Towards first order causal relativistic spin hydrodynamics

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Based On : PRC **111** (2025), 034909 and arXiv: 2503.08428

Section Outline :

Statement of Problems

Relativistic Kinetic Theory :

Relativistic Spin Hydrodynamics with ERTA :

Relativistic Spin hydrodynamics with NRTA :

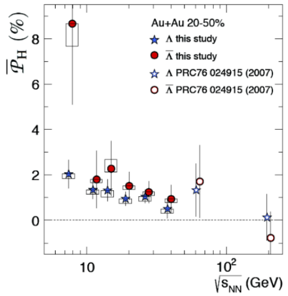
Summary and Outlook :

Particle Polarization :

- Spin polarization of hadrons in heavy-ion collisions, predicted in 2004.

  [Z. T. Liang, X. N. Wang, PRL **94**, 102301 (2005), PLB **629**, 20 (2005)]

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions,* Nature 548 62-65, 2017



First evidence of a quantum effect in (relativistic) hydrodynamics

Experimental evidence, [STAR Collaboration, Nature 548, 62 (2017), PRL 123, 132301 (2019), PRL 126, 162301 (2021)]

Theoretical models assuming equilibration of spin d.o.f. explains the data.

Particle Polarization :

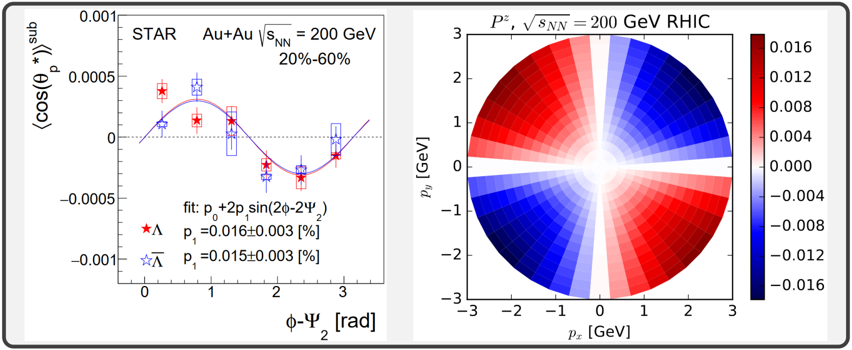


Figure 1: Observation (L) and prediction (R) of longitudinal polarization.

[Left: Phys. Rev. Lett. **123** 132301 (2019); Right: Phys. Rev. Lett. **120** 012302 (2018)]

- Inclusion of shear-induced polarization (SIP) solves the problem with extra constraints.
  [Fu et. al. Phys. Rev. Lett. 127, 142301 (2021); Becattini et. al. Phys. Lett. B 820 136519 (2021)]
- Still the resolution remains ambiguous.
  [Florkowski et. al., Phys. Rev. C 100, 054907 (2019); Phys.Rev.C 105, 064901 (2022)]
- Do dissipative forces play any role and solve the problem? [Sapna et. al., arXiv:2503.22552]

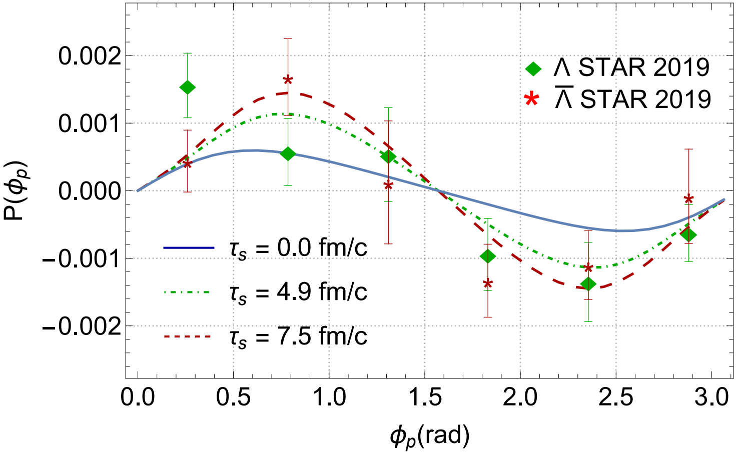


Figure 2: Polarization. $\tau_s = 7.5$ fm for $\bar{\Lambda}$ and, $\tau_s = 4.9$ fm for Λ. [S. Banerjee *et. al.*, accepted in PRC]

- τ_s is in agreement with [Hidaka et. al., PRC **109** (2024), 054909, Wagnar et. al., PRR **6**, (2024) 043103].
- Incorporation of dissipative effects is necessary.

Relativistic Dissipative Spin Hydrodynamics :

- We first note that spin-polarization originates from the rotation of fluid.

- Hence, we will have to deal with three conserved currents :

$$\partial_\mu N^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0, \qquad \partial_\lambda J^{\lambda,\mu\nu} = 0$$

where, $J = L + S$. Also, $L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$.

[See talk by Zbigniew Drogosz]

- For symmetric $T^{\mu\nu}$ we have, $\boxed{\partial_\lambda S^{\lambda,\mu\nu} = 0}$

$$N^\mu = N^\mu_{\text{eq}} + \delta N^\mu, \qquad T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \delta T^{\mu\nu}, \qquad S^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}_{\text{eq}} + \delta S^{\lambda,\mu\nu}$$

- The dissipative parts require microscopic description → Kinetic Theory.

Relativistic Dissipative Spin Hydrodynamics :

- Hydrodynamics is derived through an order-by-order gradient expansion:

$$X^{\mu_1\cdots\mu_\ell} = X_{(0)}^{\mu_1\cdots\mu_\ell} + X_{(1)}^{\mu_1\cdots\mu_\ell} + \cdots + X_{(n)}^{\mu_1\cdots\mu_\ell}$$

$$\begin{aligned} n = 0 \longrightarrow &\quad \text{Perfect fluid dynamics.} \\ n = 1 \longrightarrow &\quad \text{Navier – Stokes.} \\ n = 2 \longrightarrow &\quad \text{MIS – type theories.} \end{aligned} \quad \Big\} \text{Dissipative}$$

- Navier-Stokes theories lead to acausal theories.

  [W. A. Hiscock et. al., Annals Phys. **151** (1983) 466-496, PRD **31** (1985) 725-733, **35** (1987) 3723-3732]

- MIS theories introduce extra degrees of freedom.

- Casual first-order theories require generalized frame definition.

  [F. Bemfica et. al., PRD **98** (2018), 104064 PRL **122** (2019), 221602,
  P. Kovtun JHEP **10** (2019) 034, PRD **106** (2022), 066023]

Summary of the Problem :

- Construct relativistic spin hydrodynamics in general frame.
  - → With Extended RTA (ERTA).
  - → With Novel RTA (NRTA).

Section Outline :

Boltzmann Equation

- Spacetime Evolution

$$p^\mu \partial_\mu f + F^\mu \partial_\mu^{(p)} f = C[f]$$

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- Collision Kernel

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- Background Forces
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◦ Gravitational Forces : $F^{\mu} = -\Gamma^{\mu}_{\alpha\beta} p^{\alpha} p^{\beta}$.

◦ Electromagnetic Forces : $F^{\mu} = qF^{\mu\nu} p_{\nu}$.

◦ Mean-Field Forces : $F^{\mu} = M\,(\partial^{\mu} M)$.

Boltzmann Equation

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  - Electromagnetic Forces : $F^\mu = q F^{\mu\nu} p_\nu$.
  - Mean-Field Forces : $F^\mu = M\,(\partial^\mu M)$.

- Global Equilibrium solution:

$$f_0 = (\exp g + r)^{-1}$$

where, $r = 0, \pm 1$ and $g \equiv \sum_n \alpha_n \phi_n$

- Under local equilibrium, $\alpha_n \to \alpha_n(x^\mu)$

- For $2 \leftrightarrow 2$ collisions:

$$C[f] = \int dP\, dP'\, dK'\, \underbrace{\mathcal{W}_{\mathbf{k}\mathbf{k}' \leftrightarrow \mathbf{p}\mathbf{p}'}} \times \left(f_{\mathbf{p}} f_{\mathbf{p}'} - f_{\mathbf{k}} f_{\mathbf{k}'}\right)$$

Transition Amplitude:

The Collision Kernel

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Transition Amplitude: ⤴

- Chapman-Enskog Expansion:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}}\left(1 + \phi_{\mathbf{k}}\right)$$

The Collision Kernel

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Transition Amplitude:

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$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

- Linearized Collision Operator :

$$C[f] \to \hat{L}\phi_{\mathbf{k}} = \int dP\, dP'\, dK'\, \mathcal{W}_{\mathbf{k}\mathbf{k}' \leftrightarrow \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \times \left(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'}\right)$$

Collisional Invariants:

[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

The Conservation Laws

- Collisional invariants remain conserved during collisions.

- Each collisional invariant correspond to a conservation law.

- For a non-rotating, unpolarizable fluid :
  - $\phi \equiv 1 \longrightarrow$ Number Conservation.
  - $\phi \equiv E_{\mathbf{k}} \longrightarrow$ Energy Conservation.
  - $\phi \equiv \vec{k}\,(\sim k^{\langle\mu\rangle}) \longrightarrow$ Linear Momentum Conservation.

- Thus, a collision kernel should satisfy:

$$\hat{L}\,1 = 0, \qquad \hat{L}\,E_{\mathbf{k}} = 0, \qquad \hat{L}\,k^{\langle\mu\rangle} = 0.$$

- The linearized collision kernel satisfies the property:

$$\int dK\,\psi_{\mathbf{k}}\hat{L}\phi_{\mathbf{k}} = \int dK\,\phi_{\mathbf{k}}\hat{L}\psi_{\mathbf{k}}$$

$$\Longrightarrow \int dK\,\hat{L}\phi_{\mathbf{k}} = 0, \qquad \int dK\,k^{\mu}\,\hat{L}\phi_{\mathbf{k}} = \int dK\left(u^{\mu}E_{\mathbf{k}} + k^{\langle\mu\rangle}\right)\hat{L}\phi_{\mathbf{k}} = 0.$$

[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

New Collision Kernels

- We will work with two types linearized collision kernels.
  1. Extended Relaxation Time Approximation (ERTA):

$$\boxed{\hat{L}_{\rm ERTA}\,\phi_{\mathbf{k}} = -\frac{E_{\mathbf{k}}}{\tau_{\rm R}}\left(\phi_{\mathbf{k}} - \phi^*_{\mathbf{k}}\right) f_{0\mathbf{k}}}$$

where,

$$\phi^*_{1,\mathbf{k}} = -\frac{(k\cdot\delta u)}{T} + \frac{(E_{\mathbf{k}}-\mu)\,\delta T}{T^2} + \frac{\delta\mu}{T}$$

$$\delta u_\mu = u^*_\mu - u_\mu\,, \qquad \delta T = T^* - T, \qquad \delta\mu = \mu^* - \mu.$$

[D. Dash *et. al.*, PLB 831 (2022) 137202]

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[D. Dash *et. al.*, PLB 831 (2022) 137202]

  2. Novel Relaxation Time Approximation (NRTA):

$$\boxed{\hat{L}_{\text{NRTA}} \sim \left(-\mathbb{1} + \sum_{n=1}^{5} |\lambda_n\rangle\langle\lambda_n|\right)}$$

where, $|\lambda_n\rangle$ are degenerate, orthogonal eigenvectors of $\hat{L}_{\text{NRTA}}$.
[G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

Solving The Boltzmann Equation

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- Thus, the solution is:

$$\phi = \phi_{\rm h} + \phi_{\rm ih}$$

- The homogeneous part looks like :

$$\phi_{\rm h} = a + b_\mu k^\mu = a \cdot 1 + (b \cdot u) E_{\mathbf{k}} + b_{\langle\mu\rangle} k^{\langle\mu\rangle}.$$

- The coefficients, a, b_μ can be determined from frame and matching conditions.
- Subtracting the homogeneous part of the solution, gives freedom to choose the frame and matching conditions.

[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

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[S. R. de Groot *et. al.*, Relativistic Kinetic Theory, C. Cercignani *et. al.*, The Relativistic Boltzmann Equation]

- Both ERTA and NRTA have this feature.

$$\text{ERTA}: \quad a \to (E_{\mathbf{k}} - \mu)(\delta T/T^2) + (\delta\mu/T), \quad \text{and,} \quad b_\mu \to -(\delta u_\mu/T).$$

$$\text{NRTA}: \quad a \to \Phi_0, \quad \text{and,} \quad b_\mu \to \Phi_0^{\langle\mu\rangle}.$$

[D. Dash *et. al.*, PLB 831 (2022) 137202, G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

- Two popular approaches are considered :

1. Chapman-Enskog-like iterative solution:

$$\phi_{n,\mathbf{k}}\, f_{0\mathbf{k}} = \phi^{*}_{n,\mathbf{k}}\, f_{0\mathbf{k}} - \left(\frac{\tau_{\mathrm{R}}}{E_{\mathbf{k}}}\right)(k\cdot\partial)\, f_{(n-1)\,\mathbf{k}},$$

We will use this to solve the Extended RTA case.
[D. Dash *et. al.*, PLB 831 (2022) 137202]

2. Moment method:

$$\phi_{\mathbf{k}} = \sum_{n,\ell=0}^{\infty} \Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle}\, k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle}\, P_n^{(\ell)}(\beta E_{\mathbf{k}})$$

We will use this to solve the Novel RTA case. Here $P_n^{(\ell)}$ are orthogonal polynomials satisfying the property:

$$\frac{\ell!}{(2\ell+1)!!}\left\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})\,(k\cdot\Delta\cdot k)^{\ell}\, P_n^{(\ell)} P_m^{(\ell)}\right\rangle_{0\mathbf{k}} = A_n^{(\ell)}\delta_{nm},$$

$$A_n^{(\ell)} = \frac{\ell!}{(2\ell+1)!!}\left\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})\,(k\cdot\Delta\cdot k)^{\ell}\, P_n^{(\ell)} P_n^{(\ell)}\right\rangle_{0\mathbf{k}}$$

[G. S. Rocha *et. al.*, PRL 127 (2021), 042301]

Inclusion of Spin

- Phase-space is extended to include spin degrees of freedom :

$$f_{\mathbf{k}}(x,k) \longrightarrow f_s(x,k,s)$$

$$f_{0\mathbf{k}} \longrightarrow f_{0,s} = f_{0\mathbf{k}} \exp(s:\omega) \approx f_{0\mathbf{k}} \left[1 + \frac{1}{2}(s:\omega)\right] + \mathcal{O}\left(\omega^2\right)$$

- Homogeneous part for spin-polarizable particles :

$$\phi_{\mathrm{h}} = a + b_\mu k^\mu + c_{\mu\nu} s^{\mu\nu}$$

- The solutions are modified as :

1. Chapman-Enskog-like iterative solution (ERTA):

$$\phi_{n,s} f_{0,s} = \phi^*_{n,s}\, f_{0,s} - \frac{\tau_{\mathrm{R}}}{(u\cdot p)}\,(p\cdot\partial)\, f_{(n-1),s},$$

2. Moment method (NRTA):

$$\phi_s = \sum_{n,\ell=0}^{\infty} \left(\Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} + s_{\mu\nu}\Psi_n^{\mu\nu,\langle\mu_1\cdots\mu_\ell\rangle}\right) k_{\langle\mu_1}\cdots k_{\mu_\ell\rangle} P_n^{(\ell)}(\beta E_{\mathbf{k}})$$

[S. B., PRC **111** (2025), 034909]

Section Outline :

Statement of Problems

Relativistic Kinetic Theory :

Relativistic Spin Hydrodynamics with ERTA :

Relativistic Spin hydrodynamics with NRTA :

Summary and Outlook :

Field Redefinition - ERTA

- The thermodynamic variables (starred) are determined via field re-definitions :

$$\langle q_1\, \phi_s \rangle_0 + \langle \bar{q}_1\, \bar{\phi}_s \rangle_{\bar{0}} = 0, \qquad \langle q_2\, \phi_s \rangle_0 + \langle \bar{q}_2\, \bar{\phi}_s \rangle_{\bar{0}} = 0$$

$$\left\langle q_3\, k^{\langle\mu\rangle} \phi_s \right\rangle_0 + \left\langle \bar{q}_3\, k^{\langle\mu\rangle} \bar{\phi}_s \right\rangle_{\bar{0}} = 0, \qquad \langle q_4\, s^{\mu\nu} \phi_s \rangle_0 + \langle \bar{q}_4\, s^{\mu\nu} \bar{\phi}_s \rangle_{\bar{0}} = 0$$

  where we use the notations:

$$\langle (\cdots) \rangle_0 = \int dK\, dS\, (\cdots)\, f_{0\mathbf{k}}, \qquad \langle (\cdots) \rangle_{\bar{0}} = \int dK\, dS\, (\cdots)\, \bar{f}_{0\mathbf{k}}$$

- The thermodynamic variables are :

$$X^* = X + \delta X$$

  where, $X \equiv T, \mu, u_\mu, \omega_{\mu\nu}$.

- Then we find using the field-redefinitions (up to $\mathcal{O}(\partial)$):

$$\delta u_\mu = \beta \mathcal{C}_1 \left(\nabla_\mu \xi \right), \qquad \delta T = \mathcal{C}_2\, \theta, \qquad \delta\mu = \mathcal{C}_3\, \theta,$$

$$\delta\omega^{\mu\nu} = \mathcal{D}^{\mu\nu}_{\Pi}\, \theta + \mathcal{D}^{\mu\nu\gamma}_{n} \left(\nabla_\gamma \xi \right) + \mathcal{D}^{\mu\nu\alpha\beta}_{\pi}\, \sigma_{\alpha\beta} + \mathcal{D}^{\mu\nu\gamma\alpha\beta}_{\Sigma} \left(\nabla_\gamma \omega_{\alpha\beta} \right).$$

- Even with arbitrary frame and matching conditions, we had to use, $D \to \nabla$.

- While the first-order theory is still acausal, we can now have, $\tau_{\mathrm{R}}(x, p, s)$.

Constitutive Equations & Dissipative Currents - ERTA

- Under a general field re-definition the constitutive relations are given by :

$$N^{\mu} = (n_0 + \delta n)\, u^{\mu} + n^{\mu},$$
$$T^{\mu\nu} = (\mathcal{E}_0 + \delta\mathcal{E})\, u^{\mu}u^{\nu} - (\mathcal{P} + \delta\mathcal{P})\, \Delta^{\mu\nu} + 2h^{(\mu}u^{\nu)} + \pi^{\mu\nu},$$
$$S^{\lambda,\mu\nu} = S_0^{\lambda,\mu\nu} + \delta S^{\lambda,\mu\nu}.$$

- The dissipative currents are :

$$\delta n = \nu\,\theta \qquad \delta\mathcal{E} = e\,\theta, \qquad \delta\mathcal{P} = \rho\,\theta,$$
$$n^{\mu} = \kappa_n^{\mu\nu}\,(\nabla_{\nu}\xi)\,, \qquad h^{\mu} = \kappa_h^{\mu\nu}\,(\nabla_{\nu}\xi)\,,$$
$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$
$$\delta S^{\lambda,\mu\nu} = B_{\Pi}^{\mu\nu}\,\theta + B_n^{\mu\nu\gamma}\,(\nabla_{\gamma}\xi) + B_{\pi}^{\mu\nu\alpha\beta}\,\sigma_{\alpha\beta} + B_{\Sigma}^{\mu\nu\gamma\alpha\beta}\,(\nabla_{\gamma}\omega_{\alpha\beta})$$

- The expressions of the transport coefficients can be obtained assuming:

$$\tau_{\mathrm{R}}(x,p,s) = \tau_{\mathrm{eq}}(x)\,(\beta\cdot p)^{\ell_1}\,(u\cdot s)^{\ell_2}$$

[S. B., PRC **111** (2025), 034909]

Frame Invariant Transport Coefficients - ERTA

- The entropy production is given by,

$$\partial_\mu \mathcal{H}_\mu = -\beta \Pi \theta - \mathcal{Q}^\mu \left(\nabla_\mu \xi\right) + \beta \pi^{\mu\nu} \sigma_{\mu\nu} - \mathcal{S}^{\lambda\mu\nu} \left(\nabla_\lambda \omega_{\mu\nu}\right)$$

- The frame-invariant transport coefficients are:

$$\Pi = -\zeta\, \theta = \delta \mathcal{P} - \left(\frac{\chi_b}{\beta}\right) \delta \mathcal{E} + \left(\frac{\chi_a}{\beta}\right) \delta n,$$

$$\mathcal{Q}^\mu = \kappa^{\mu\nu} \left(\nabla_\nu \xi\right) = n^\mu - \left(\frac{n_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) h^\mu,$$

$$\pi^{\mu\nu} = 2\eta\, \sigma^{\mu\nu},$$

$$\mathcal{S}^{\lambda\mu\nu} = \beta_\Sigma^{\lambda\mu\nu\gamma\alpha\beta} \left(\nabla_\gamma \omega_{\alpha\beta}\right) = \frac{1}{2} \left(u^\rho D_\Sigma^{\alpha\beta\lambda\mu\nu} \delta S_{\rho,\alpha\beta} - \delta S^{\lambda,\mu\nu}\right)$$

[S. B., PRC **111** (2025), 034909]

Section Outline :

The Linearized collision kernel - NRTA

- We work with $\mu = 0$, as NRTA is not built for pair production and annihilation.
- We have 10 collisional invariants and hence : $\hat{L}\phi_s \equiv -\mathbb{1} + \sum_{n=1}^{10} |\lambda_n\rangle \langle\lambda_n|\phi_s\rangle$
- The orthonormal basis are :

$$|\lambda_1\rangle = \frac{E_{\mathbf{k}}}{\sqrt{\left\langle \left(E_{\mathbf{k}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0}},$$

$$|\lambda_{2-4}\rangle = \frac{k^{\langle\mu\rangle}}{\sqrt{(1/3)\left\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})\, k_{\langle\alpha\rangle} k^{\langle\alpha\rangle}\right\rangle_0}},$$

$$|\lambda_{5-7}\rangle = \sqrt{\frac{3}{\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})(\widetilde{s}\cdot\widetilde{s})\rangle_0}}\left[\widetilde{s}^{\,\mu} - \frac{\left\langle \left(E_{\mathbf{k}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^{\,\mu}\right\rangle_0}{\left\langle \left(E_{\mathbf{k}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{k}} - \frac{\left\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})\, k^{\langle\gamma\rangle}\widetilde{s}^{\,\mu}\right\rangle_0 k_{\langle\gamma\rangle}}{\left\langle (1/3)\,(E_{\mathbf{k}}/\tau_{\mathrm{R}})\, k^{\langle\alpha\rangle} k_{\langle\alpha\rangle}\right\rangle_0}\right],$$

$$|\lambda_{8-10}\rangle = \sqrt{\frac{3}{\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})(\widetilde{s}:\widetilde{s})\rangle_0}}\left[\widetilde{s}^{\,\mu\nu} - \frac{\left\langle \left(E_{\mathbf{k}}^2/\tau_{\mathrm{R}}\right)\widetilde{s}^{\,\mu\nu}\right\rangle_0}{\left\langle \left(E_{\mathbf{k}}^3/\tau_{\mathrm{R}}\right)\right\rangle_0} E_{\mathbf{k}} - \frac{\left\langle (E_{\mathbf{k}}/\tau_{\mathrm{R}})\, k^{\langle\mu\rangle}\widetilde{s}^{\,\mu\nu}\right\rangle_0 k_{\langle\mu\rangle}}{\left\langle (1/3)\,(E_{\mathbf{k}}/\tau_{\mathrm{R}})\, k^{\langle\alpha\rangle} k_{\langle\alpha\rangle}\right\rangle_0}\right].$$

where, $\widetilde{s}^{\,\mu} = \Delta^{\mu\alpha} u^{\beta} s_{\alpha\beta}$, and $\widetilde{s}^{\,\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta} s_{\alpha\beta}$.

- Then we get,

$$\hat{L}E_{\mathbf{k}} = 0, \qquad \hat{L}k^{\langle\mu\rangle} = 0, \qquad \hat{L}s^{\mu\nu} = 0.$$

[S. B., 2503.08428]

Field Redefinition - NRTA

- We work with $\mu = 0$, as NRTA is not built for pair production and annihilation.
- Now we use a new set of notations :

$$\langle(\cdots)\rangle_{\text{o}} = \int dK\, dS\, (\cdots)\, f_{\text{o}s}, \qquad \langle(\cdots)\rangle_{\text{o}\mathbf{k}} = \int dK\, (\cdots)\, f_{\text{o}\mathbf{k}}$$

- The thermodynamic variables (starred) are determined via field redefinitions:

$$\int dK dS\, q_1 \phi_s\, f_{\text{o}\mathbf{k}} = 0, \qquad \int dK dS\, q_2 k^{\langle\mu\rangle} \phi_s\, f_{\text{o}\mathbf{k}} = 0, \qquad \int dK dS\, q_3 s^{\mu\nu} \phi_s\, f_{\text{o}\mathbf{k}} = 0$$

- The solution takes the form:

$$\phi_s = \phi_{\mathbf{p}} + s : \psi_{\mathbf{p}}.$$

with

$$\phi_{\mathbf{p}} = \sum_{n \in \mathbb{S}_0^{(\ell)}} \sum_{\ell=0}^{\infty} \Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle} p_{\langle\mu_1} \cdots p_{\mu_\ell\rangle} P_n^{(0,\ell)}, \qquad \text{and,} \qquad \psi_{\mathbf{p}}^{\mu\nu} = \sum_{n \in \mathbb{S}_1^{(\ell)}} \sum_{\ell=0}^{\infty} \Psi_n^{\mu\nu,\langle\mu_1\cdots\mu_\ell\rangle} p_{\langle\mu_1} \cdots p_{\mu_\ell\rangle} P_n^{(1,\ell)},$$

$$\frac{\ell!}{(2\ell+1)!!} \int dP\, (E_{\mathbf{p}}/\tau_{\text{R}})\, (p \cdot \Delta \cdot p)^\ell\, P_m^{(j,\ell)} P_n^{(j,\ell)} f_{\text{o}\mathbf{p}} = A_n^{(j,\ell)} \delta_{mn}.$$

[S. B., 2503.08428]

Solution

- The homogeneous parts of the solution are :

$$\Phi_0 = -\sum_{n\in\mathbb{S}_0^{(0)}-\{0\}} \frac{\left\langle q_1\, P_n^{(0,0)}\right\rangle_{0\mathbf{p}}}{\left\langle q_1\, P_0^{(0,0)}\right\rangle_{0\mathbf{p}}}\Phi_n, \qquad \Phi_0^{\langle\mu\rangle} = -\frac{\left\langle q_2\,(p\cdot\Delta\cdot p)\,P_1^{(0,1)}\right\rangle_{0\mathbf{p}}}{\left\langle q_2\,(p\cdot\Delta\cdot p)\,P_0^{(0,1)}\right\rangle_{0\mathbf{p}}}\Phi_1^{\langle\mu\rangle},$$

$$\Psi_0^{\mu\nu} = -\left[\frac{\left\langle q_3 P_1^{(1,0)}\right\rangle_{0\mathbf{p}}}{\left\langle q_3 P_0^{(1,0)}\right\rangle_{0\mathbf{p}}}\right]\Psi_1^{\mu\nu}.$$

- Using these, one can obtain the expressions of all $\Phi_n^{\langle\mu_1\cdots\mu_\ell\rangle}$ and $\Psi_n^{\mu\nu,\langle\mu_1\cdots\mu_\ell\rangle}$.

[S. B., 2503.08428]

Section Outline :

Summary and Outlook :

- **Summary :**
  1. ERTA provides a simple solution to make the relaxation time dependent on p and s.
  2. ERTA allows us to have a general choice of frame and matching conditions.
  3. ERTA does not lead to first-order causal theory of spin hydrodynamics.
  4. NRTA gives the option of constructing first-order causal spin-hydrodynamics.
  5. NRTA cannot describe a system with pair production and annihilation.

- **Outlook :**
  1. Construction of spin-BDNK theory is required.
  2. Perform a linear mode analysis to verify the causality and stability of the theory.

Thank you.

Back Up :

- The frame-invariant transport coefficients are:

$$f_i \equiv \pi_i - \left(\frac{\partial \mathcal{P}_0}{\partial \mathcal{E}_0}\right)_{n_0} \varepsilon_i + \left(\frac{\partial \mathcal{P}_0}{\partial n_0}\right)_{\mathcal{E}_0} \nu_i,$$
$$l_i \equiv \gamma_i - \left(\frac{n_0}{\mathcal{E}_0 + \mathcal{P}_0}\right) \theta_i.$$

- Out of the 16 parameters, only three one-derivative transport coefficients $\zeta(f_1, f_2, f_3),\ \eta,\ \kappa(l_1, l_2)$ are independent.

Inclusion of Dissipation :

- Full collision kernel for Novel RTA with spin :
  [S. B. arXiv:2503.08428]

$$\hat{L}\phi_s = -\left(\frac{E_{\mathbf{p}}}{\tau_{\rm R}}\right) f_{0s}\Bigg\{\phi_s - \frac{\big\langle\big(E_{\mathbf{p}}^2/\tau_{\rm R}\big)\,\phi_s\big\rangle_0}{\big\langle\big(E_{\mathbf{p}}^3/\tau_{\rm R}\big)\big\rangle_0}E_{\mathbf{p}} - \frac{\big\langle\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\mu\rangle}\phi_s\big\rangle_0}{\big\langle(1/3)\,\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\alpha\rangle}p_{\langle\alpha\rangle}\big\rangle_0}p_{\langle\mu\rangle}$$
$$-\left[\langle(E_{\mathbf{p}}/\tau_{\rm R})\,\widetilde{s}_\mu\phi_s\rangle_0 - \frac{\big\langle\big(E_{\mathbf{p}}^2/\tau_{\rm R}\big)\,\widetilde{s}_\mu\big\rangle_0}{\big\langle\big(E_{\mathbf{p}}^3/\tau_{\rm R}\big)\big\rangle_0}\big\langle\big(E_{\mathbf{p}}^2/\tau_{\rm R}\big)\,\phi_s\big\rangle_0 - \frac{\big\langle\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\gamma\rangle}\widetilde{s}_\mu\big\rangle_0}{\big\langle(1/3)\,\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\alpha\rangle}p_{\langle\alpha\rangle}\big\rangle_0}\big\langle\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p_{\langle\gamma\rangle}\phi_s\big\rangle_0\right]$$
$$\times\left[\widetilde{s}^\mu - \frac{\big\langle\big(E_{\mathbf{p}}^2/\tau_{\rm R}\big)\,\widetilde{s}^\mu\big\rangle_0}{\big\langle\big(E_{\mathbf{p}}^3/\tau_{\rm R}\big)\big\rangle_0}E_{\mathbf{p}} - \frac{\big\langle\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\rho\rangle}\widetilde{s}^\mu\big\rangle_0}{\big\langle(1/3)\,\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\beta\rangle}p_{\langle\beta\rangle}\big\rangle_0}p_{\langle\rho\rangle}\right]\frac{1}{\langle(1/3)\,(E_{\mathbf{p}}/\tau_{\rm R})\,(\widetilde{s}\cdot\widetilde{s})\rangle_0}$$
$$-\left[\langle(E_{\mathbf{p}}/\tau_{\rm R})\,\widetilde{s}_{\mu\nu}\phi_s\rangle_0 - \frac{\big\langle\big(E_{\mathbf{p}}^2/\tau_{\rm R}\big)\,\widetilde{s}_{\mu\nu}\big\rangle_0}{\big\langle\big(E_{\mathbf{p}}^3/\tau_{\rm R}\big)\big\rangle_0}\big\langle\big(E_{\mathbf{p}}^2/\tau_{\rm R}\big)\,\phi_s\big\rangle_0 - \frac{\big\langle\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\gamma\rangle}\widetilde{s}_{\mu\nu}\big\rangle_0}{\big\langle(1/3)\,\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\alpha\rangle}p_{\langle\alpha\rangle}\big\rangle_0}\big\langle\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p_{\langle\gamma\rangle}\phi_s\big\rangle_0\right]$$
$$\times\left[\widetilde{s}^{\mu\nu} - \frac{\big\langle\big(E_{\mathbf{p}}^2/\tau_{\rm R}\big)\,\widetilde{s}^{\mu\nu}\big\rangle_0}{\big\langle\big(E_{\mathbf{p}}^3/\tau_{\rm R}\big)\big\rangle_0}E_{\mathbf{p}} - \frac{\big\langle\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\rho\rangle}\widetilde{s}^{\mu\nu}\big\rangle_0}{\big\langle(1/3)\,\big(E_{\mathbf{p}}/\tau_{\rm R}\big)\,p^{\langle\beta\rangle}p_{\langle\beta\rangle}\big\rangle_0}p_{\langle\rho\rangle}\right]\frac{1}{\langle(1/3)\,(E_{\mathbf{p}}/\tau_{\rm R})\,(\widetilde{s}:\widetilde{s})\rangle_0}\Bigg\},$$

Inclusion of Dissipation :

- Decomposition of conserved current :

$$T^{\mu\nu} = \varepsilon\, u^\mu u^\nu - P\,\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} = \langle p^\mu p^\nu \rangle\,,$$
$$S^{\lambda,\mu\nu} = u^\lambda \mathcal{S}^{\mu\nu} + \left(u^\mu \Delta^{\nu\lambda} - u^\nu \Delta^{\mu\lambda}\right)\Sigma + u^\mu \Sigma_{(s)}^{\langle\lambda\nu\rangle} - u^\nu \Sigma_{(s)}^{\langle\lambda\mu\rangle} + u^\mu \Sigma_{(a)}^{\lambda\nu} - u^\nu \Sigma_{(a)}^{\lambda\mu} + \Sigma^{\lambda,\mu\nu} = \left\langle p^\lambda s^{\mu\nu} \right\rangle,$$

- Transport Coefficients are given by :

$$\delta\varepsilon = \sum_{n\in\mathbb{S}_0^{(0)}} \Phi_n \left\langle E_{\mathbf{p}}^2 P_n^{(0,0)} \right\rangle_{0\mathbf{p}}, \qquad \delta P = -\sum_{n\in\mathbb{S}_0^{(0)}} \Phi_n \left\langle (1/3)\,(p\cdot\Delta\cdot p)\, P_n^{(0,0)} \right\rangle_{0\mathbf{p}},$$
$$q^\mu = \sum_{n\in\mathbb{S}_0^{(1)}} \Phi_n^{\langle\mu_1\rangle} \left\langle (1/3)\, E_{\mathbf{p}}\, (p\cdot\Delta\cdot p)\, P_n^{(0,1)} \right\rangle_{0\mathbf{p}}, \qquad \pi^{\mu\nu} = \Phi_0^{\langle\mu\nu\rangle} \left\langle (2/15)\,(p\cdot\Delta\cdot p)^2 \right\rangle_{0\mathbf{p}},$$
$$\delta\mathcal{S}^{\mu\nu} = \sum_{n\in\mathbb{S}_1^{(0)}} \Psi_n^{\mu\nu} \left\langle E_{\mathbf{p}} P_n^{(1,0)} \right\rangle_{0\mathbf{p}}, \qquad \delta\Sigma_{(s)}^{\langle\mu\nu\rangle} = u_\alpha \Delta_{\rho\gamma}^{\mu\nu} \Psi_0^{\alpha\rho,\langle\gamma\rangle} \langle (1/3)\,(p\cdot\Delta\cdot p)\rangle_{0\mathbf{p}},$$
$$\delta\Sigma_{(a)}^{\mu\nu} = \sum_{n\in\mathbb{S}_1^{(0)}} u_\alpha \Psi_n^{\alpha[\mu} u^{\nu]} \left\langle E_{\mathbf{p}} P_n^{(1,0)} \right\rangle_{0\mathbf{p}} + u_\alpha \Psi_0^{\alpha[\mu,\langle\nu\rangle]} \langle (1/3)\,(p\cdot\Delta\cdot p)\rangle_{0\mathbf{p}}, \qquad \delta\Sigma^{\lambda,\mu\nu} = \Psi_0^{\langle\mu\rangle\langle\nu\rangle,\langle\lambda\rangle} \langle (p\cdot\Delta\cdot p)\rangle_{0\mathbf{p}},$$

[S. B. arXiv:2503.08428]