^\dagger(t_0)|\Omega\rangle$$

Generalized eigenvalue problem

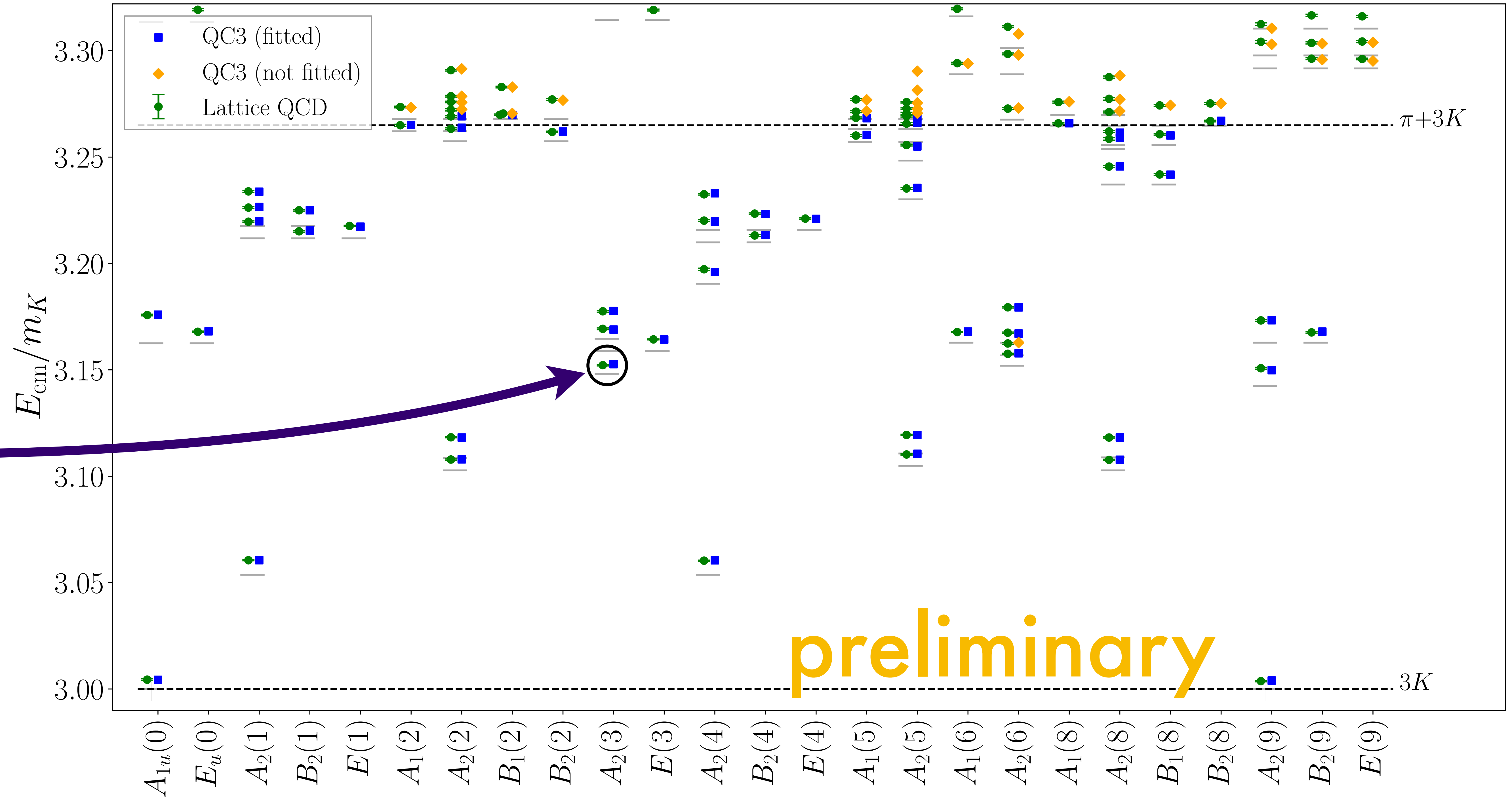
$$\widehat{C}(t)\,v_n(t,t_0) = \lambda_n(t,t_0)\,\widehat{C}(t_0)\,v_n(t,t_0)$$

and other tricks



Spectrum of $3K^+$

Hanlon, Hörz, Morningstar, Skinner



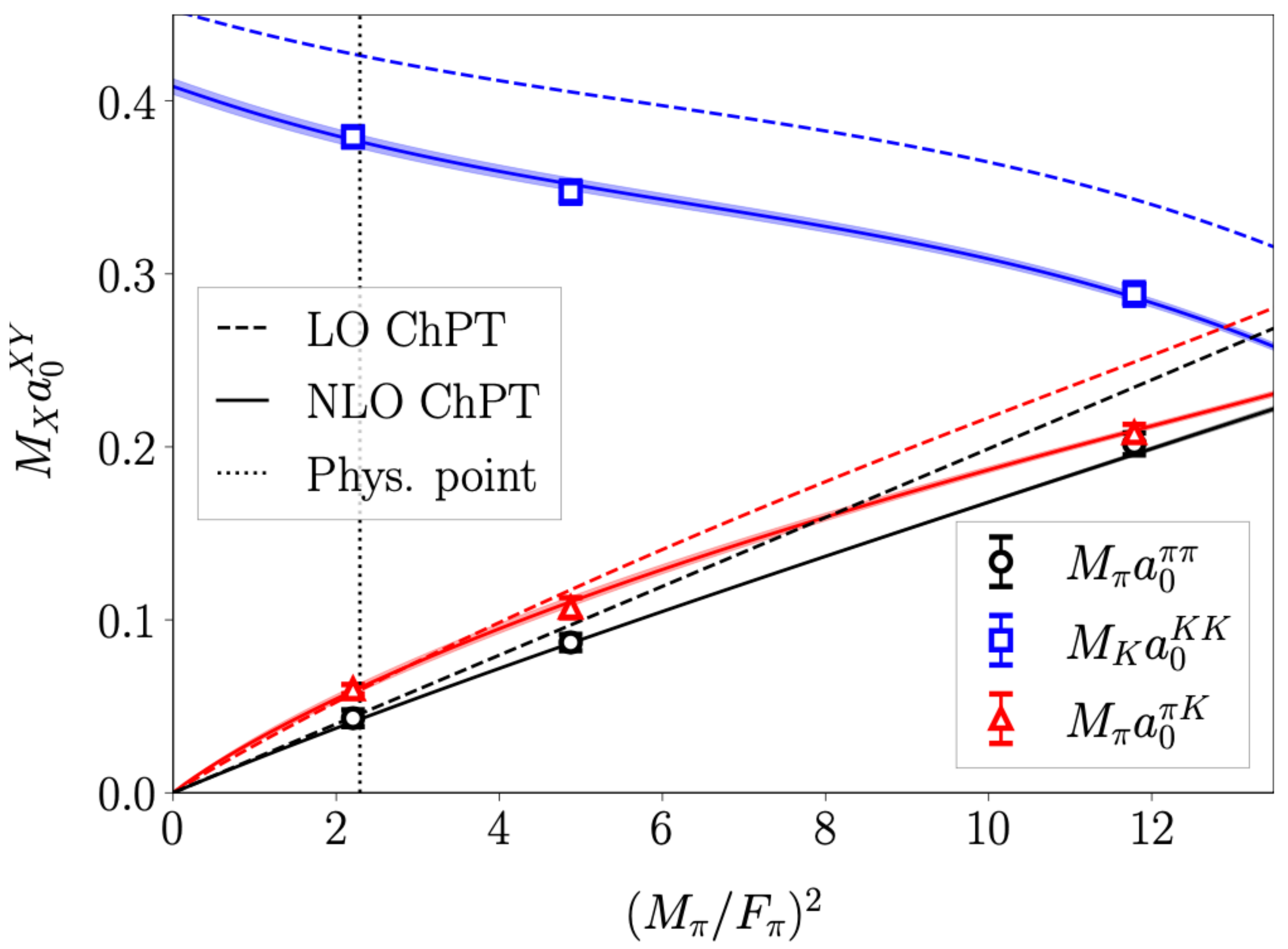
Fitting the two- and three-body K matrices*

Models of the two-body and three-body interactions
Correlated fits of several systems

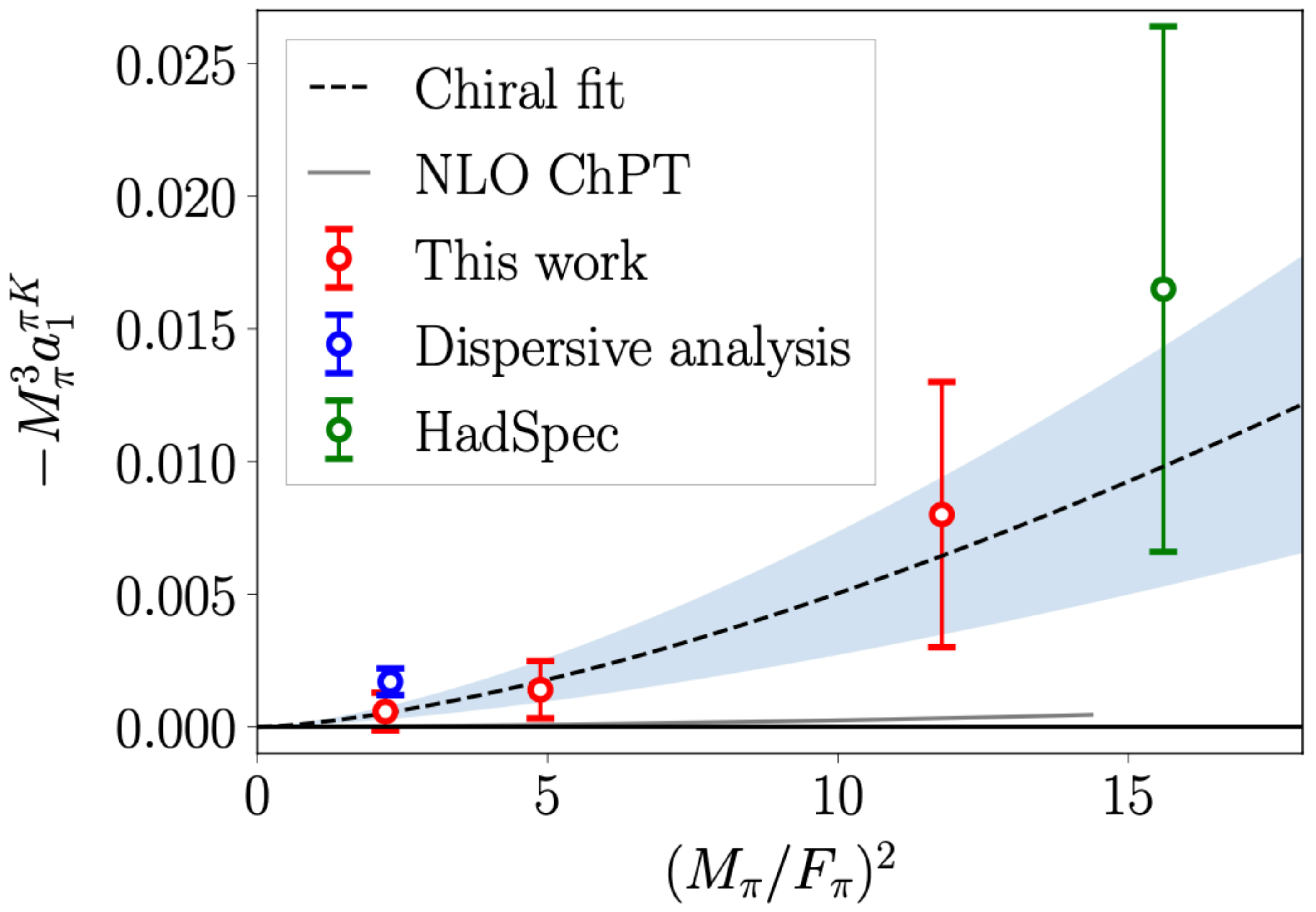
Two−body QC

Three−body QC

Draper, Romero-López, Sharpe

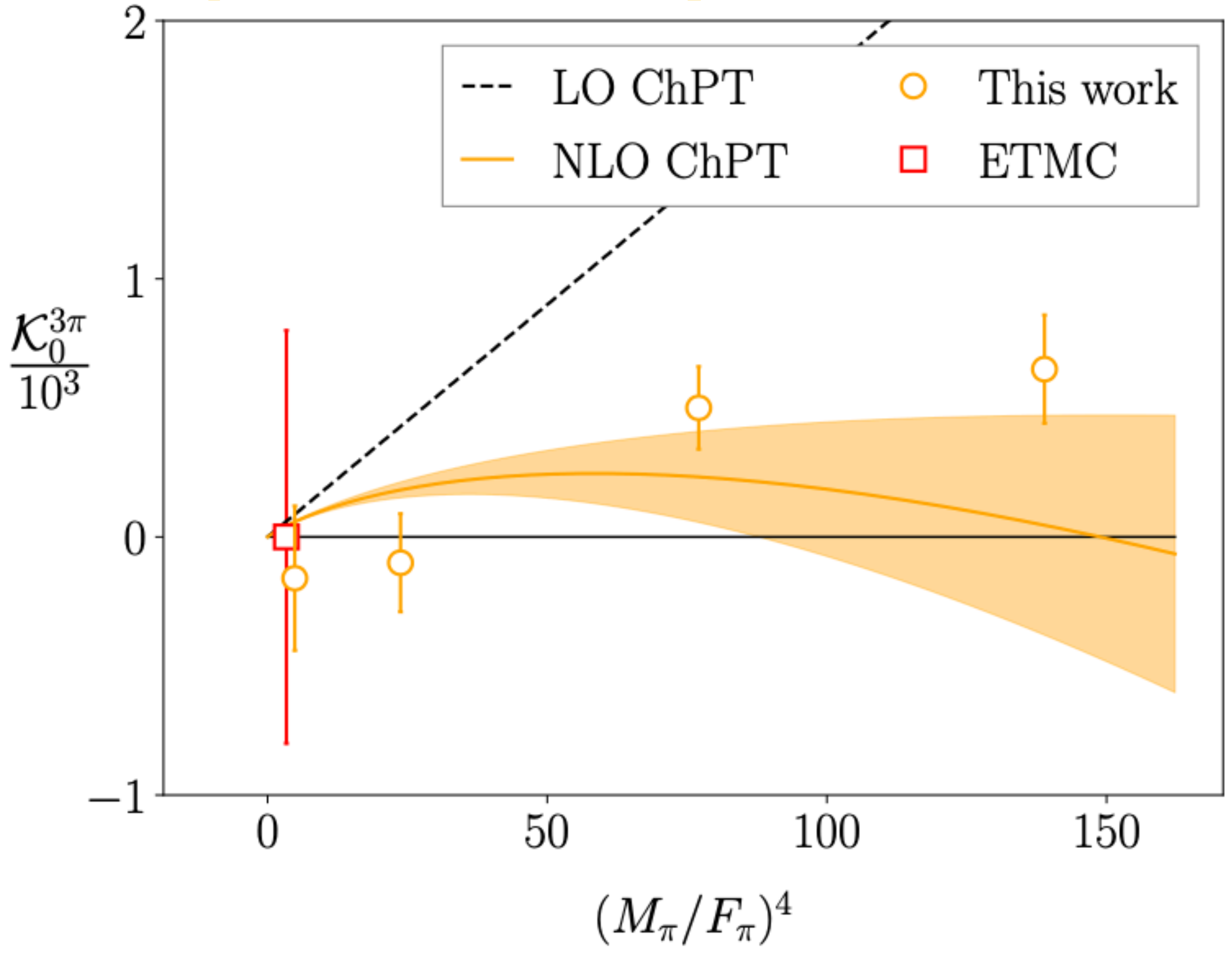


s−wave scattering lengths



p−wave πK scattering length

preliminary



Threshold expansion

$$M^2\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_2\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B \qquad \Delta = \frac{s - E_{\mathrm{th}}^2}{E_{\mathrm{th}}^2}$$

Generalizing to DDπ

$J^P = 1^+$

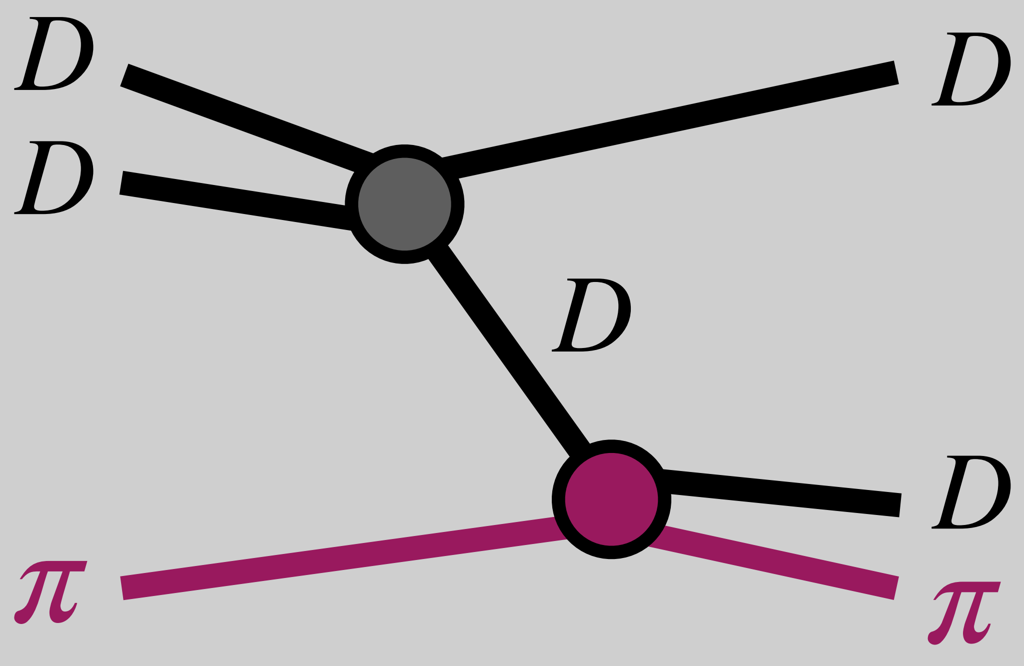
Dawid, Romero-López, Sharpe, arXiv:2409.17059

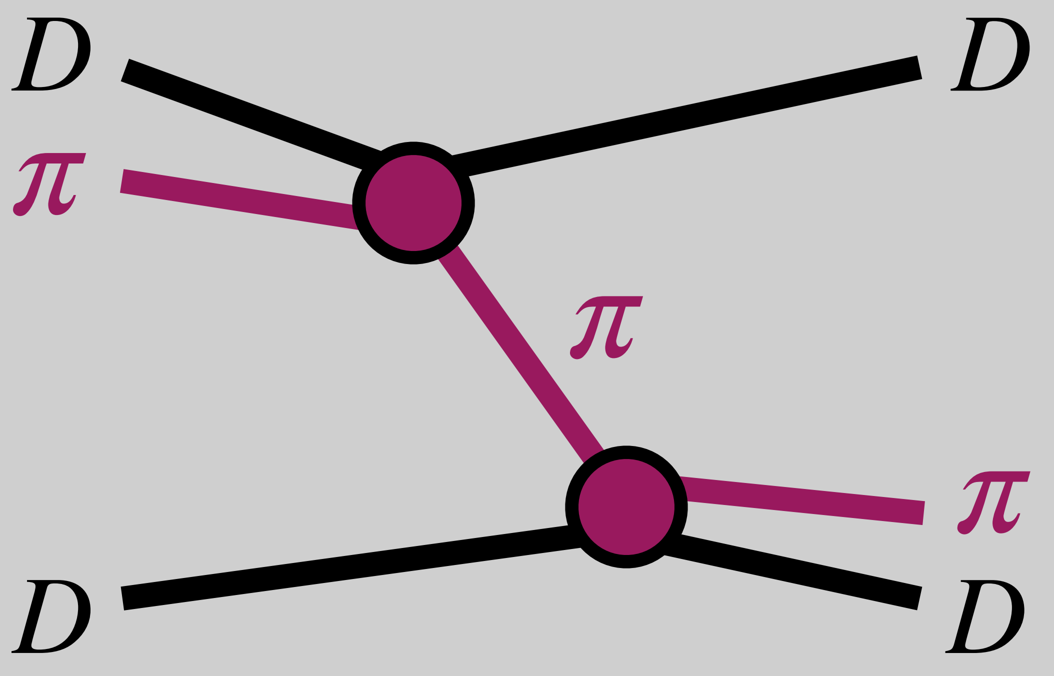
$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 G \mathcal{D}$$

The amplitude becomes a matrix describing coupled-channel scattering between pairs and spectators of different angular momenta (PW mixing allowed)

$$\begin{array}{cccccc} & (D\pi)D & (D\pi)D & (D\pi)D & (DD)\pi & \\ \left[\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} \mathcal{M}_3(^1P_1|^1P_1) \\ \mathcal{M}_3(^3S_1|^1P_1) \\ \mathcal{M}_3(^3D_1|^1P_1) \\ \mathcal{M}_3(^1P_1|^1P_1) \end{array} & \begin{array}{c} \mathcal{M}_3(^1P_1|^3S_1) \\ \mathcal{M}_3(^3S_1|^3S_1) \\ \mathcal{M}_3(^3D_1|^3S_1) \\ \mathcal{M}_3(^1P_1|^3S_1) \end{array} & \begin{array}{c} \mathcal{M}_3(^3P_1|^3D_1) \\ \mathcal{M}_3(^3S_1|^3D_1) \\ \mathcal{M}_3(^3D_1|^3D_1) \\ \mathcal{M}_3(^1P_1|^3D_1) \end{array} & \begin{array}{c} \mathcal{M}_3(^1P_1|^1P_1) \\ \mathcal{M}_3(^3S_1|^1P_1) \\ \mathcal{M}_3(^3D_1|^1P_1) \\ \mathcal{M}_3(^1P_1|^1P_1) \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] & \begin{array}{c} (D\pi)D \\ (D\pi)D \\ (D\pi)D \\ (DD)\pi \end{array} \end{array}$$

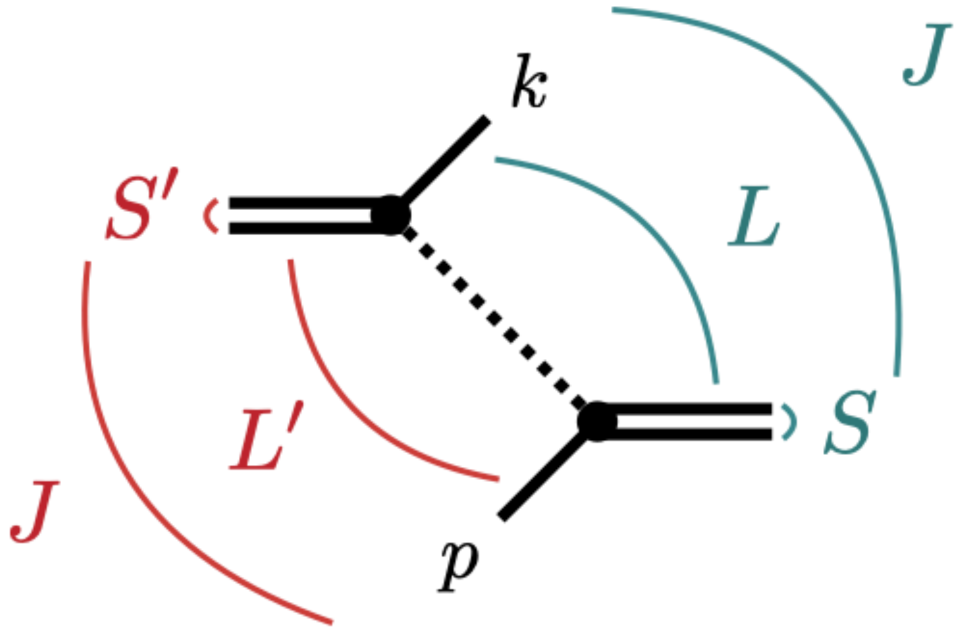
Driving processes





$$\propto -\frac{4\pi\, Y^*_{s'\lambda'}(\hat{\boldsymbol{p}}^\star)\, Y_{s\lambda}(\hat{\boldsymbol{k}}^\star)}{(P-p-k)^2 - m^2 + i\epsilon}$$

$$G(^{2S'+1}L'_J|^{2S+1}L_J) \; =$$



$$\propto \frac{1}{4pk} \log\left(\frac{1+z(p,k;E)}{1-z(p,k;E)}\right)$$

Partial-wave projection of the one-particle exchange in three-body scattering amplitudes
Jackura, Briceño, PRD 109, 096030 (2024)

Including three-body forces $J^P = 1^+$

Implementing the three-particle quantization condition for ππK and related systems
Blanton, Romero-López, Sharpe, JHEP 02 (2022) 098

Dawid, Romero-López, Sharpe, arXiv:2409.17059

$$\mathcal{T} = \mathcal{K}_3 - \mathcal{K}_3\, \rho\, \mathcal{L}\, \mathcal{T}$$

Matrix-integral equation governed by the symmetric three-body K matrix and two-body rescatterings.

$$\mathcal{K}_3^{(ij)}(p,k) = \sum_a \mathcal{K}_{L,a}^{(i)}(p)\, \mathcal{K}_{R,a}^{(j)}(k)$$

$$\mathcal{T} = \mathcal{K}_L^T\, [1+\mathcal{I}]^{-1}\, \mathcal{K}_R$$

Solution of another integral equation is unnecessary for certain models of the three-body K matrix

Threshold expansion

$$\mathcal{K}_3 = \mathcal{K}_3^{\mathrm{iso},0} + \mathcal{K}_3^{\mathrm{iso},1}\Delta + \mathcal{K}_3^B \Delta_2^S + \mathcal{K}_3^E t'_{22}$$

$$\Delta = \frac{s-(2m_D+m_\pi)^2}{(2m_D+m_\pi)^2} \qquad t'_{22} = \frac{(p_2-p'_2)^2}{(2m_D+m_\pi)^2}$$

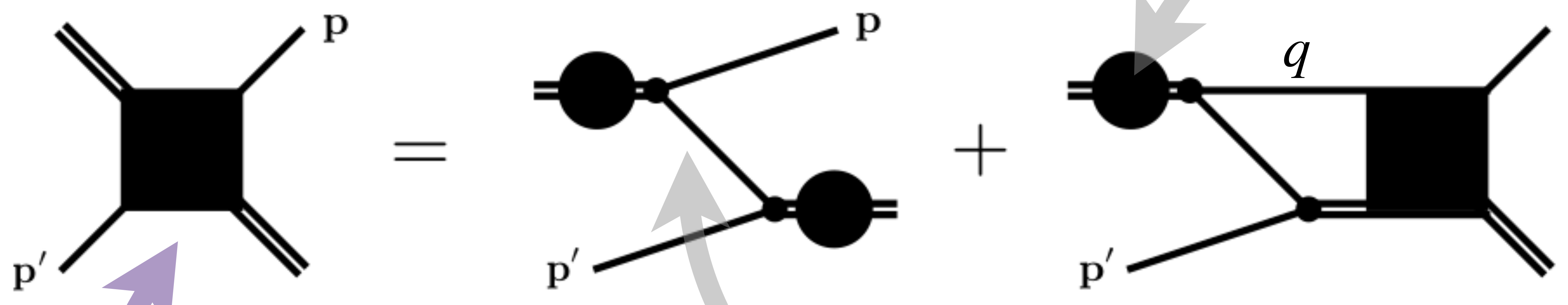
The last term contributes, for instance,

$$\mathcal{K}_3({}^3S_1|{}^3S_1) = \frac{2}{27}\mathcal{K}_3^E\, q_p^\star\, q_k^\star\, (\gamma_p+2)(\gamma_k+2)$$

Relative two−body momentum in a pair

Boost to pair's rest frame

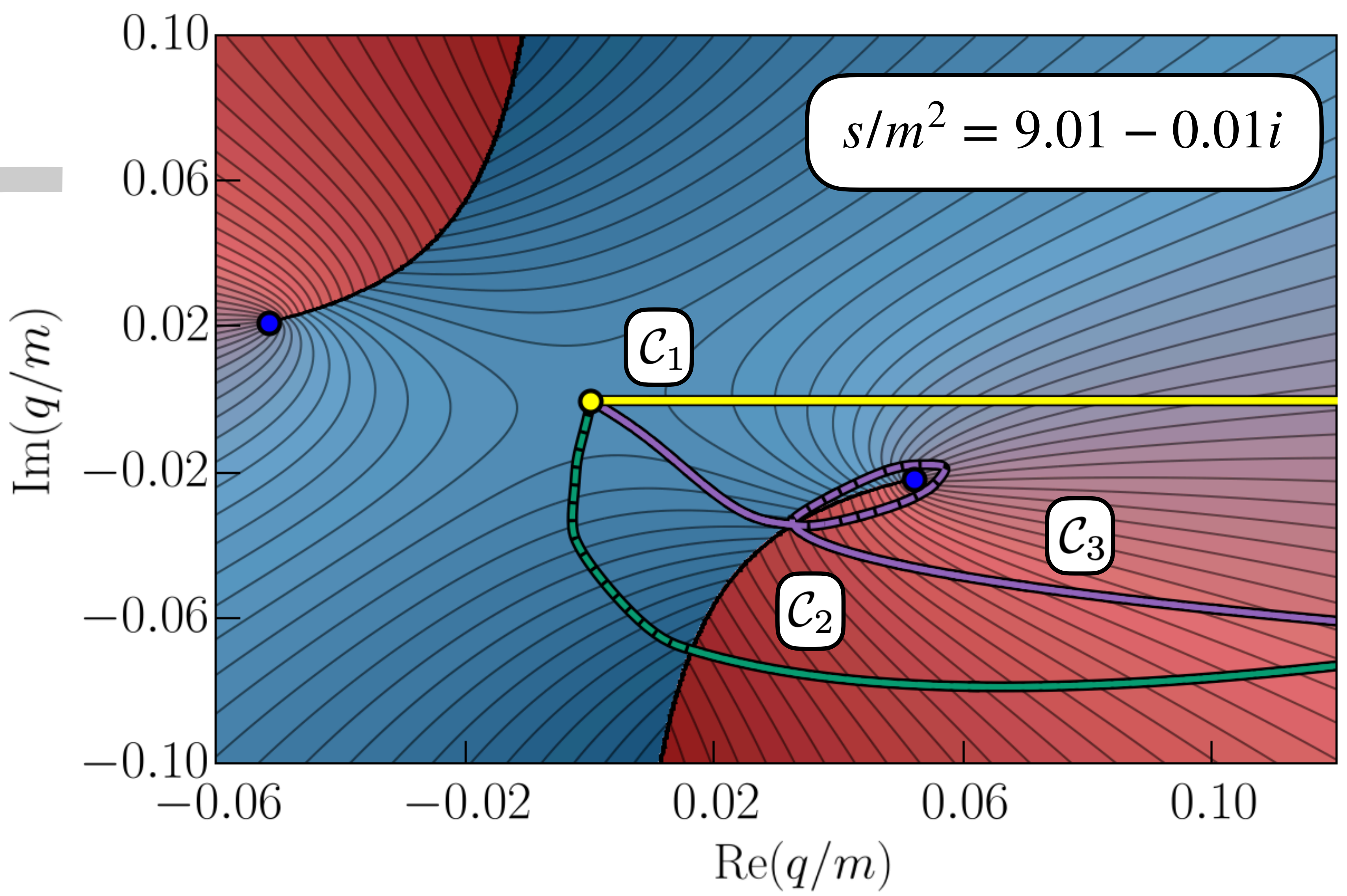
Analytic continuation

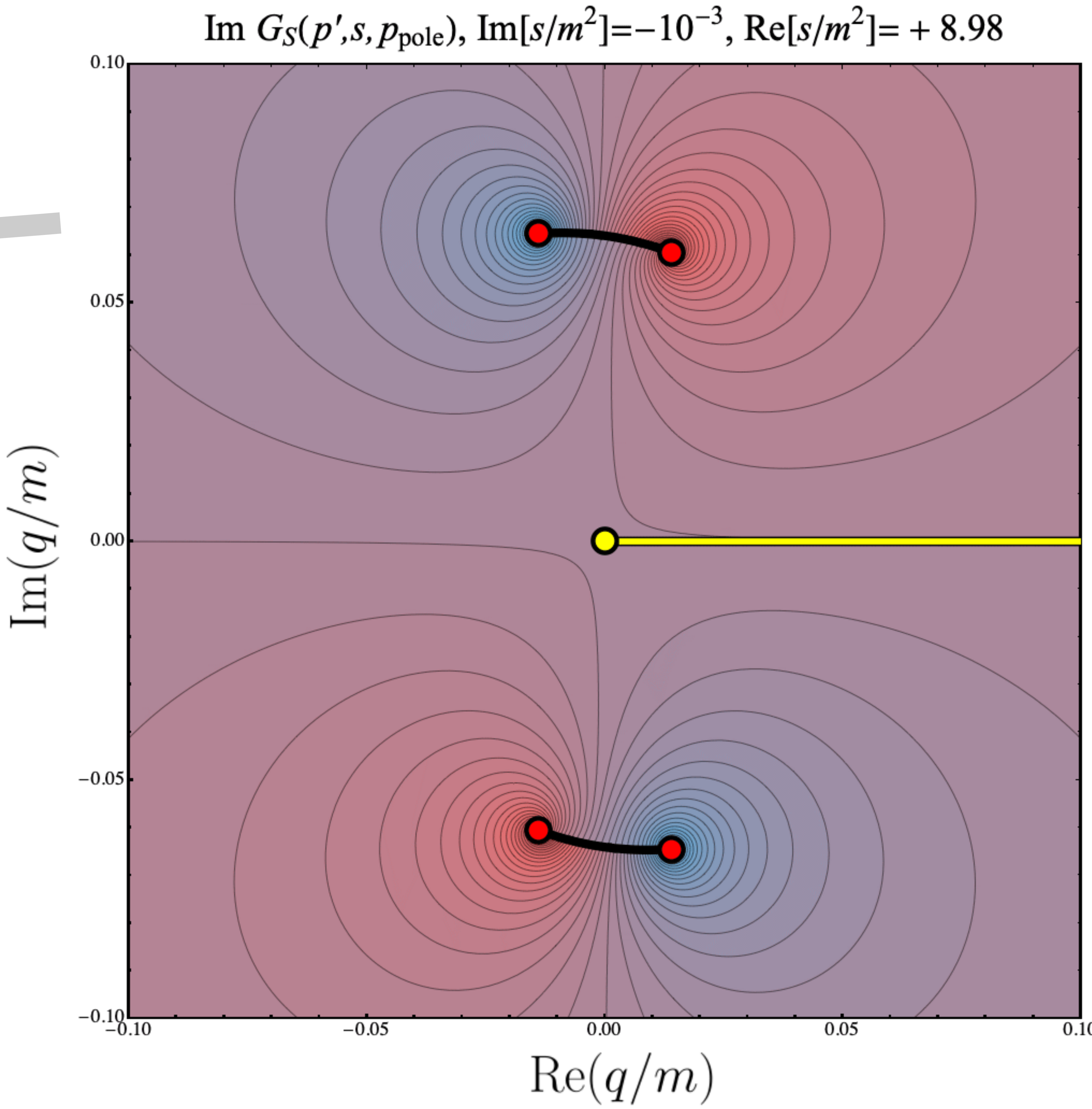


Two-body amplitude

Singularities

One-particle exchange



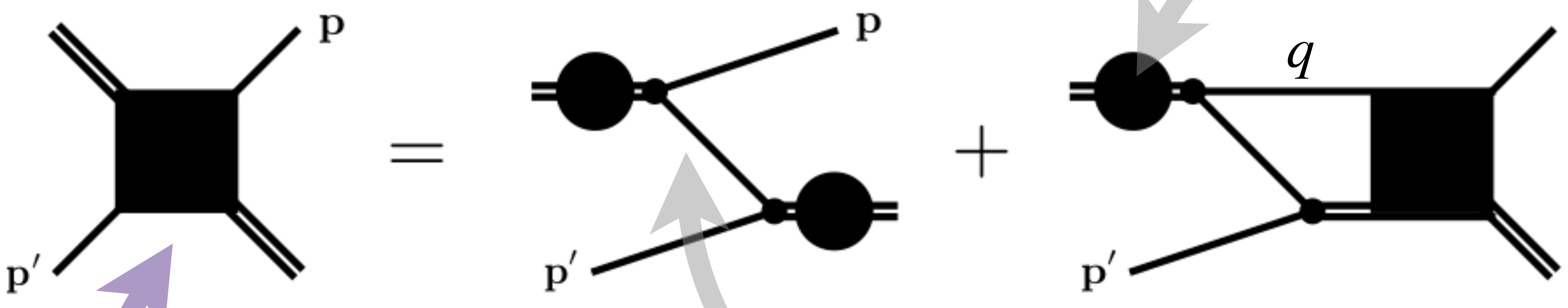


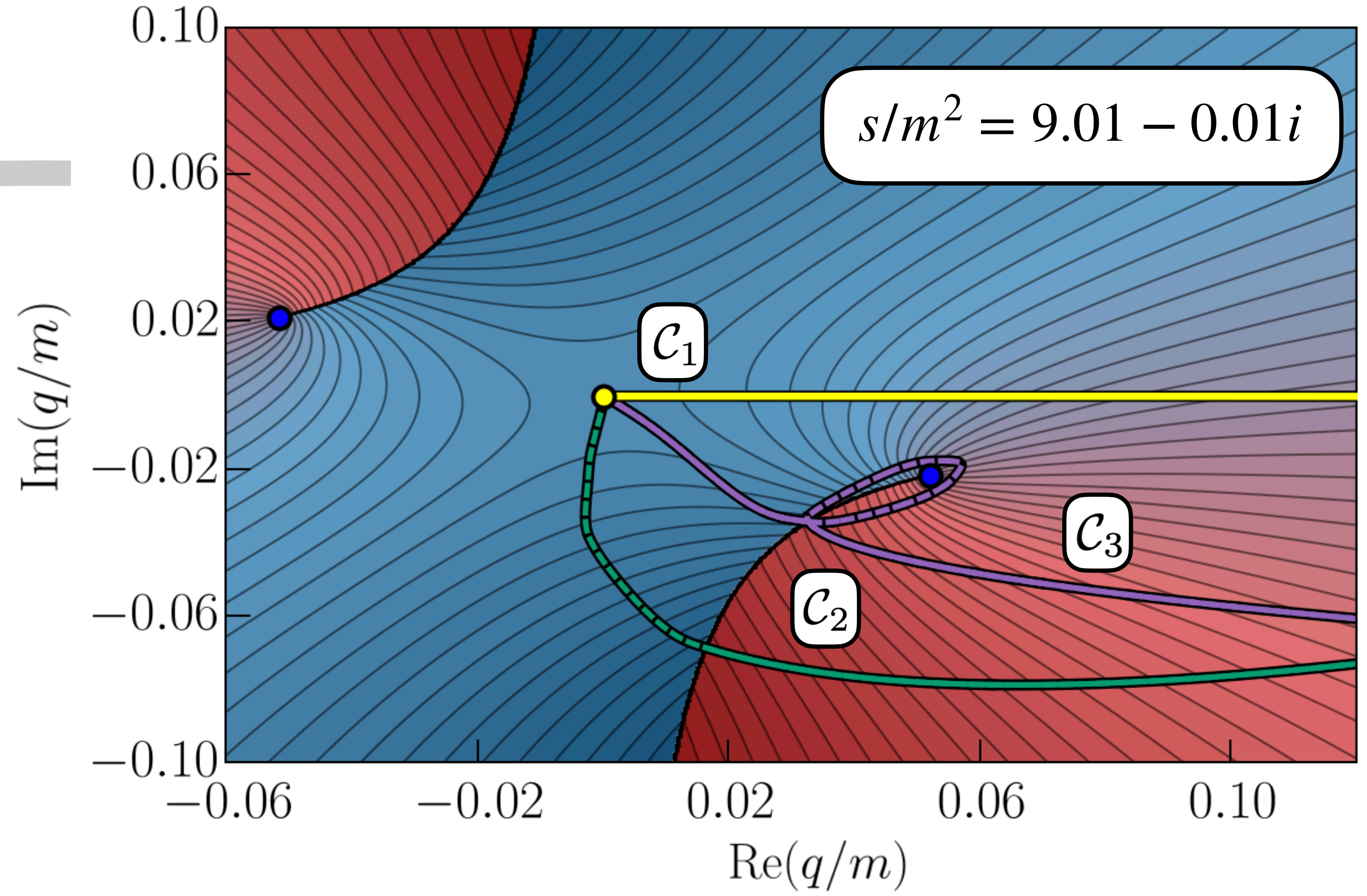
In a nutshell

- We need to avoid crossing the singularities while integrating
- Achieved by contour deformations, addition of discontinuities, …
- Multi-valuedness of the amplitude originates from collisions of the contour with: poles (two-body threshold) and branch points (three-body threshold),
- Riemann sheets defined by a monodromy

Analytic continuation

Two-body amplitude





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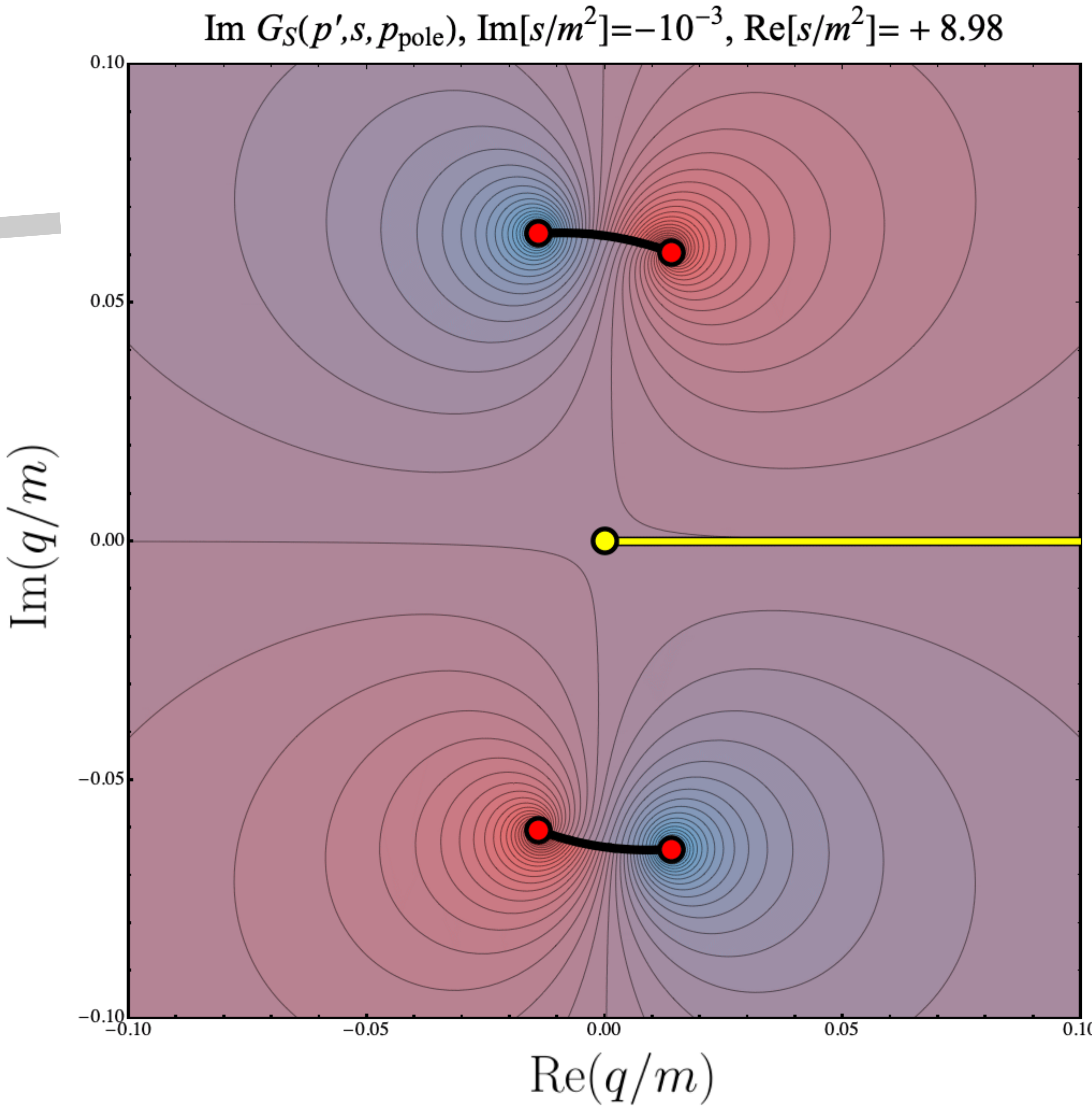

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- We need to avoid crossing the singularities while integrating
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Self-consistency of the deformed contour

zrobic 05.03.2024

Singularities of the ladder equation

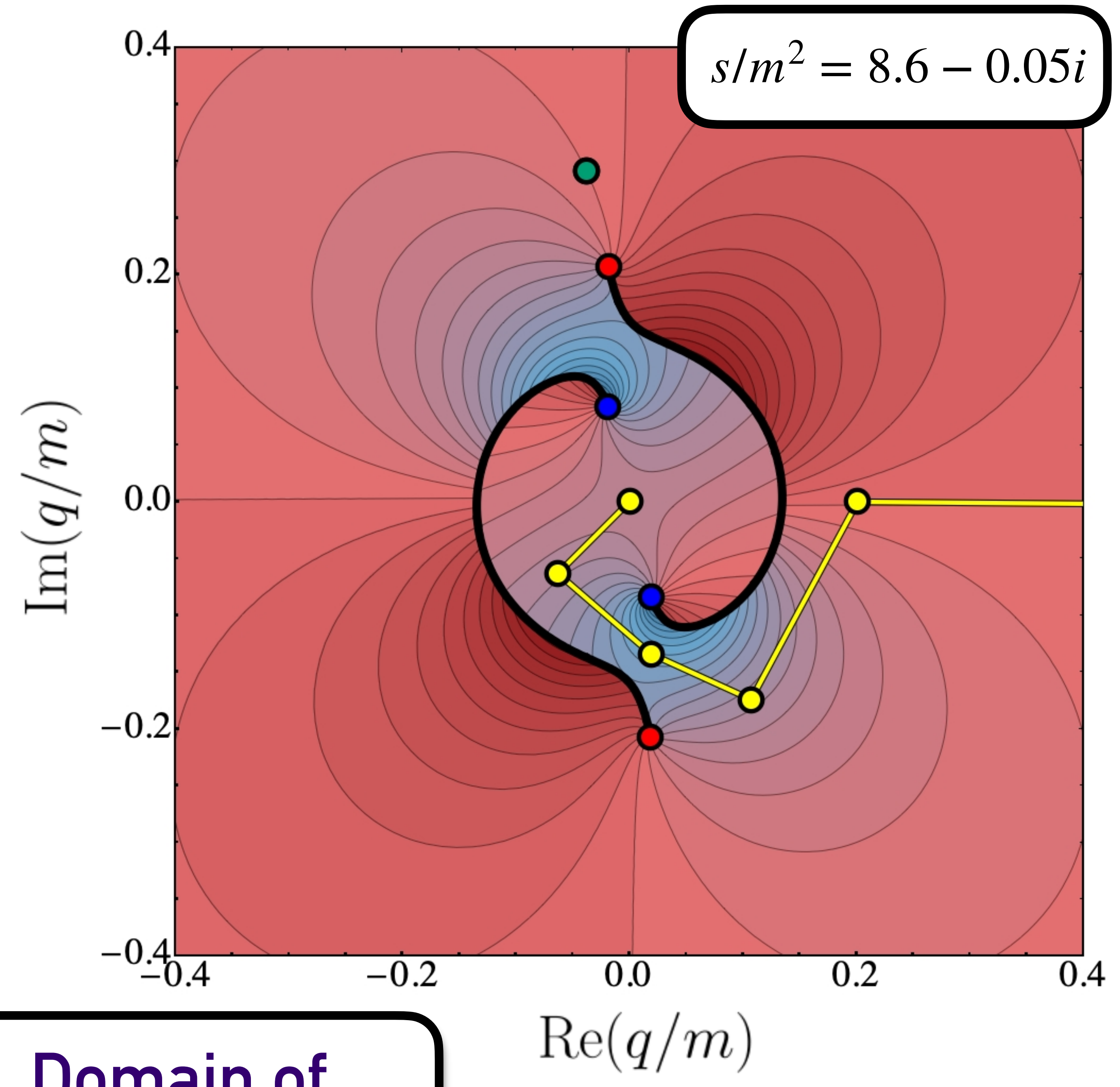
Deformed contour

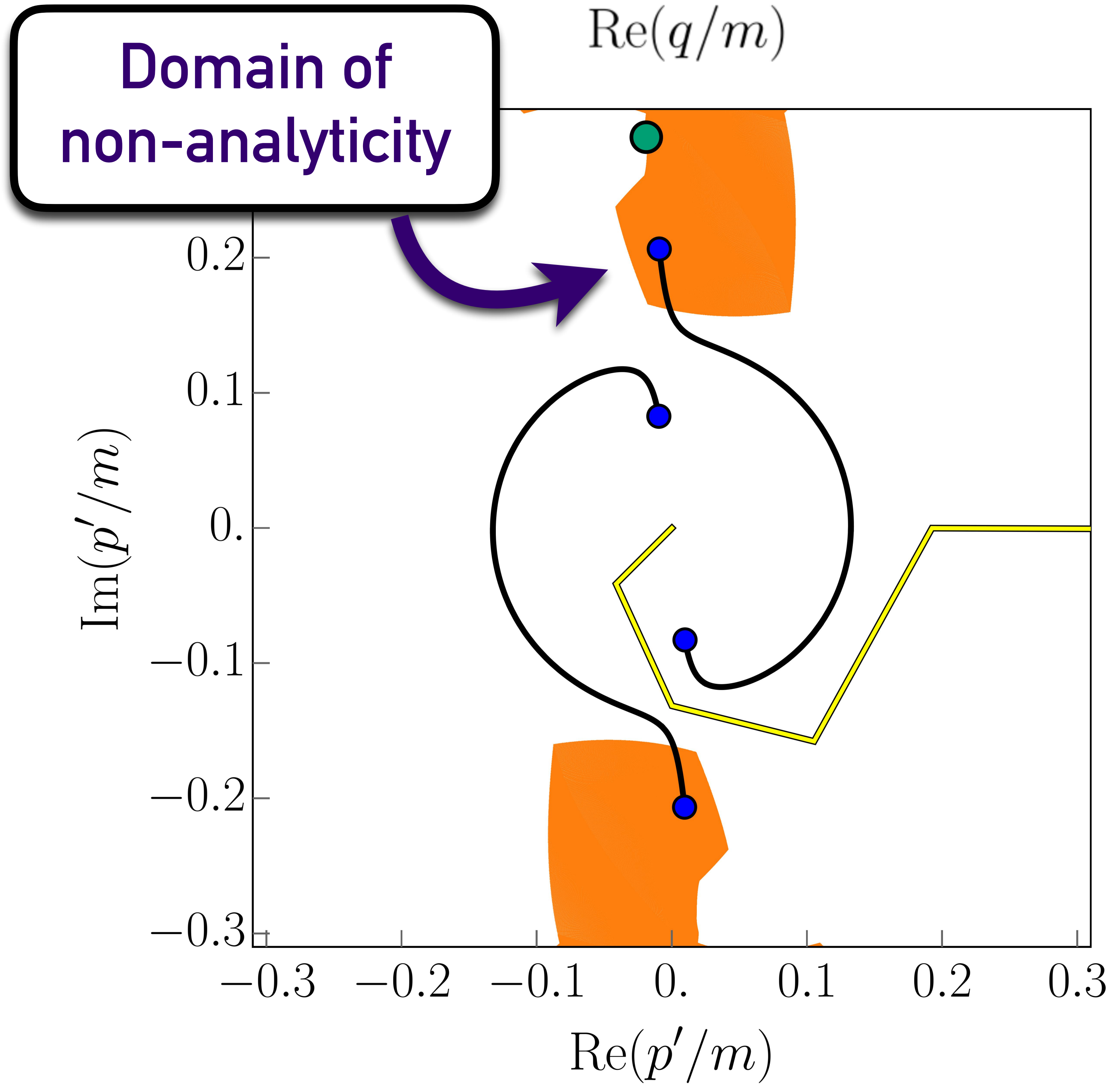
$$d(p',s,p) = -G(p',s,p) - \boxed{\int_0^{q_{\max}} \frac{dq\,q^2}{(2\pi)^2\omega_q}\, G(p',s,q)\mathcal{M}_2(q,s)}\, d(q,s,p)$$

Integration kernel

Extrapolation to the desired momentum $p' \longrightarrow q_{\varphi b}$

Addition of discontinuity to the integration kernel $\Delta(p',s,p) \propto 2\pi i$



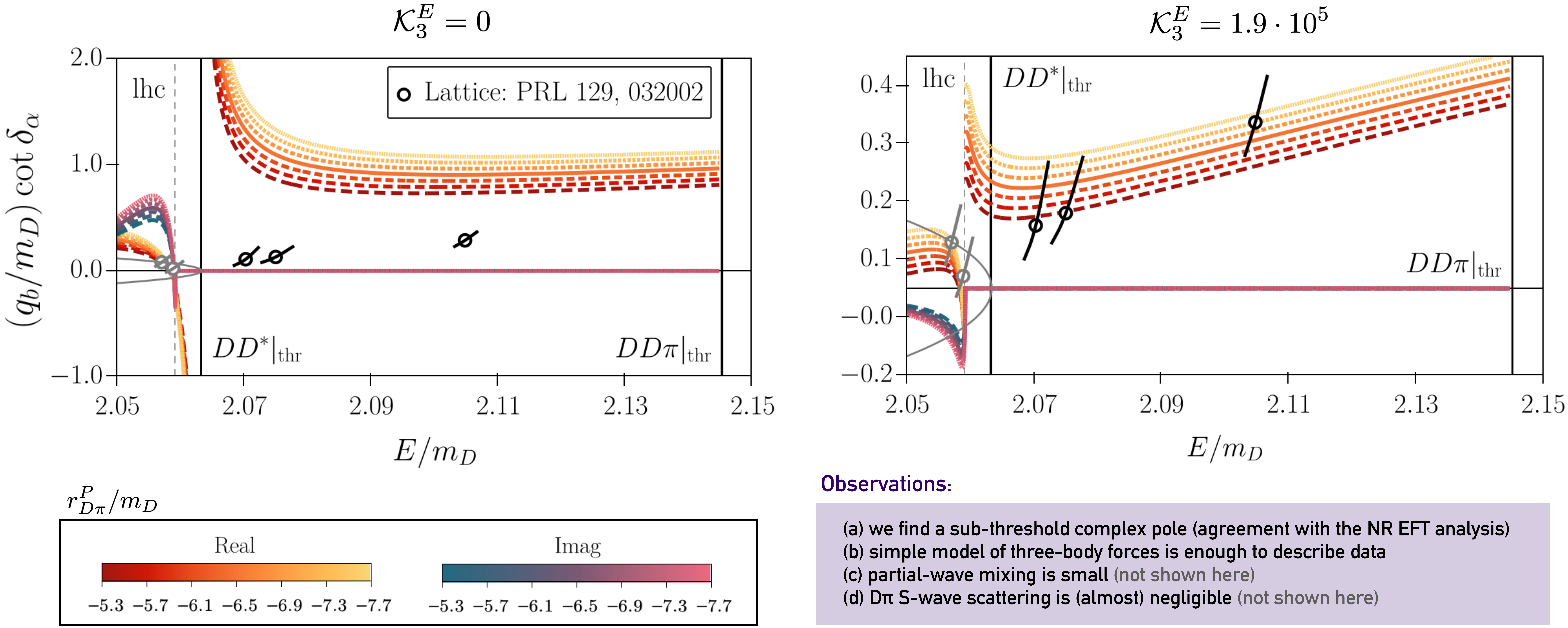


Partial-wave mixing amplitude

Blatt–Biederharn parametrization

$$\begin{bmatrix} \mathcal{M}_{DD^*}({}^3S_1|{}^3S_1) & \mathcal{M}_{DD^*}({}^3S_1|{}^3D_1) \\ \mathcal{M}_{DD^*}({}^3D_1|{}^3S_1) & \mathcal{M}_{DD^*}({}^3D_1|{}^3D_1) \end{bmatrix}$$

$$q_b^{-\ell'} \left[\mathcal{K}^{-1}_{DD^*}\right]_{\ell',\ell} q_b^{-\ell} = \begin{pmatrix} \cos(\epsilon) & -\frac{1}{q_b^2}\sin(\epsilon) \\ q_b^2 \sin(\epsilon) & \cos(\epsilon) \end{pmatrix} \begin{pmatrix} q_b \cot(\delta_\alpha) & 0 \\ 0 & q_b^5 \cot(\delta_\beta) \end{pmatrix} \begin{pmatrix} \cos(\epsilon) & q_b^2 \sin(\epsilon) \\ -\frac{1}{q_b^2}\sin(\epsilon) & \cos(\epsilon) \end{pmatrix}$$



Observations:

(a) we find a sub-threshold complex pole (agreement with the NR EFT analysis)
(b) simple model of three-body forces is enough to describe data
(c) partial-wave mixing is small (not shown here)
(d) Dπ S-wave scattering is (almost) negligible (not shown here)

Partial-wave mixing amplitude

Blatt$-$Biederharn parametrization

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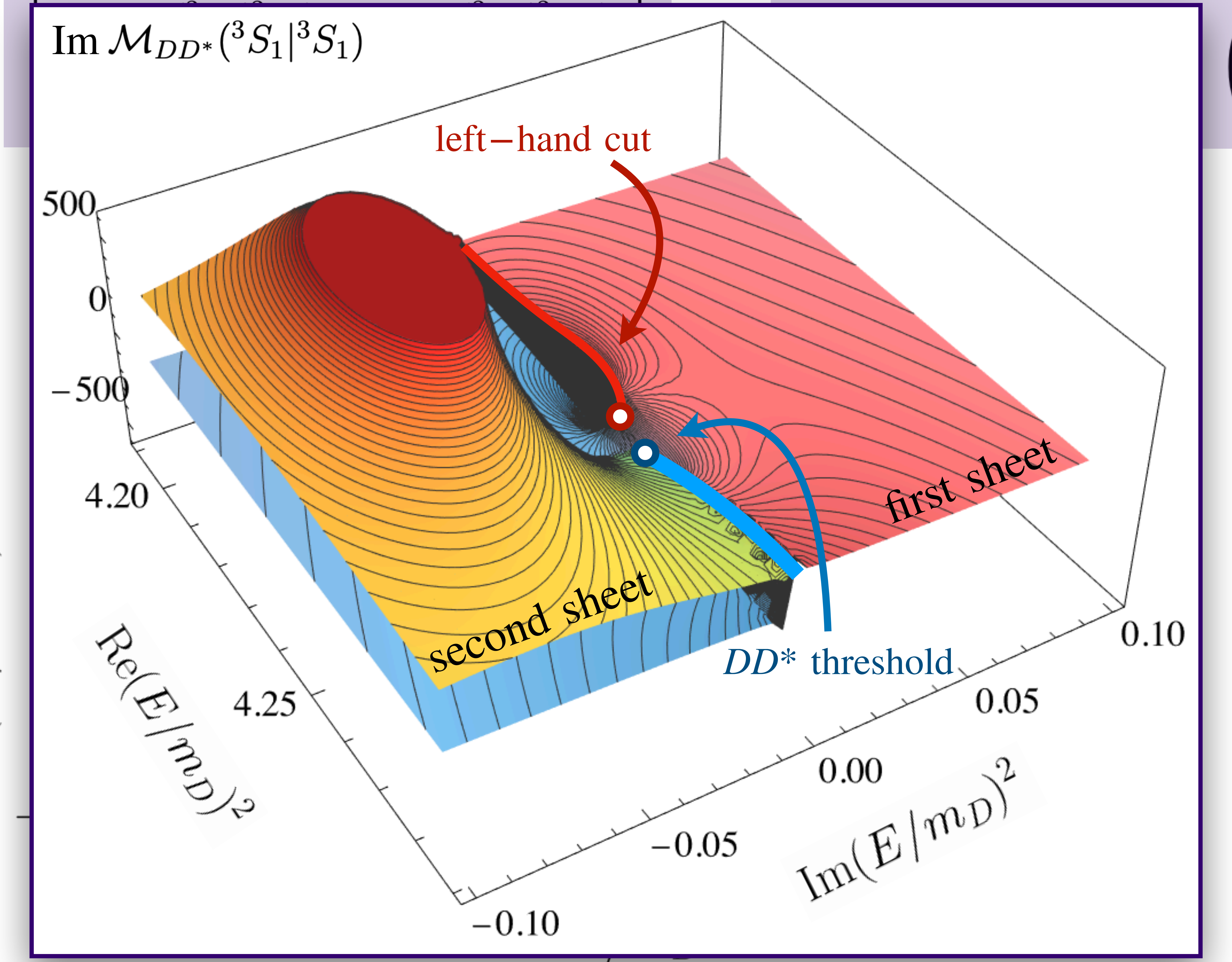

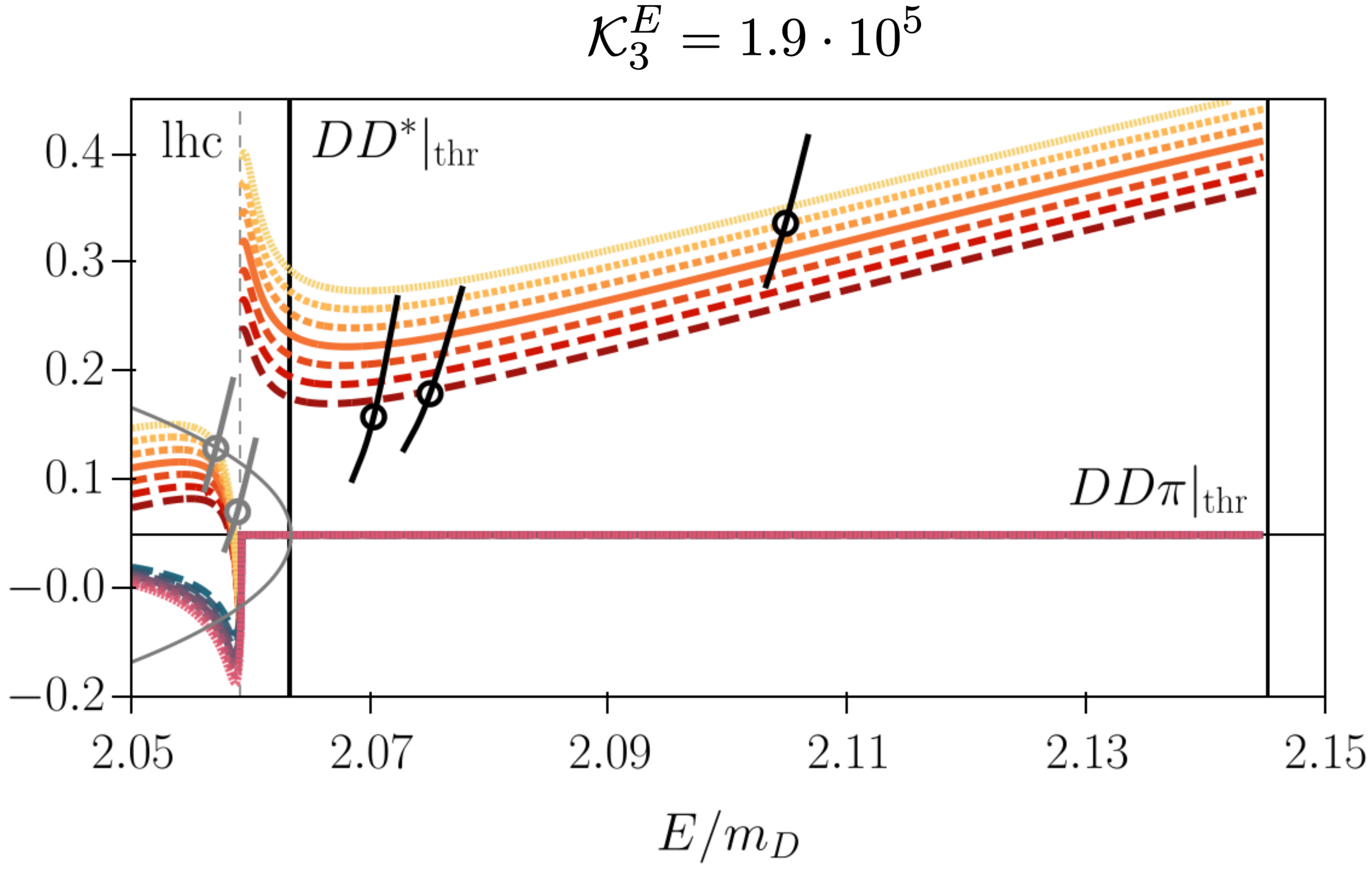

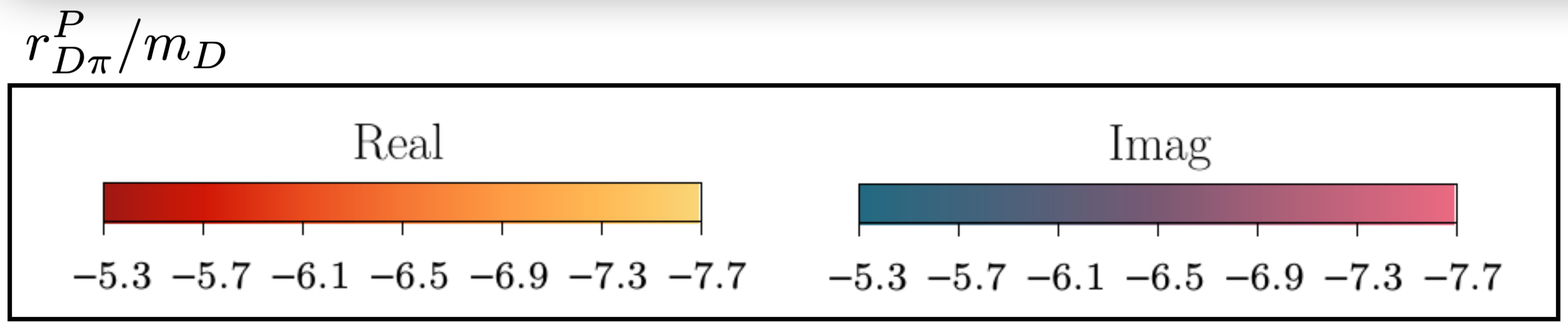

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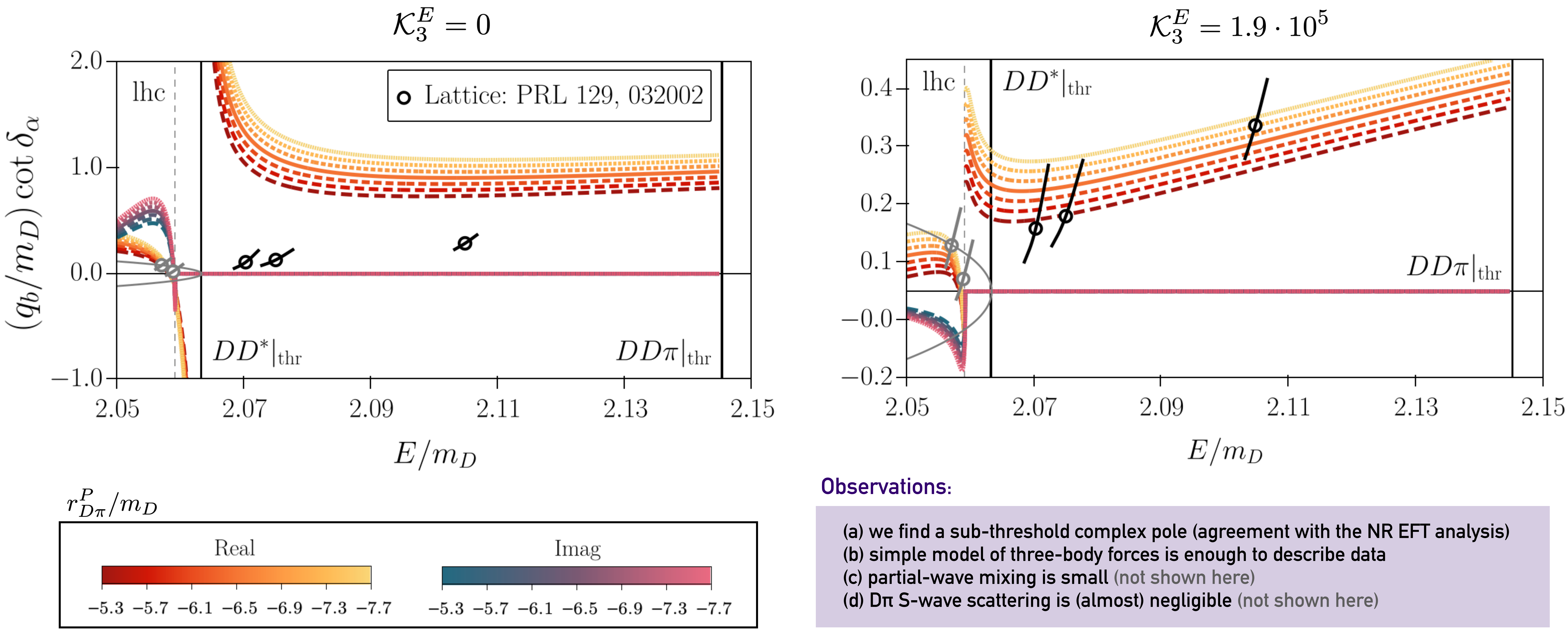

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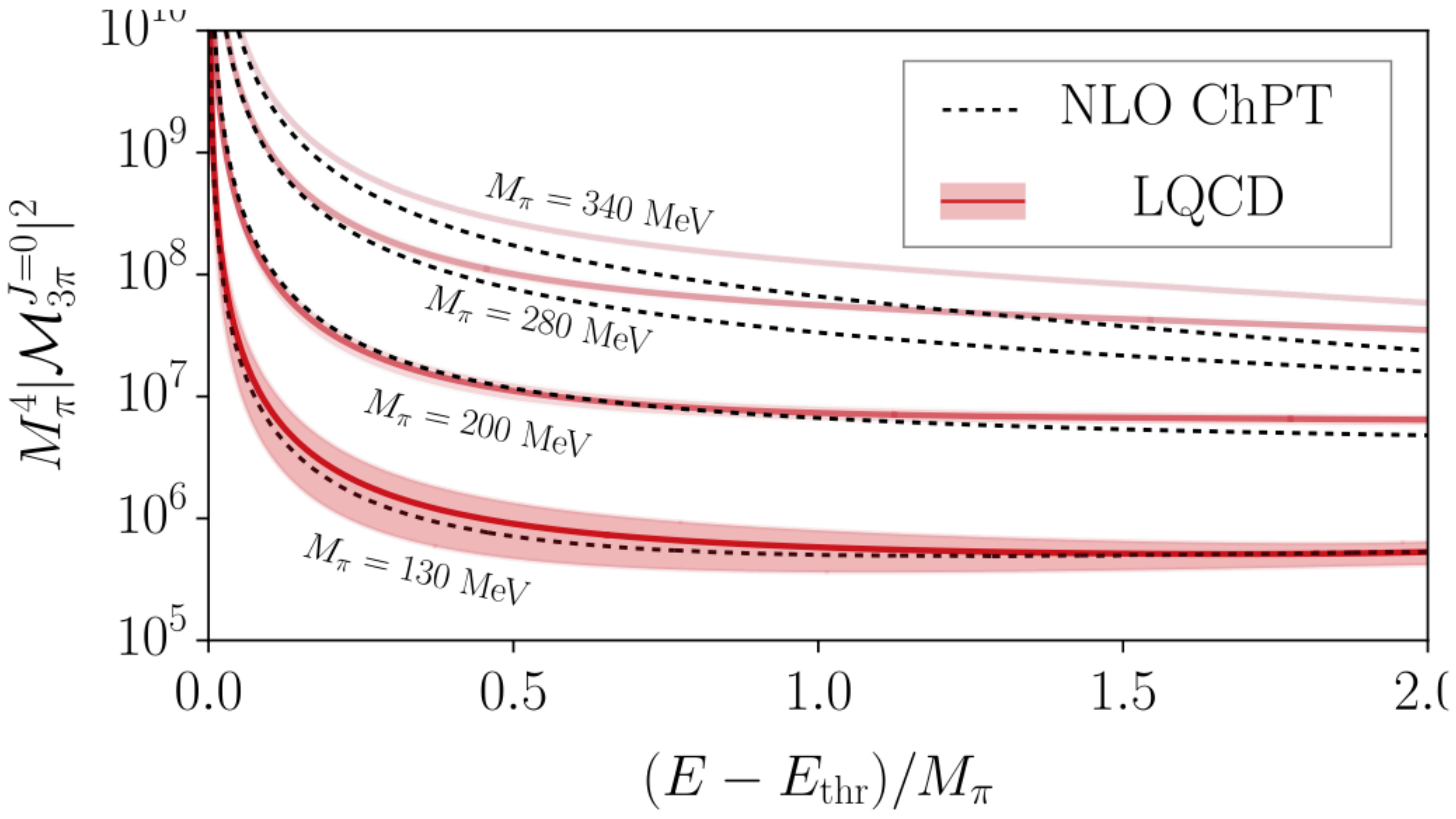
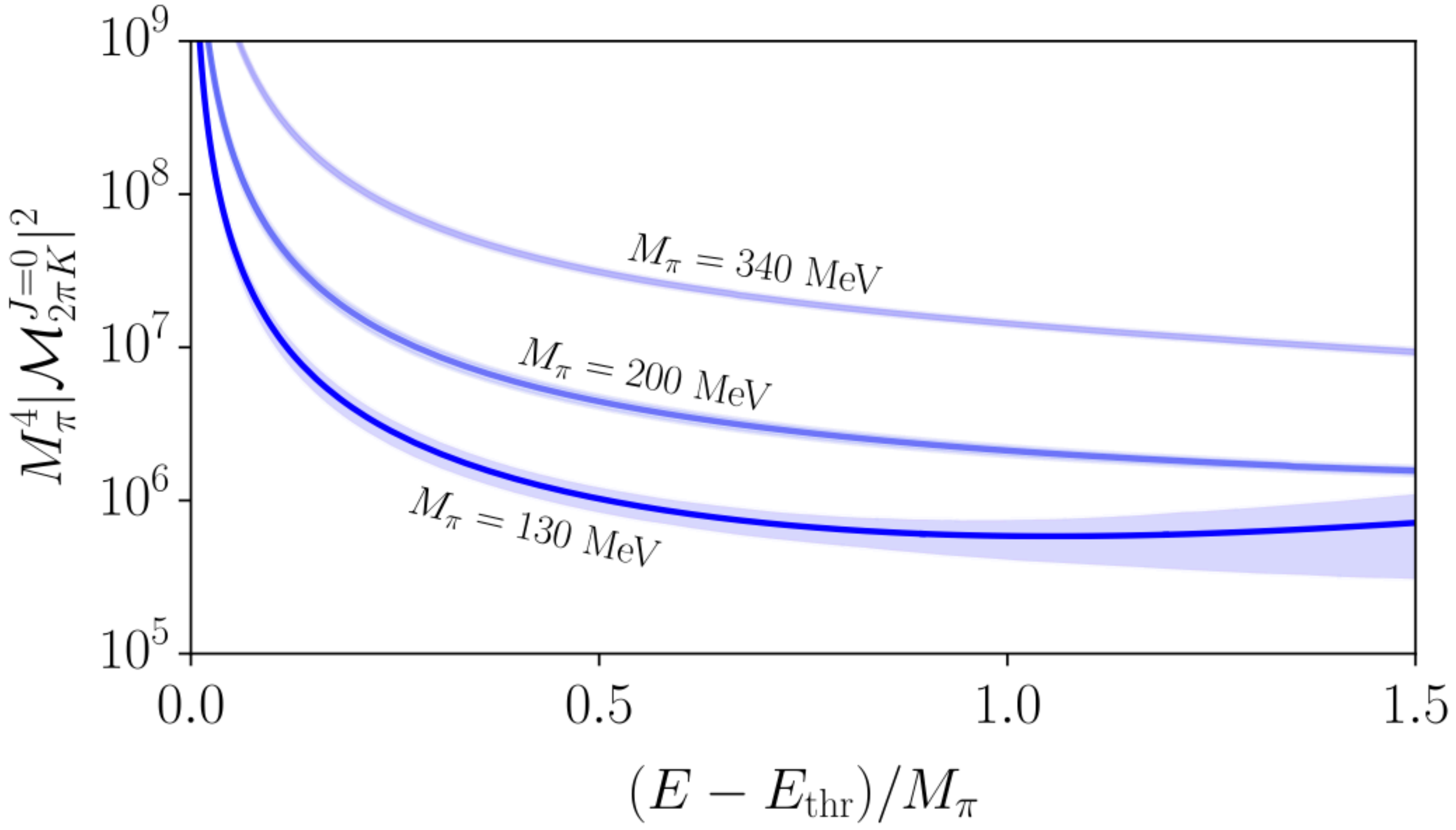
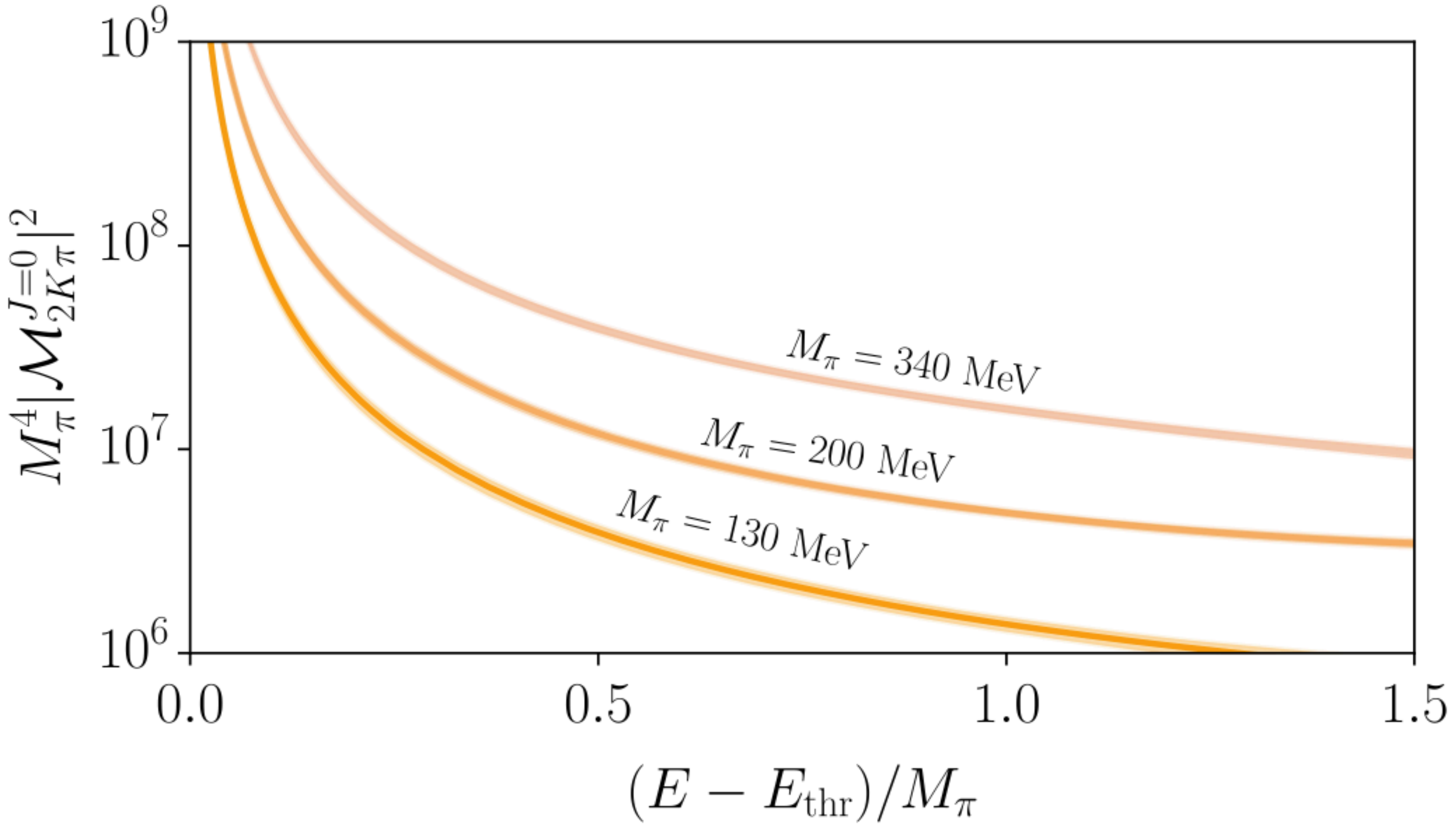


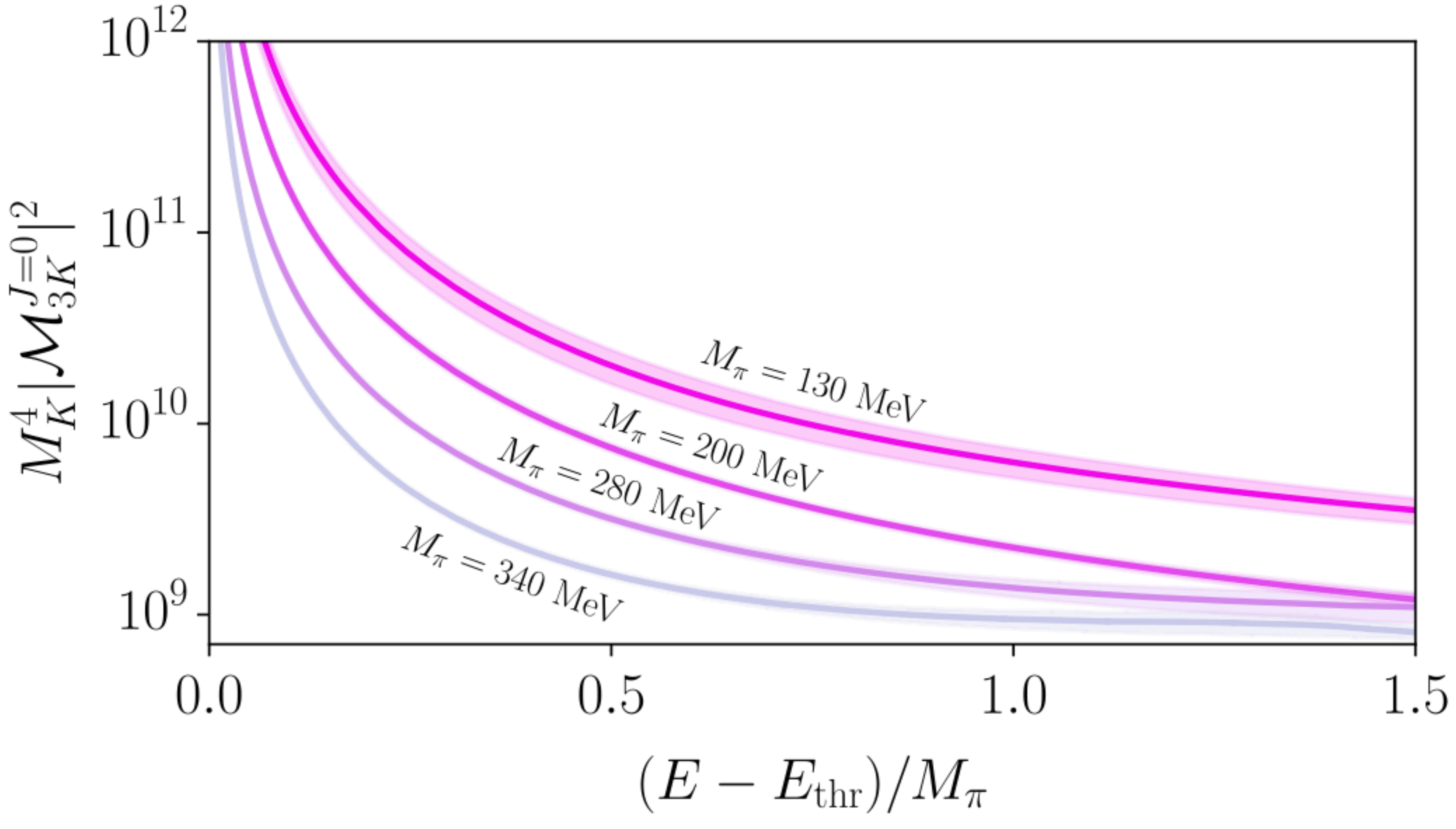
(a) 3π amplitude



(b) $\pi\pi K$ amplitude



(c) $KK\pi$ amplitude



(d) $3K$ amplitude