Lorentz symmetry violating Lifshitz-type field theories

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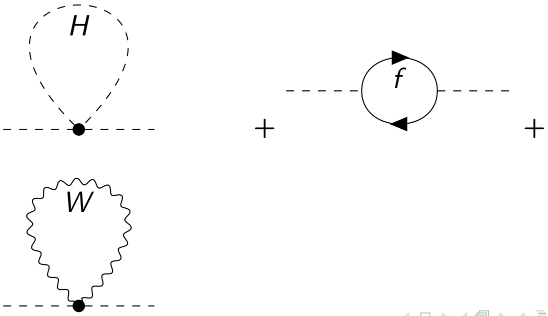
[1] E.Rizza, D.Zappala', "Lorentz symmetry violating Lifshitz-type field theories" Modern Physics Letters A Vol. 37, No. 30, 2250203 - arXiv:2209.11060

Outline

- Introduction & Motivations
- Scalar fields
- Fermionic fields
- Gauge fields
- Hierarchy problem
- Conclusions

- The perturbative expansion of quantum field theories inevitably leads to the appearance of Feynman diagrams containing loops which diverge at large k
- Example: main contributions to the full scalar propagator of the Higgs come from the one-loop corrections respectively given by the Higgs self-interaction, the fermionic loop and the loop of a Gauge field:



- Consider the action for scalar theory with interaction $g_4\phi^4$

$$\mathcal{S} = \int d^3x dt \left((\partial_\mu \phi)^2 + m^2\phi^2 + g_4\phi^4 \right)$$

- Cut-off regularization method by introducing a Λ cutoff $\rightarrow$ evaluate the divergent character of tadpole graph (first order):

$$-i\Pi_s = -i\frac{g_4}{32\pi^2}\left(\Lambda^2 - m^2 \log \frac{\Lambda^2 + m^2}{m^2} \right)$$

- $G(k_0, \vec{k}) = \frac{1}{k_0^2 - \vec{k}^2 - m^2}$
- Renormalization procedure through the introduction of counter terms

Introduction & Motivations

- Smoothen the UV divergences and improve the renormalizability of a QFT by an enhanced number of space and time derivatives of the field
- Presence of a novel Renormalization Group fixed point known in condensed matter as Lifshitz points

R. Hornreich, M. Luban, and S. Shtrikman, Phys.Rev.Lett. 35, 1678 (1975).

- Especially important in the context of renormalizable anisotropic gravitational theory $\rightarrow$ Horava-Lifshitz gravity

P. Horava, Phys. Rev. D 79, 084008 (2009). [arXiv:0901.3775].

- We divide the coordinates in two groups, $\vec{x} = (\vec{x}_{\|}, \vec{x}_{\perp})$ that respectively belong to a m and $d - m$ dimensional (with $m \leq d$) subspace whose scaling behavior is different.

$$\mathcal{O}(l^{\theta}\vec{x}_{\|}, l\vec{x}_{\perp}) = l^{-\Delta_{\mathcal{O}}}\mathcal{O}(\vec{x})$$

with θ different from unity.

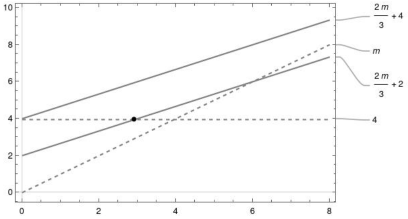
- Anisotropic scaling law near the ultraviolet fixed point

$$t = b^{z}t', \vec{x} = b\vec{x}'$$

- Choice $z = 3$ dictated by the requirement of maximizing the powers of momentum in the propagator and maintain the presence of a Gaussian-Lifshitz point.

- The upper critical dimension (UCD) $d_u(m)$ obtained by $[\lambda_4] = 0 \rightarrow d_u(m) = 4 + \frac{2m}{3}$.
- The lower critical dimension (LCD) $d_l(m)$ obtained by $[\phi] = 0$ $\rightarrow d_l(m) = 2 + \frac{2m}{3}$.
- In the region between these two lines of the (m, d) plane we can find non-trivial Lifshitz points.

d



m

Figure: *md* plane showing upper and lower critical dimension for $z = 3$.

- The derivative sector of the action:

$$S_D = \int d^3x dt \left\{ \frac{1}{2} (\partial_t \phi)^2 - \sum_{k=1}^{z} \sum_{i=1}^{3} \frac{a_k}{2M^{2k-2}} \left([\partial_i]^k \phi \right)^2 \right\}$$

where z is the anisotropic critical exponent.

- The dispersion relation reads:

$$E^2 = \vec{k}^2 \left[a_1 + a_2 \left(\frac{k}{M} \right)^2 + a_3 \left(\frac{k}{M} \right)^4 \right] + g_2 M^2$$

- Astrophysical observations pose the energy scale of Lorentz violating effects above 10^{10} GeV

J. Ellis, N. Mavromatos, D.V. Nanopoulos, A.S. Sakharov Astron. Astrophys. 402, 409 (2003). [arXiv:astro-ph/0210124].

- Theories that contain more than two time derivatives are affected by the Ostrogradski instability, associated with violation of unitarity

R. P. Woodard, Lect. Notes Phys. 720, 403 (2007). [astro-ph/0601672].

Scalar Fields

- The scaling dimension of the field $[\phi]_s = \frac{3-z}{2}\Big|_{z=3} = 0$ determines the structure of the interaction sector:

$$S_I = -\int d^3xdt \sum_{n=2}^{\infty} \frac{g_n \phi^n}{n! M^{(n-4)}} +$$

$$+ \sum_{k=1}^{3} \sum_{i=1}^{3} \left[\sum_{m=1}^{\infty} \frac{w_{m,k} \phi^m}{M^{2k-2+m}} \right] \left([\partial_i]^k \phi \right)^2$$

- We neglect terms like:

$$w_{ms} \phi^m (\partial_i^s \phi \partial_i^s \phi)$$

- Two interacting scalar fields show a potentially detectable difference of their speed of light where logarithmic corrections $\propto w_{ms}$ modify the "speed of light" a_1.
- Experimentally forbidden $\rightarrow$ unless one fixes a fine-tuning between the bare couplings but unnatural.

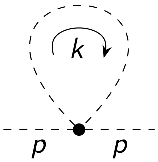
R. Iengo, J. G. Russo, and M. Serone, Renormalization group in Lifshitz-type theories, JHEP 11

- The degree of divergence of a loop diagram D_Λ is:

$$D_\Lambda = 6\left(1 - \sum_n V_n\right)$$

where V_n is the number of vertices with n legs (proportional to g_n).

- Diagrams (a) are those with one vertex, an arbitrary number of external legs, and an arbitrary number of tadpoles:



- Propagator $\rightarrow G(k_0, \vec{k}) = \frac{1}{k_0^2 - a_3\vec{k}^6 - a_2\vec{k}^4 - a_1\vec{k}^2 - g_2}$

Scalar Fields

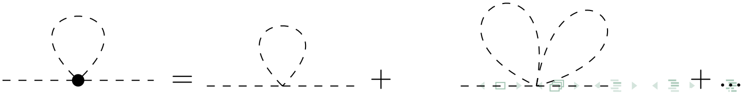
- It can be regularized by adopting the non-Lorentz invariant cutoff on the modulus of the tri-momentum.
- We integrate the tadpole between two momentum scales Λ_1 and Λ_2 , with $\Lambda_1 \gg \Lambda_2 \gg M$

$$I_1\left(\Lambda_1, \Lambda_2, M\right) = \frac{M^2}{\sqrt{a_3}(2\pi)^2} \ln\left(\frac{\Lambda_1}{\Lambda_2}\right) + O\left(\frac{M^4}{\Lambda_2^2}, \frac{M^2\Lambda_2^2}{\Lambda_1^2}\right)$$

- The quantum corrections of a generic coupling g_n yield the following series:

$$g_n\left(\Lambda_2\right) = \sum_{m=0}^{\infty} \frac{g_{n+2m}\left(\Lambda_1\right)}{m!} \left[\frac{I_1\left(\Lambda_1, \Lambda_2, M\right)}{2M^2}\right]^m$$

- Divergent diagram series for the 4-point vertex:



Scalar Fields

- If one assumes that the theory has only one coupling, i.e. $g_n = g$ for all n:

$$g(\Lambda_2) = g(\Lambda_1)\exp\left[\frac{I_1(\Lambda_1, \Lambda_2, M)}{2M^2}\right] \simeq g(\Lambda_1)\left(\frac{\Lambda_1}{\Lambda_2}\right)^{1/(8\pi^2)}$$

- In the limit $\Lambda_1 \to \infty$, at fixed $g(\Lambda_2)$, $g(\Lambda_1) \to 0$
- The theory shows a Liouville-like potential and quantum corrections are exactly summable, giving an asymptotically free theory.

Fermionic Fields

- The higher derivative part of the $3+1$ dimensional action, with $z=3$, is

$$S_F = \int d^3x dt \bar{\psi} \left[i\gamma^0 \partial_0 - \left(b_1 + \frac{\partial_j \partial^j}{M^2} \right) (i\gamma^i \partial_i) - m_f \right] \psi$$

- The scaling dimension of the fermion field turns out to be equal to its canonical dimension, $[\psi]_s = 3/2$.
- If we assume a Yukawa-like interaction with the scalar excitations, the renormalizable interaction sector of the action is:

$$S_Y = -\int d^3x dt \sum_{n=1}^{\infty} y_n \frac{\bar{\psi}\psi\phi^n}{n! M^{n-1}}$$

- We did not include $2\mathrm{n}$-fermion vertices. The only vertex that is renormalizable is the 4 -fermion vertex. Through Hubbard-Stratonovich transformation reduce to a Yukawa-like interaction.

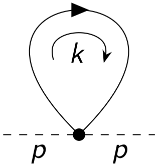
Fermionic Fields

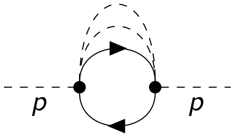
- The degree of divergence of the diagrams generated by S_Y yields:

$$D_\Lambda = 6 - 3\sum_n Y_n - \frac{3}{2}E_f$$

where Y_n is the number of vertices proportional to y_n and E_f is the number of external fermionic legs.

- Diagrams (b) with a fermionic tadpole if $E_f = 0$ and $\sum_n Y_n = 1$ and Diagrams (c) with two vertices, a fermionic loop and n scalar loops $E_f = 0$ and $\sum_n Y_n = 2$:





- Yukawa sector where all couplings are equal: $y = y_n$, the sum of all diagrams contributing to the renormalization of y gives:

$$y(\Lambda_2) = y(\Lambda_1) \exp\left[\frac{I_1(\Lambda_1, \Lambda_2, M)}{2M^2}\right] \equiv y(\Lambda_1)\,\mathcal{E}$$

i.e. the coupling y is asymptotically free.

- The renormalization of the scalar sector is more involved. By keeping different g_n for different n and by retaining the leading divergences only, we find:

$$g_n(\Lambda_2) = \sum_{m=0}^{\infty} \frac{g_{n+2m}(\Lambda_1)}{m!}\left[\frac{I_1(\Lambda_1, \Lambda_2, M)}{2M^2}\right]^m - c_n y(\Lambda_1)^2\,\mathcal{E}^4 I_1 \tag{1}$$

where $c_n > 0$ is associated with the combinatorial weight of the diagrams in (c)

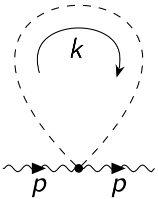
Heuristic argument:

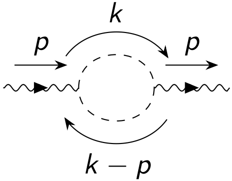
- The stability of the scalar potential requires $g_n \geq 0$, at least for all n larger than some $\bar{n}$, then for $n \geq \bar{n}$ the first term in the rhs must be larger than the second
- The coupling y is asymptotically free and it grows when the momentum scale is lowered
- In case the g_n are not sufficiently large at the scale Λ_1 to ensure a negative β-function, then g_n could become negative at the infrared scale Λ_2, yielding an unstable potential
- This is avoided if the values of $g_n(\Lambda_1)$ are sufficiently large, at least for all g_n with $n \geq \bar{n}$.

- The inclusion of gauge fields can be realized by defining the appropriate covariant derivative $\widehat{D}_\mu = \widehat{\partial}_\mu - iqA_\mu$ where

$$\widehat{\partial}_0 = \partial_0, \quad \widehat{\partial}_i = \left(1 + \frac{\partial_j \partial^j}{M^2}\right) \partial_i$$

- Accordingly, the generalized electromagnetic tensor is $\widehat{F}_{\mu\nu} = \widehat{\partial}_\mu A_\nu - \widehat{\partial}_\nu A_\mu$.
- One-loop contribution to the photon propagator correction:





Gauge Fields

- $\widehat{D}_\mu$, which is non-linear in the derivatives $\rightarrow$ under gauge transformations we have gauge violating contributions to the amplitudes, proportional to powers of $\frac{k^2}{M^2}$.
- Different higher derivative formulation , that is gauge invariant, appears in the form $(D_h D_j F_{ik})^2$ (where D_i are the space components of the standard covariant derivative), which contains terms analogous to those proportional to $w_{m,k}$.

P. Horava, Phys. Lett. B 694, 172 (2011). [arXiv:0811.2217].

- The effect of the dynamics of A_μ in Eq. (1) is, from an effective point of view, equivalent to the presence of additional scalar degrees of freedom, with the consequence of modifying only some coefficients in (1) but not the overall structure of the equation.

Hierarchy problem

- We get a finite non-vanishing correction to the scalar mass from the momentum region between Λ_1 and Λ_2, proportional to the scale $g_4 M^2$, that is very large if compared for instance to the Higgs square mass.
- The fermionic contribution to the correction of scalar square mass $g_2 M^2$ occurs with opposite sign with respect to the scalar.
- An exact cancellation of the two would be quite unnatural.

Conclusions

By discarding from the action all momentum dependent vertex operators, we find:

- Drastic reduction of the degree of divergence of the diagrams from quadratic to logarithmic (tadpole).
- Renormalizable scalar self-interaction and Yukawa-like couplings are asymptotically free (under the hypothesis of stability of the potential).
- Correspondence with Liouville's theory in the hypothesis of equal couplings at the scale M.
- Incompatibility with Gauge symmetry, recovered as an emergent law energy symmetry (below M).

Thanks for the attention

Back-up slides

Relevant and marginal operators in a Lorentz invariant theory of a single scalar field

Dimension	Relevant operators	Marginal operators
$d=2$	ϕ^{2k} for all $k \geq 0$	$(\partial\phi)^2, \ \ \phi^{2k}(\partial\phi)^2$ for all $k \geq 0$
$d=3$	ϕ^{2k} for $k=1,2$	$(\partial\phi)^2, \ \ \phi^6$
$d=4$	ϕ^2 for ≤ 3	$(\partial\phi)^2, \ \ \phi^4$
$d>4$	ϕ^2 for $0 \leq k \leq 3$	$(\partial\phi)^2$

Figure: Relevant and marginal operators in a Lorentz invariant theory of a single scalar field in various dimensions and whose terms of the Lagrangian are invariant for the global symmetry group $\mathcal{G} = Z_2$.

- In the case of ASI, we divide the coordinates in two groups, $\vec{x} = (\vec{x}_{||}, \vec{x}_{\perp})$ that respectively belong to a m and $d-m$ dimensional (with $m \leq d$) subspace whose scaling behavior is different.

$$\mathcal{O}(l^{\theta}\vec{x}_{||}, l\vec{x}_{\perp}) = l^{-\Delta_{\mathcal{O}}}\mathcal{O}(\vec{x})$$

  with θ different from unity.
- We can start from a generalization of the Landau model ϕ^4:

$$\Gamma[\phi] = \int d^{d-m}x_{\perp} d^m x_{||} \Big(W_{||}(\partial_{||}^2\phi)^2 + W_{\perp}(\partial_{\perp}^2\phi)^2 + \\ + \frac{Z_{||}}{2}(\partial_{||}\phi)^2 + \frac{Z_{\perp}}{2}(\partial_{\perp}\phi)^2 + V(\phi)\Big)$$

  where $V(\phi) = m^2|\phi|^2 + \lambda_4|\phi|^4$.

- A critical line is observed when $Z_{||} > 0$ that respectively correspond to a disordered and an ordered phase, while for $Z_{||} < 0$ a critical value of the minimum of V separates the disordered phase from a modulated one with an oscillating ground state. The Lifshitz point, where the three phases meet, is located at $Z_{||} = 0$.

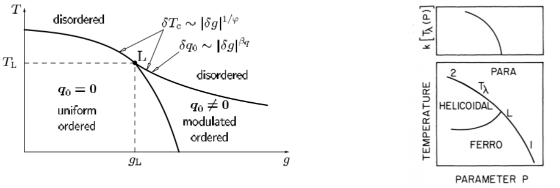
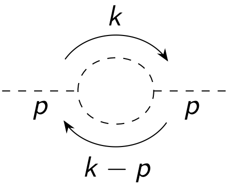
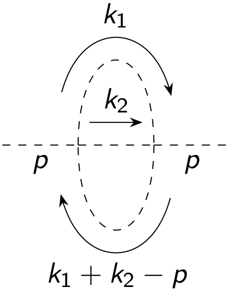


Figure: Schematic phase diagram with a Lifshitz point L and a typical example of a system that exhibits this type of behavior.

Scalar Fields - Flow of the derivative terms

- The coefficients $a_{1,2,3}$ take contributions from diagrams with at least two vertices:

$R_1 \propto$ 

$R_2 \propto$ 

- Corrections given by:

$$\delta a_j = \frac{1}{(2j)!} \frac{\partial^{2j}}{(\partial|\vec{p}|)^{2j}} \left[\frac{i g_3^2}{2} R_1 - \frac{g_4^2}{6} R_2 \right] \Bigg|_{p=0}$$

Scalar Fields - Flow of the couplings

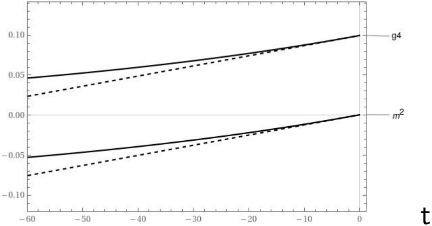
- For the evolution of the couplings $g_n(\mu)$ with μ going toward the IR region, we can write a set of first-order coupled differential equations:

$$\begin{cases} -\mu\partial_\mu g_4(\mu) & = \frac{g_6(\mu)}{2}\frac{\mu^3}{(2\pi)^2}\frac{1}{D(\mu)} - \frac{3}{4}g_4^2(\mu)\frac{\mu^3}{(2\pi)^2}\frac{1}{D(\mu)^3} \\ -\mu\partial_\mu g_3(\mu) & = \frac{g_5(\mu)}{2}\frac{\mu^3}{(2\pi)^2}\frac{1}{D(\mu)} - \frac{3}{4}g_4(\mu)g_3(\mu)\frac{\mu^3}{(2\pi)^2}\frac{1}{D(\mu)^3} \\ -\mu\partial_\mu g_6(\mu) & = \frac{g_8(\mu)}{2}\frac{\mu^3}{(2\pi)^2}\frac{1}{D(\mu)} - \frac{15}{4}g_4(\mu)g_6(\mu)\frac{\mu^3}{(2\pi)^2}\frac{1}{D(\mu)^3} \end{cases}$$

where we denote $D(\mu) = \sqrt{a_3(\mu)\mu^6 + a_2(\mu)\mu^4 + a_1(\mu)\mu^2 + g_2(\mu)}$ and set the mass scale $M = 1$.

- UV flow of the coupling g_4 and m^2 with respect to $t = \ln(M/\mu)$ with μ the running scale:

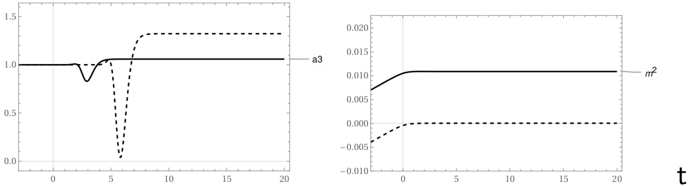
D. Zappala, Eur. Phys. J. C 82, 341 (2022). [arXiv:2111.08385].



- Truncated series up to the term $\bar{n} = 6$ (dashed lines) $\rightarrow$ logarithmic trend
- Series in the limit $\bar{n} \rightarrow \infty$ (solid lines) $\rightarrow$ Exponential trend
- Boundaries fixed at $t = 0$ where $g_4 = 0.1$ and $m^2 = 0.001$

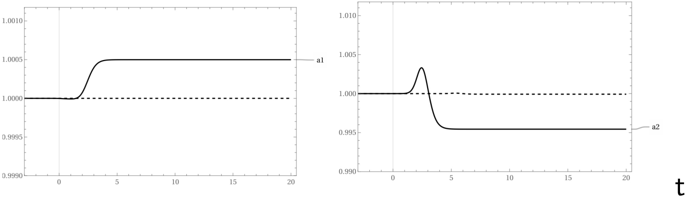
Scalar Fields - Flow of the couplings

- Flow of the coupling a_3 and m^2 in the intermediate region.



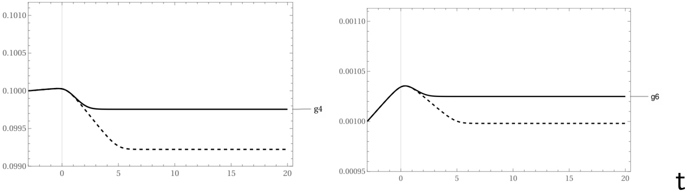
- Boundaries fixed at $t = -3$ where $g_4 = g_3 = 0.1$ and $m^2 = 0.007$ in one case (solid lines) and $g_4 = 0.1$, $g_3 \simeq 10^{-17}$ and $m^2 \simeq -0.004$ in the other (dashed lines).
- Appropriate fine-tuning to avoid the emergence of singularities in the analysis of RG flow equations of parameter a_3.

- Flow of the coupling a_1 and a_2 in the intermediate region.



- Flow of a_1 (left panel) and a_2 (right panel) in the region $t \in [-3, 20]$ with the same initial conditions adopted in the previous cases.

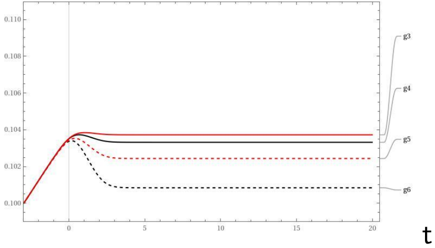
- Flow of the coupling g_4 and g_6 in the intermediate region.



- Boundaries fixed at $t = -3$ where $g_4 = 0.1$ and $g_6 = 0.001$.

Scalar Fields - Flow of the couplings

- Flow of the coupling $g_{3,4,5,6}$ in the intermediate region in the hypothesis of equal couplings at the scale M.



- Boundaries fixed at $t = -3$ where g_3 (red line), g_4 (black), g_5 (red dashed) and g_6 (black dashed) are all equal to $g_{3,4,5,6} = 0.1$.