Maximising survival time of an engine-equipped spacecraft between spatial hypersurfaces, as applied to the Schwarzschild spacetime

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Agenda

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Worldlines under the event horizon

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Generalisation

Some preliminary information

The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric line element of the two sphere. The coordinate ranges were originally
$t \in (-\infty, +\infty), r > 2M, \theta \in (0, \pi), \phi \in (0, 2\pi)$.

Killing vector fields

The Schwarzschild spacetime has multiple Killing vectors: a three-element group for rotational symmetry, and ∂_t. We use the spherical symmetry to constrain motion to the equator, and are left with two constants of geodesic motion:

- $-u^\mu (\partial_t)_\mu = \left(\frac{2M}{r} - 1\right)^{-1} u^t = e$, which is energy per unit mass measured at spatial infinity.
- $u^\mu (\partial_\phi)_\mu = \frac{u^\phi}{r^2} = L$, which is angular momentum per unit mass.

The problem and the rationale for research

“What would happen to me if I fell into a black hole?”

A common question asked of physicists by laymen or students.

Descriptions of the process abound, from spaghettification by tidal forces, to showing how the singularity is inevitable and what was previously a spatial coordinate becomes timelike.

However, in some textbooks...

- *Spacetime and Geometry* – “[...] once you enter the event horizon, you are utterly doomed. This is worth stressing; not only can you not escape back to [region above the event horizon], you cannot even stop yourself from moving in the direction of decreasing r, since this is simply the timelike direction. [...] **Since proper time is maximized along a geodesic, you will live the longest if you don’t struggle, but just relax as you approach the singularity.**”
- *Gravity* – aside to an exercise about longest possible proper time under event horizon: “One of the author’s students characterized this result as ‘The more you struggle, the shorter your life.’ ”

This isn't actually correct!

- Timelike geodesics maximalise proper time between two causally connected events, however...
- ...the final destination of our journey (the singularity) is not a single event, but a degenerate hypersurface!
- This means applying acceleration allows us to deviate from our initial geodesic connecting $(t_0, 2M, \theta_0, \phi_0)$ and $(t_1, r \to 0, \theta_1, \phi_1)$. Since the new terminal event $(t_2, r \to 0, \theta_2, \phi_2)$ can be completely different, the worldline can be longer.

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The Schwarzschild metric, adapted for calculation under the event horizon

$$ds^2 = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left(\frac{2M}{r} - 1\right) dt^2 + r^2 d\Omega^2, \quad (2)$$

with coordinates other than r having the range $0 < r < 2M$ and the other coordinates the same as in (1). Underneath the horizon in these coordinates, $-r$ is the time function (fulfills the requirement that ∇f is timelike past pointing). From now on we will reorder coordinates to the form (r, t, θ, ϕ) so that they conform to the metric signature $(-, +, +, +)$.

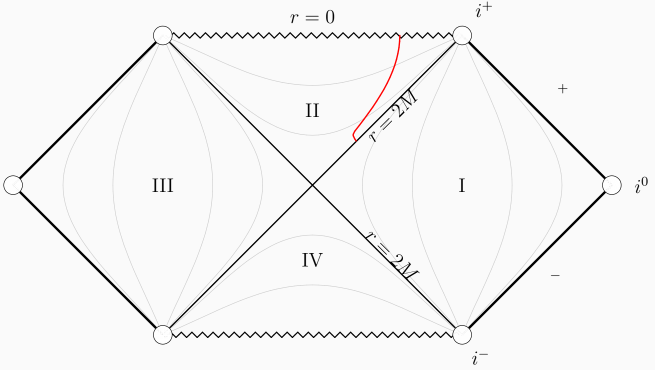


Figure 1: Carter-Penrose diagram for the maximally extended Schwarzschild spacetime. Region II is the region under the event horizon, which we will explore in this paper. The red worldline is a schematic example of an accelerated worldline which starts with some e_0 at the event horizon but, through acceleration by means of a rocket, soon starts travelling along a line of constant t.

Statement of the problem

Problem: An astronaut, through an unfortunate accident or a grisly death wish, crosses the event horizon of a M mass Schwarzschild black hole. At $r = 2M$, they have initial energy per unit mass e_0 and angular momentum per unit mass L_0. Their spacecraft is equipped with an engine capable of generating a 3-acceleration of magnitude α in the observer's instantaneous frame of reference.

How should the astronaut fire the engines to travel on the worldline that maximalises proper time among all possible worldlines under these constraints, and how much proper time τ do they have before their demise?

Why even bother?

- The error is extremely subtle, so it is instructive to realise how it works
- Cool example to explain to young physicists or laymen
- Strictly speaking, one could decide to fall into a black hole and perform science under the horizon – possible location of the final POTOR conference? :)
- This simple spacetime may shed some light on other spacetimes, as we will show at the end

Previous work

- Most textbooks – show that singularity is unavoidable, and the upper bound of survival time
- Burnett (1995) – description of upper bounds in general terms
- Lewis, Kwan (2007) – showing the misconception and numerical proof for radial infall
- Cieślik, Mach (2022) – description of geodesic motion in terms of Weierstrass elliptic integrals
- Toporensky, Zaslavskii (various between 2019-2023) – showing solutions for instantaneous acceleration, and maximalising the 'outside universe' time

Worldlines under the event horizon

Upper bound: the optimal geodesic observer

The proper time under the horizon is

$$\tau = \int_0^{2M} dr \left[e^2 + \left(1 + \frac{L^2}{r^2}\right)\left(\frac{2M}{r} - 1\right)\right]^{-\frac{1}{2}} . \tag{3}$$

This time is maximised when $e = L = 0$ for all $r < 2M$, and is equal to

$$\tau_{\rm opt} = \int_0^{2M} dr \left[\frac{2M}{r} - 1\right]^{-\frac{1}{2}} = M\pi \tag{4}$$

for an observer with 4-velocity $u^\mu_{\rm opt} = (-\sqrt{\frac{2M}{r} - 1}, 0, 0, 0)$. Note: no actual observer will ever fall into the black hole like this.

Equation of motion and normalisation conditions

The EOM is

$$a^\mu = u^\alpha u^\mu_{;\alpha} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta. \tag{5}$$

The normalisation conditions are

- orthogonality of 4-acceleration and 4-velocity $a^\alpha u_\alpha = 0$,
- normalisation of 4-acceleration $a^\alpha a_\alpha = \alpha^2$,
- normalisation of 4-velocity $u^\alpha u_\alpha = -1$.

Radial infall

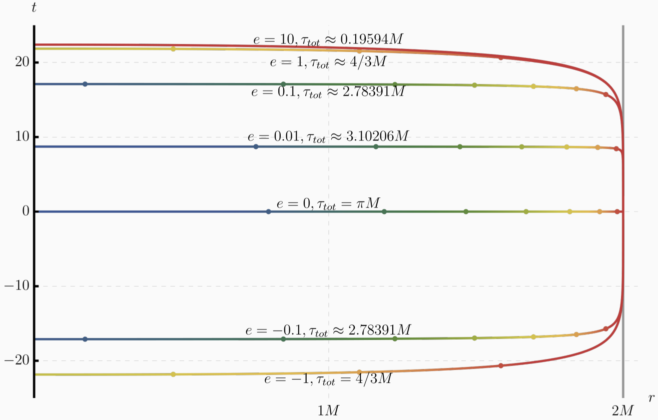


Figure 2: The geodesics of radial infall $L_0 = 0$ of massive particles into a black hole, starting $5 * 10^{-5}M$ under the event horizon. t is chosen to start at 0. Colors go from red at low proper times to blue at high proper times; the individual dots on each geodesics signify $\frac{\pi M}{8}$ intervals as measured on the clock carried by the falling observer. Clearly visible is the different total proper time τ_{tot} spent before hitting the singularity, depending on the initial constant e_0.

Taking the absolute derivative of the expression for e written as a tensor equation with respect to proper time gives

$$\frac{De}{d\tau} = \frac{D}{d\tau}(u_\mu(\partial_t)^\mu) = \frac{Du_\mu}{d\tau}(\partial_t)^\mu + \frac{D(\partial_t)^\mu}{d\tau}u_\mu. \tag{6}$$

Using the EOM (5) and the normalisation conditions, we get

$$\frac{de}{dr}u^r = -g_{tt}\alpha g_{tt}^{-1}u^r, \tag{7}$$

and ultimately

$$\frac{de}{dr} = -\alpha. \tag{8}$$

After integration of (8), we get

$$e(r) = -\alpha r + 2M\alpha + e_0. \tag{9}$$

Inserted back into (3):

$$\tau = \int_0^{2M} dr \left[(-\alpha r + 2M + e_0)^2 - \left(1 - \frac{2M}{r}\right) \right]^{-\frac{1}{2}}. \tag{10}$$

After substitution $u \to r^{-1}$:

$$\tau = \frac{1}{\sqrt{2M}} \int_{\frac{1}{2M}}^{\infty} \frac{du}{u\sqrt{u^3 + (2\alpha^2 M + 2\alpha e_0 + \frac{{e_0}^2}{2M} - \frac{1}{2M})u^2 - (2\alpha^2 + \alpha\frac{e_0}{2M})u + \frac{\alpha^2}{2M}}}. \tag{11}$$

Equation (11) can be represented as

$$\tau = \frac{1}{\sqrt{2M}} \int_{\frac{1}{2M}}^{\infty} \frac{du}{u\sqrt{(u-u_1)(u-u_2)(u-u_3)}}, \tag{12}$$

where u_i are functions of M, e_0, α.

To obtain a Carlson symmetric form elliptic integral, we only need to bring the lower bound to 0 by substitution $u' \to u - \frac{1}{2M}$:

$$\tau = \frac{2}{3} \frac{R_J\left(\frac{1}{2M} - u_1, \frac{1}{2M} - u_2, \frac{1}{2M} - u_3, \frac{1}{2M}\right)}{\sqrt{2M}}, \tag{13}$$

which is the analytic expression for the proper time along the accelerating curve from $r = 2M$ to $r = 0$ in infall with no angular momentum.

We are interested in reaching the optimal geodesic. Therefore, sometimes we should cease our acceleration before reaching $r = 0$. From the equation (9) we can obtain the moment of reaching the optimal geodesic

$$r_{\text{opt}} = 2M + \frac{e_0}{\alpha}, \tag{14}$$

and use a substitution $u' \to u - \frac{1}{r_{\text{opt}}}$ on (12) with a different lower bound $\frac{1}{r_{\text{opt}}}$ to obtain an integral that we can subtract from integral (13):

$$\tau(r_{\text{opt}}) = \frac{2}{3}\frac{R_J\left(\frac{1}{2M}-u_1, \frac{1}{2M}-u_2, \frac{1}{2M}-u_3, \frac{1}{2M}\right) - R_J\left(\frac{1}{r_{\text{opt}}}-u_1, \frac{1}{r_{\text{opt}}}-u_2, \frac{1}{r_{\text{opt}}}-u_3, \frac{1}{r_{\text{opt}}}\right)}{\sqrt{2M}}. \tag{15}$$

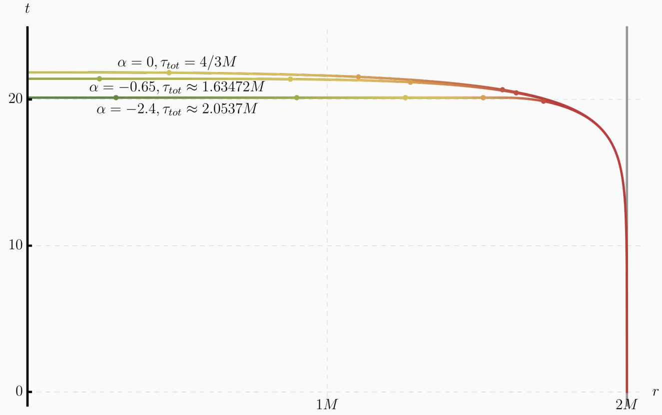
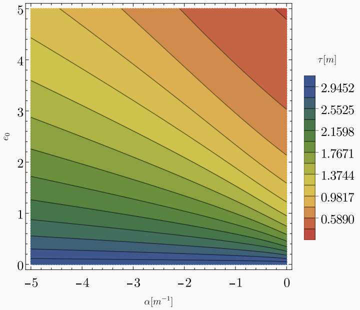
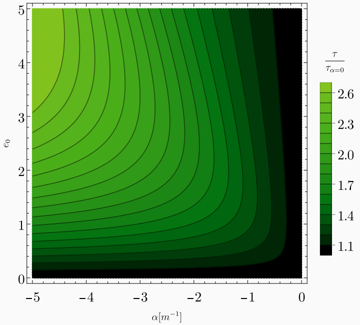


Figure 3: Curves of accelerated motion after infall with $L_0 = 0, e_0 = 1$ for various values of acceleration α. If we reach the optimal geodesic, we cease accelerating further, and this way, the more acceleration we have, the more we can live.



(a) Contour plot of proper time for radial infall depending on the parameters α and e_0. Colors retain meaning of proper time τ from worldline graphs. Contours are separated by $\frac{\pi}{16}$.



(b) Contour plot of the fraction $\frac{\tau}{\tau_{\alpha=0}}$ representing the relative gain thanks to using engines.

Figure 4: Plots of proper time and time gained for accelerated motion in a black hole with $M = 1m$.

Infall with angular momentum only

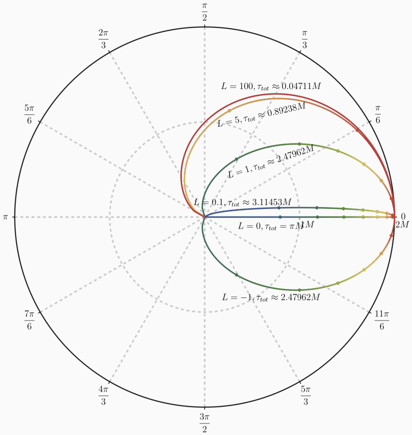


Figure 5: The geodesics of the infall with $e_0 = 0$ of massive particles into a black hole, starting $5 * 10^{-5}M$ under the event horizon. Colors of the individual curves go from red at early proper times to blue at late proper times; the individual dots on each geodesics signify $\frac{\pi M}{8}$ intervals as measured on the clock carried by the falling observer. Clearly visible is the different total proper time τ_{tot} spent under the event horizon before hitting the singularity, depending on the initial constant L_0.

Similar analysis as before gives

$$\frac{dL}{dr} = -\frac{\alpha r}{\sqrt{\frac{2M}{r} - 1}}. \tag{16}$$

This time, the metric components do not cancel neatly, and the behavior of L under acceleration is radically different. Integrating:

$$L(r) = L_0 + \alpha \left(\frac{1}{2} r(3M + r)\sqrt{\frac{2M}{r} - 1} + 3M^2 \arctan \sqrt{\frac{2M}{r} - 1} \right). \tag{17}$$

The equation for proper time does not admit an analytical integration, so we use numerical integration.

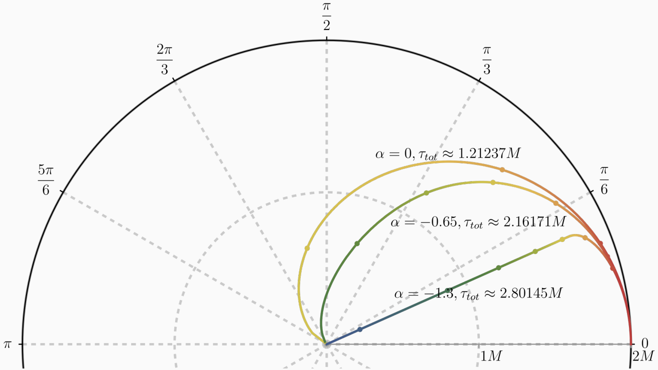
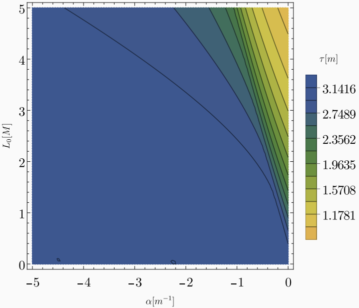
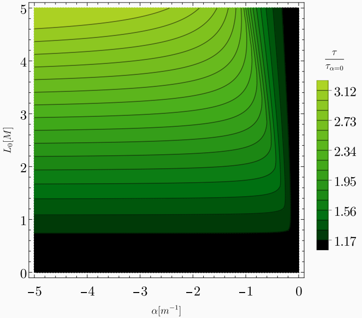


Figure 6: Curves of accelerated motion after infall with $L_0 = 3.5, e_0 = 0$ for various values of acceleration α. If we reach the optimal geodesic, we cease accelerating further, and this way, the more acceleration we have, the more we can live.



(a) Contour plot of proper time for infall depending on the parameters α and L_0. Colors retain meaning of proper time τ from worldline graphs. Contours are separated by $\frac{\pi}{16}$.



(b) Contour plot of the fraction $\frac{\tau}{\tau_{\alpha=0}}$ representing the relative gain thanks to using engines.

Figure 7: Plots of proper time and time gained for accelerated motion in a black hole with $M = 1m$.

General case

In the general case, we have a degree of freedom (angle at which we will fire the engines). To handle this, we instead prove how we need to thrust first.

Theorem (Schwarzschild black hole survival time maximisation principle)

Assuming a rocket equipped astronaut falls towards the singularity with some 3-velocity $\vec{u}$ in relation to the optimal observer frame, to move on the worldline that maximises proper time, they must continuously thrust with all available engine power with a 3-acceleration $\vec{a}$ directed opposite to $\vec{u}$ until their 4-velocity matches the optimal observer.

Proof.

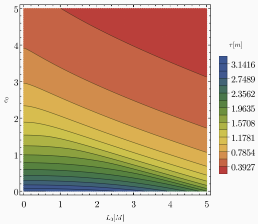
At each point along our accelerated worldline, we can find a frame attached to the optimal observer. One can boost between these frames using Lorentz boosts:

$$\gamma = g_{\mu\nu} u^\mu u^\nu_{\text{opt}} = \sqrt{\frac{L^2}{r^2} + 1 + e^2 \left(\frac{2M}{r} - 1\right)^{-1}}, \tag{18}$$

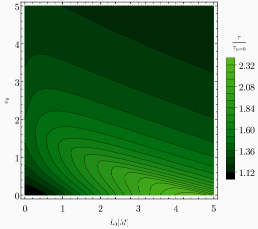
and the ratio of proper time for the optimal worldline to the proper time for the accelerated worldline is $\frac{d\tau_{\text{opt}}}{d\tau} = \gamma$. In the limit, one obtains $\tau = \int_0^{M\pi} \gamma^{-1} d\tau_{\text{opt}}$, which is maximalised when γ is minimised. That happens when

$$\frac{d\gamma}{d\tau_{\text{opt}}} = \gamma^3 \vec{v} \cdot \vec{a} \tag{19}$$

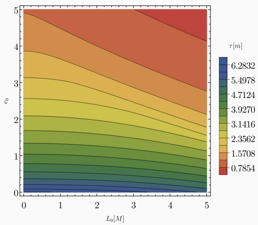
is negative and has maximal absolute value, which is when 3-acceleration is applied opposite to 3-velocity. $\square$



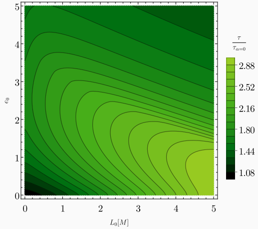
(a) $M = 1m$, τ.



(b) $M = 1m$, gain.



(c) $M = 2m$, τ.



(d) $M = 2m$, gain.

Figure 8: Contour plots of proper time and time gained for infall depending on the parameters e_0 and L_0 for various values of M.

Gains for ‘realistic’ situations

Potential values of parameters

- M: the larger, the better, as the behavior is dependent on the product $M\alpha$.
- α: using a booth filled with water, humans can survive up to $24g$; with liquid breathing, potentially $100g$ or even $1000g$ is possible.
- e_0 and L_0: either radial infall $e_0 = 1$, or e_0 around 0.8 and L_0 around $\sqrt{12M}$ are pretty realistic.

M	starting parameters	α	τ	gain
$M = 4.297 * 10^6 M_\odot$ (SagA*)	$e_0 = 1, L_0 = 0$	0	$28.6467s$	N/A
		$1000g$	$28.6537s$	0.00025
	$e_0 = 0.8, L_0 = \sqrt{12}M$	0	$16.156s$	N/A
		$1000g$	$16.160s$	0.00025
$M = 6.5 * 10^9 M_\odot$ (M87*)	$e_0 = 1, L_0 = 0$	0	$12.037h$	N/A
		$100g$	$12.491h$	0.0377
		$1000g$	$16.139h$	0.341
	$e_0 = 0.8, L_0 = \sqrt{12}M$	0	$6.788h$	N/A
		$100g$	$7.065h$	0.0407
		$1000g$	$12.537h$	0.847
$M = 10^{11} M_\odot$ (PhoenixA)	$e_0 = 1, L_0 = 0$	0	$7.7160d$	N/A
		$10g$	$8.1671d$	0.0585
		$1000g$	$15.065d$	0.952
	$e_0 = 0.8, L_0 = \sqrt{12}M$	0	$4.352d$	N/A
		$10g$	$4.633d$	0.0646
		$1000g$	$15.407d$	2.541

Generalisation

It turns out the problem is solvable this way because of the symmetries in the Schwarzschild spacetime in equatorial motion.

The theorem generalises to other manifolds with high symmetry (though details are too long for a 15 minute talk!).

This allows use as a more general tool when considering proper time for accelerated worldlines.

Some of the spacetimes where this is usable include:

- Minkowski spacetime for a foliation $t = \mathit{const}$.
- Big Bang spacetime, perhaps with a hypothetical cataclysm like a Big Rip as the final hypersurface.
- Reissner–Nordström spacetime constrained to equatorial motion between outer and inner horizons.
- The Bianchi type I cosmological models and the Kasner metric.

Summary

- „Do not go gentle into that good night."
- In large black holes and assuming some technological leaps, survival time can be extended by a lot for large black holes
- Research ended up revealing a principle useful in other contexts.
- Perhaps the venue is not so good for POTOR – fewer jogging trails :)
- Article up on arxiv: `https://arxiv.org/abs/2405.03510`

Thank you!