64th Cracow School of Theoretical Physics

Non-Gaussianities in the PBH formation and induced GWs

Shi Pi

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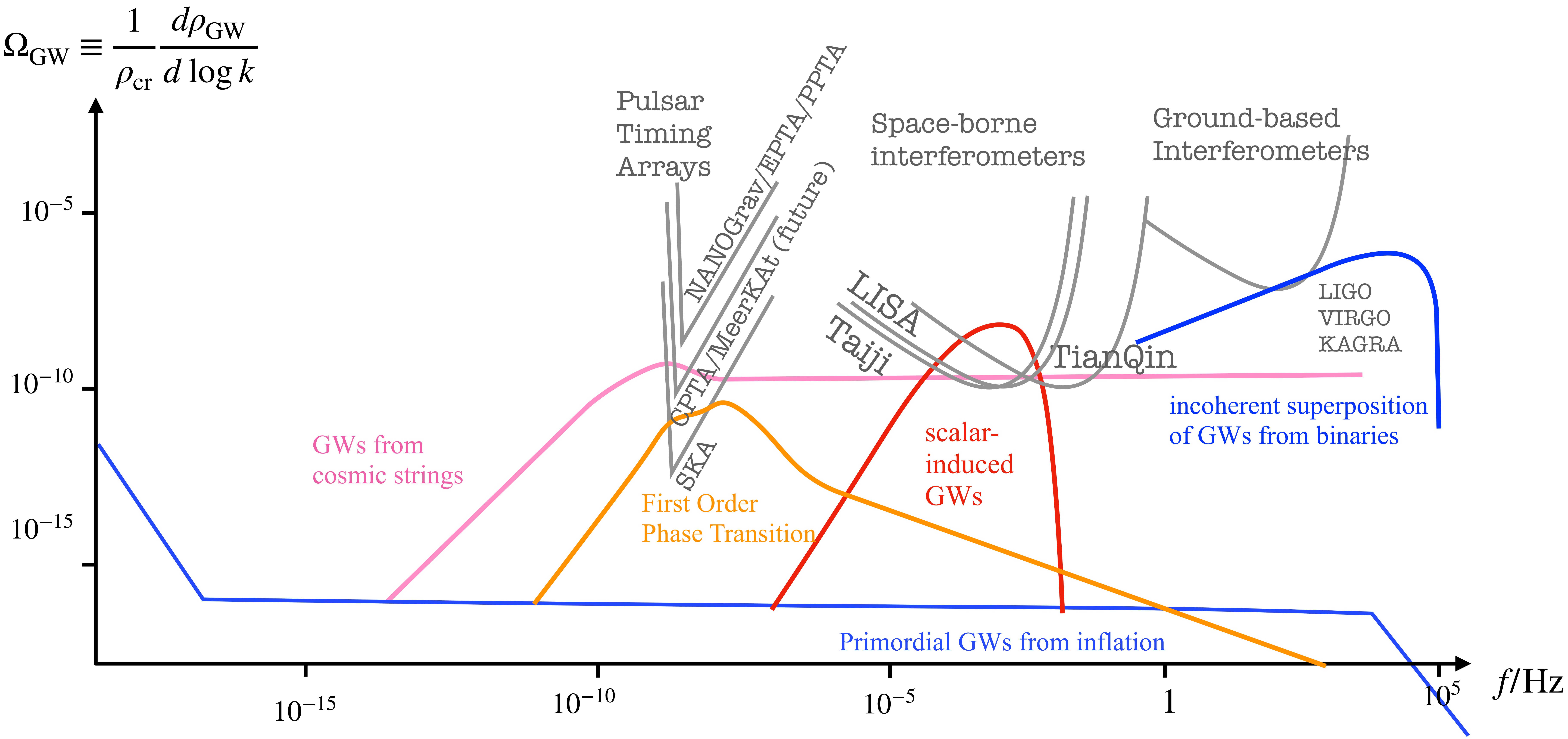
Based on
Rong-Gen Cai, SP, Misao Sasaki, PRL, 122, 201101 (1810.11000)
SP and Misao Sasaki PRD 108, L101301 (2112.12680)
SP and Jianing Wang, JCAP 06 (2023) 018 (2209.14183)
SP and Misao Sasaki, PRL, 131, 011002 (2211.13932)
SP, 2404.06151, *Primordial Black Holes* Chapter 8

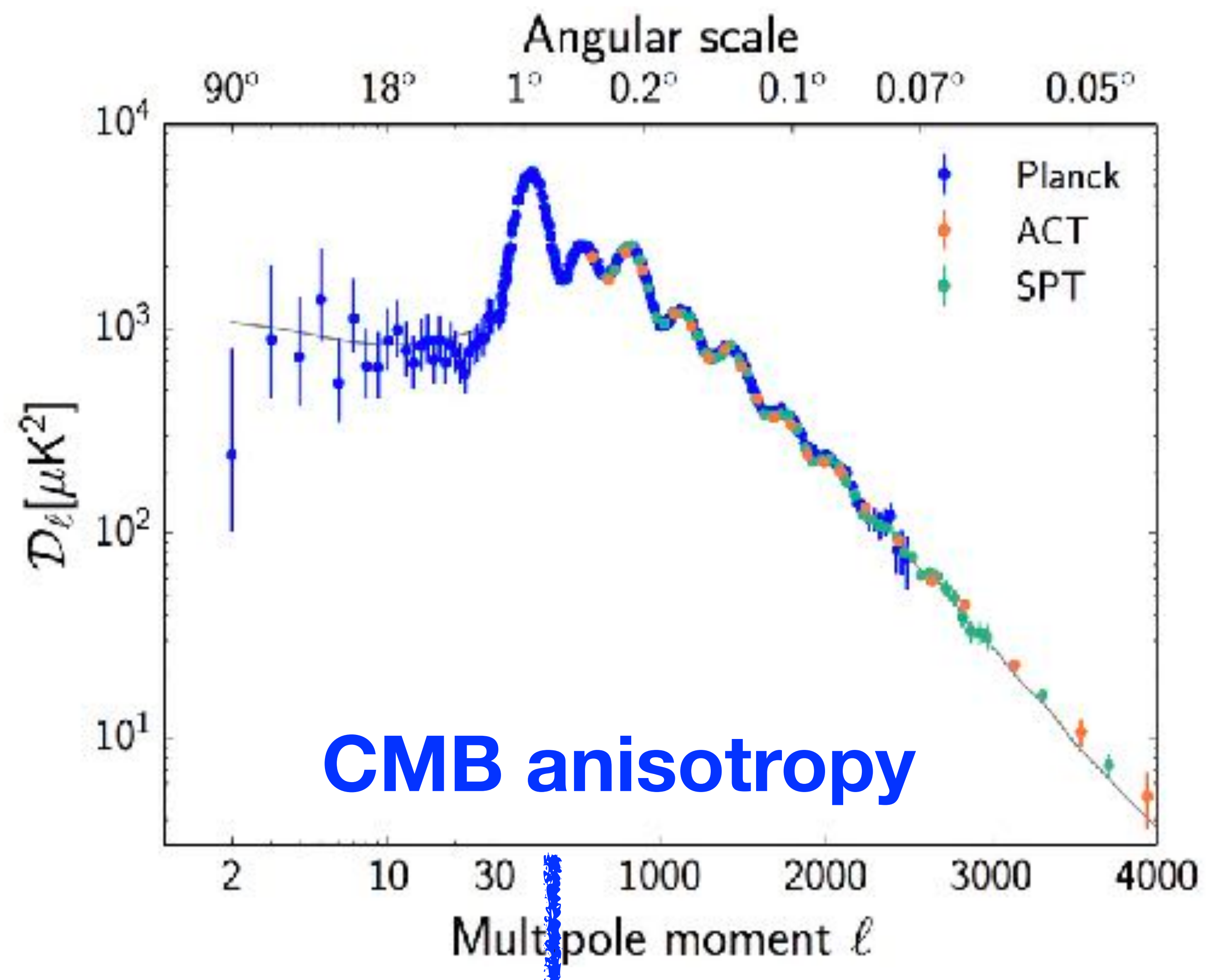
Zakopane, 18th June 2024

CONTENT

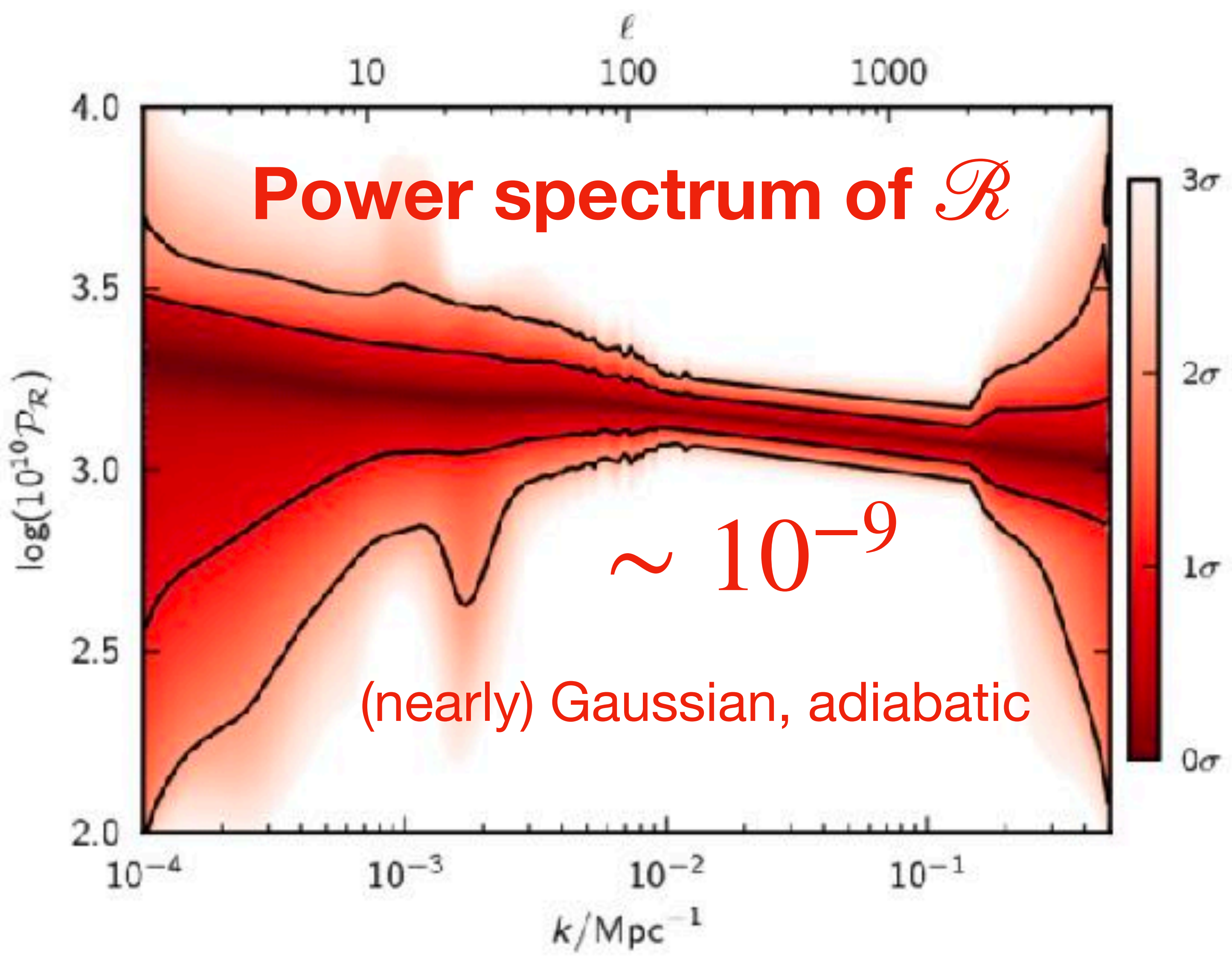
- Introduction
- Non-Gaussianity impact on PBH formation
- Origin of non-Gaussianity in inflation models
- Prediction in mHz and nHz

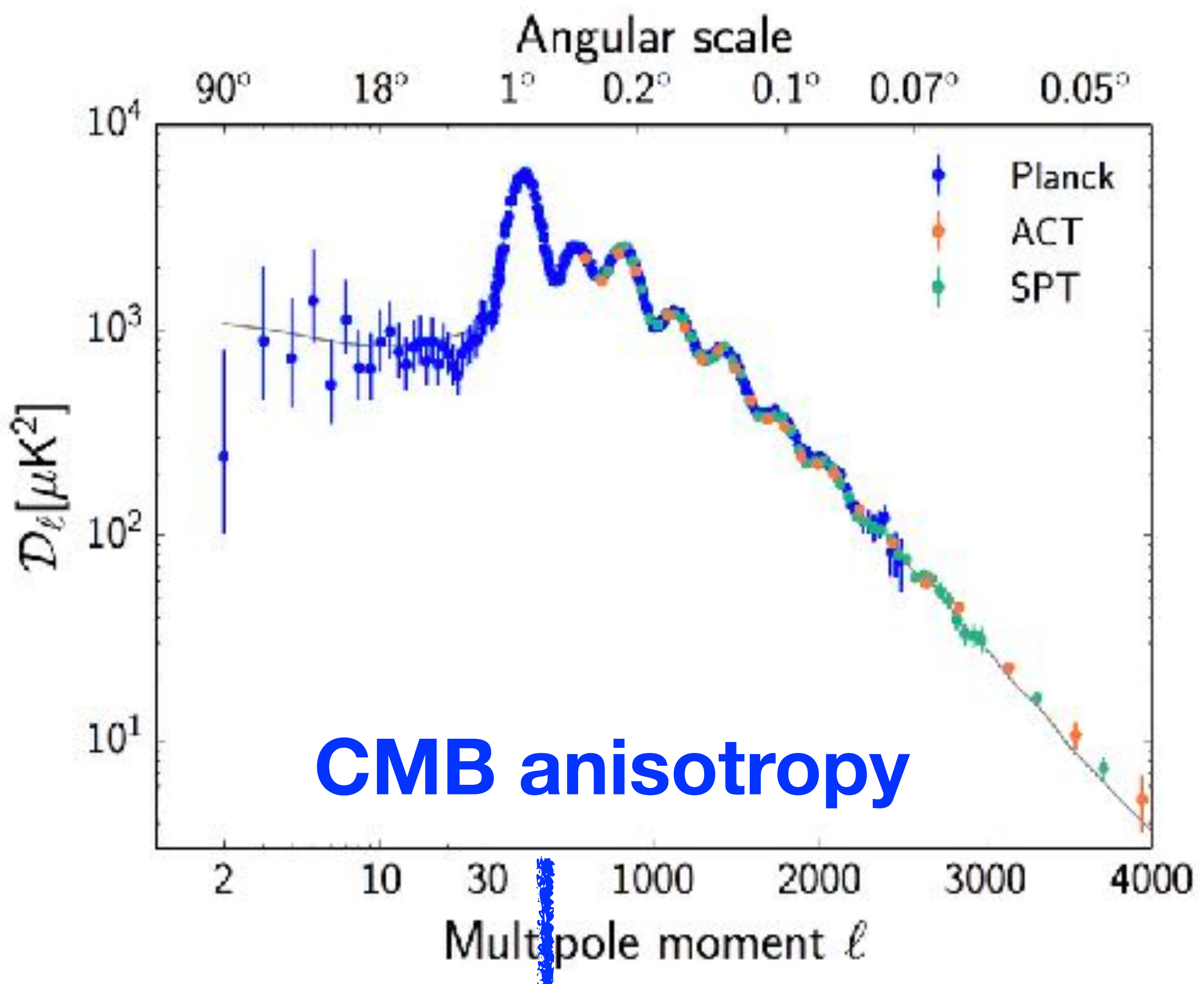
Stochastic GWs



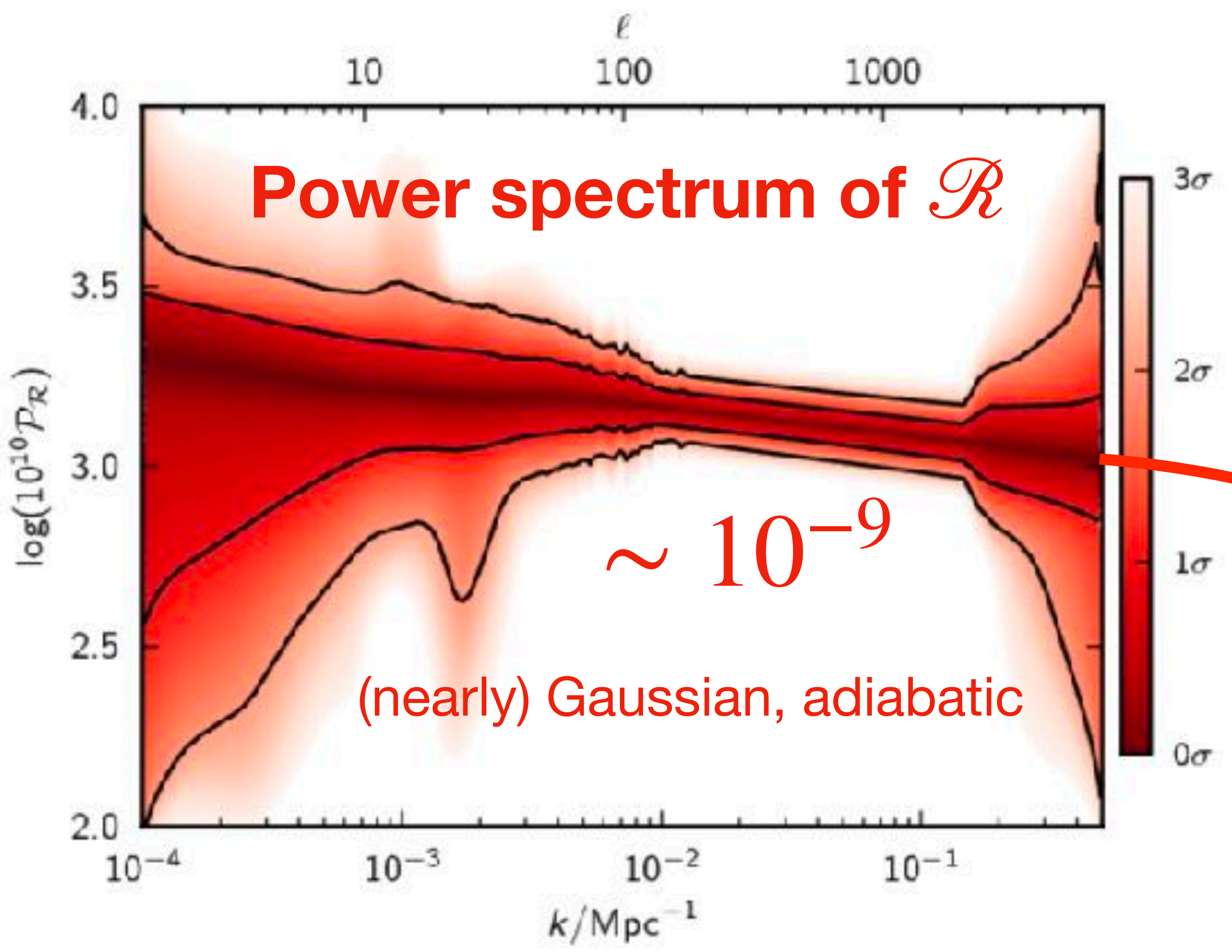


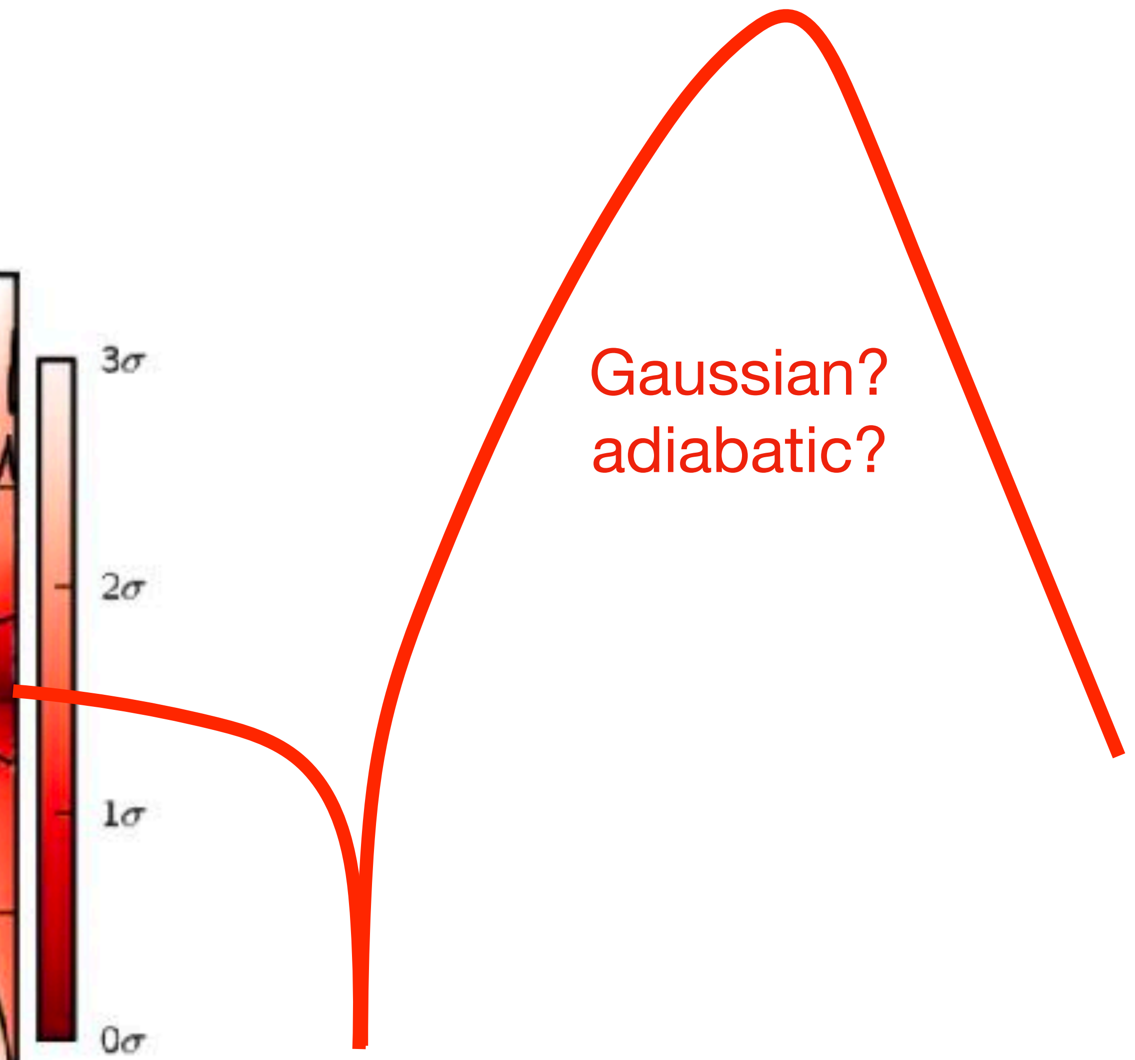
Reconstruction

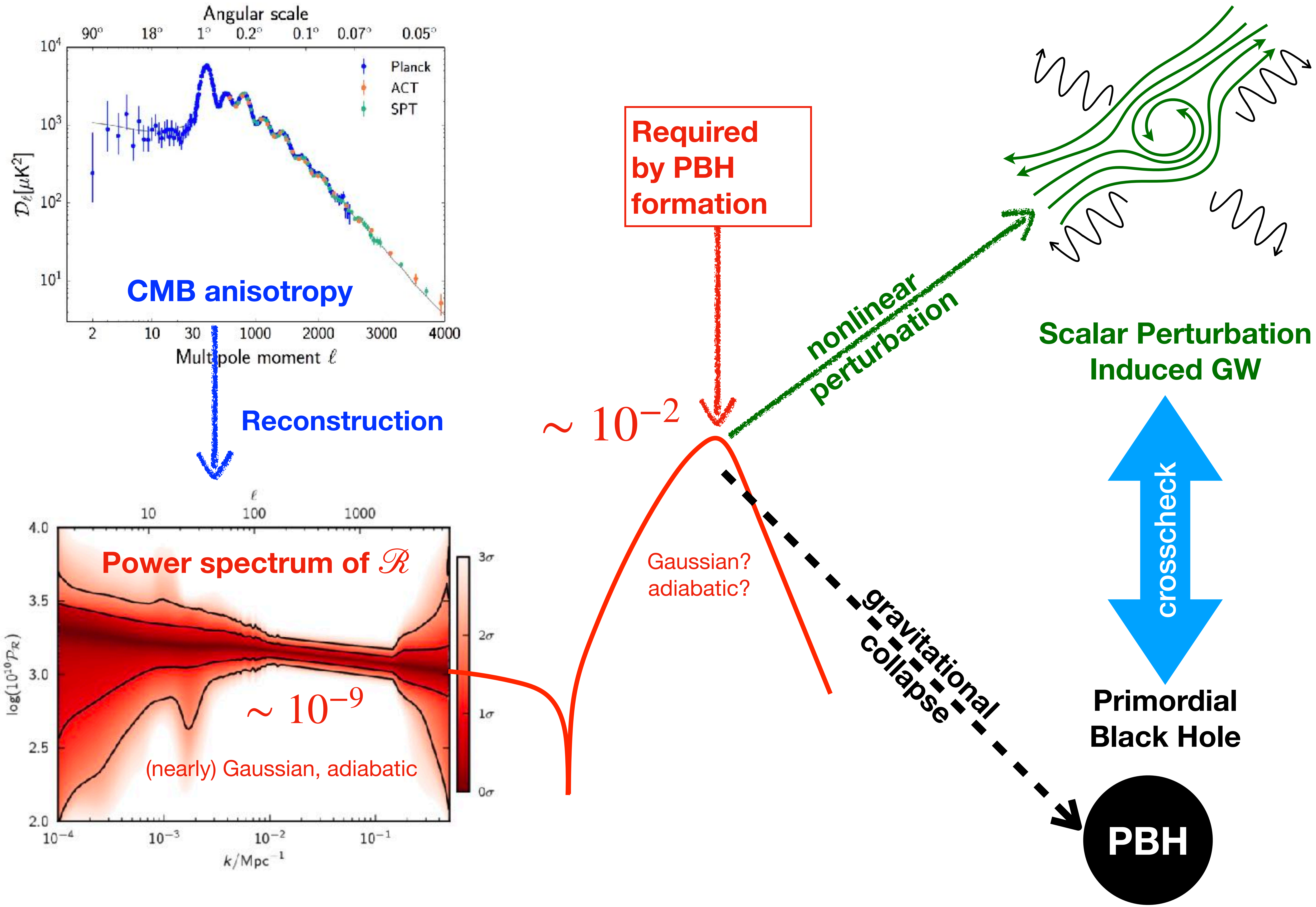




Reconstruction



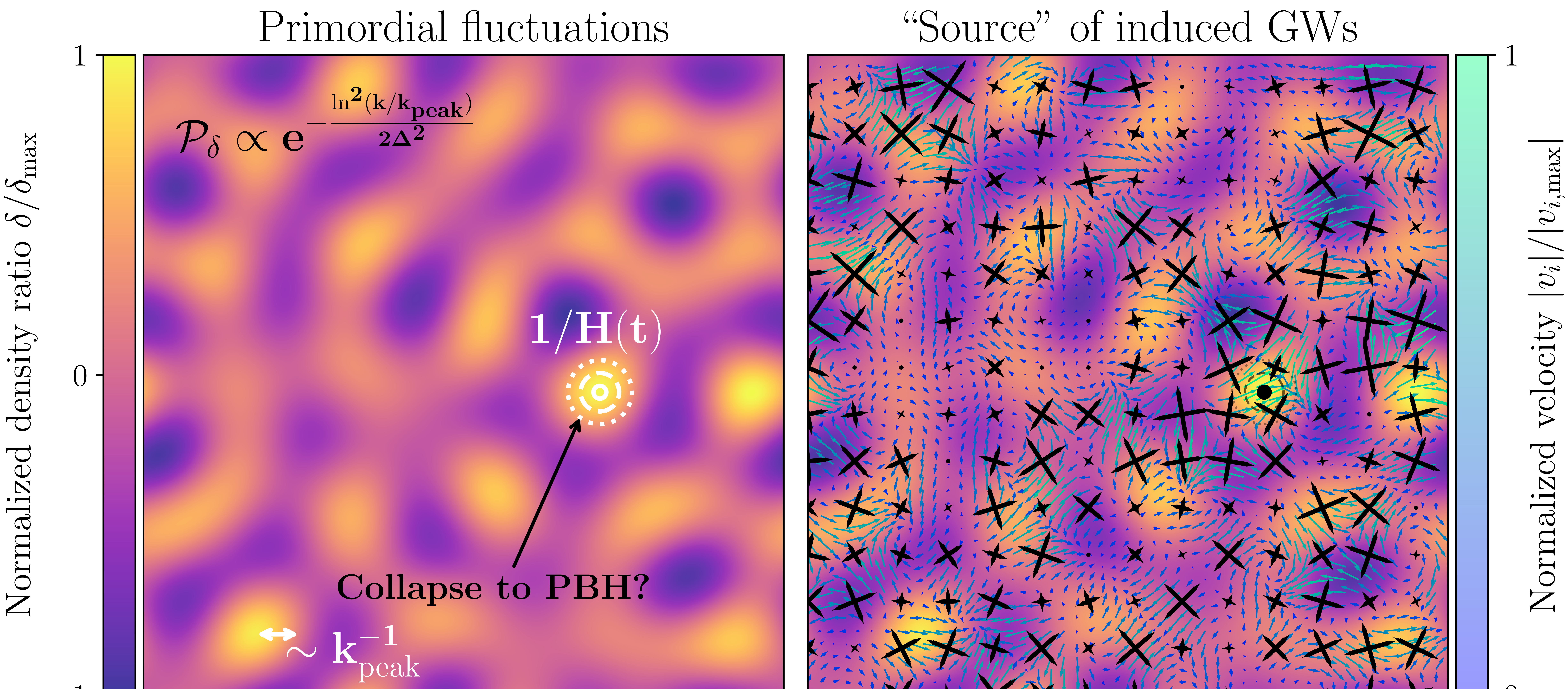




Matarrese et al, PRD 47, 1311;
PRL 72, 320; PRD 58, 043504
Ananda et al, gr-qc/0612013
Bauman et al, hep-th/0703290

Zeldovich & Novikov 1966
Hawking 1971
Carr & Hawking 1974


Primordial Black Holes
Stochastic Gravitational Waves

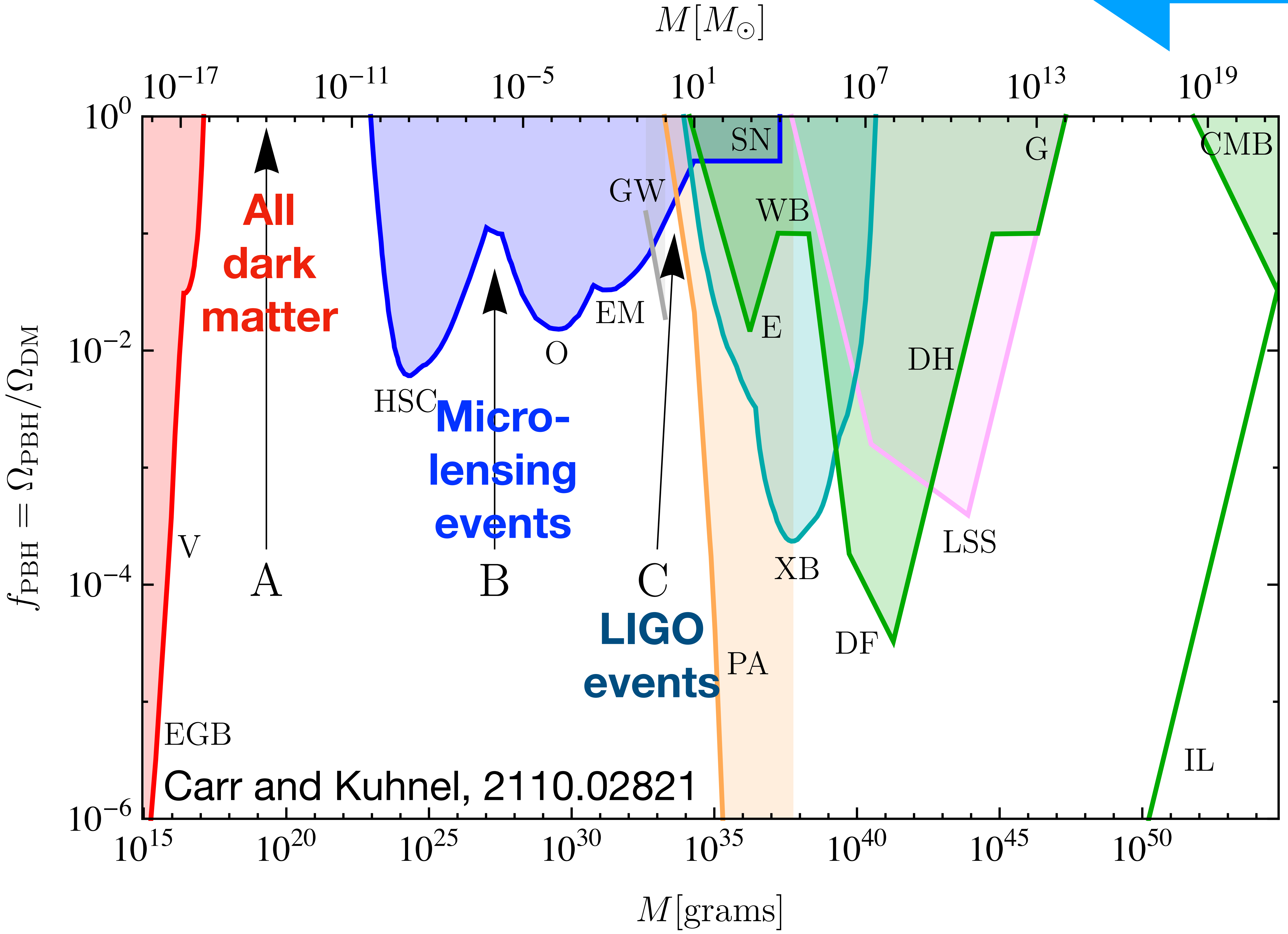


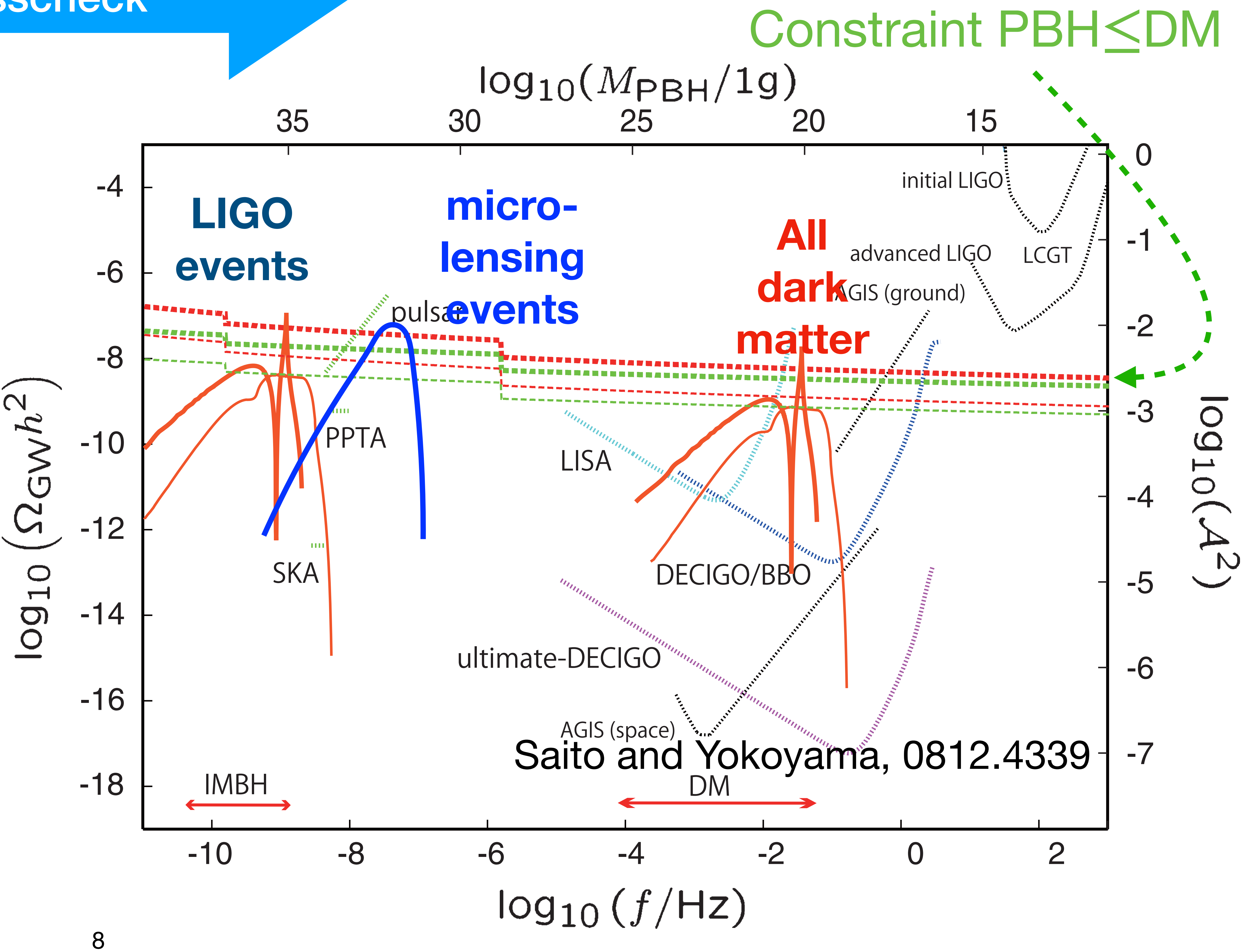
Domenech, 2402.17388, PBH book, Chapter 18

PBH-IGW crosscheck

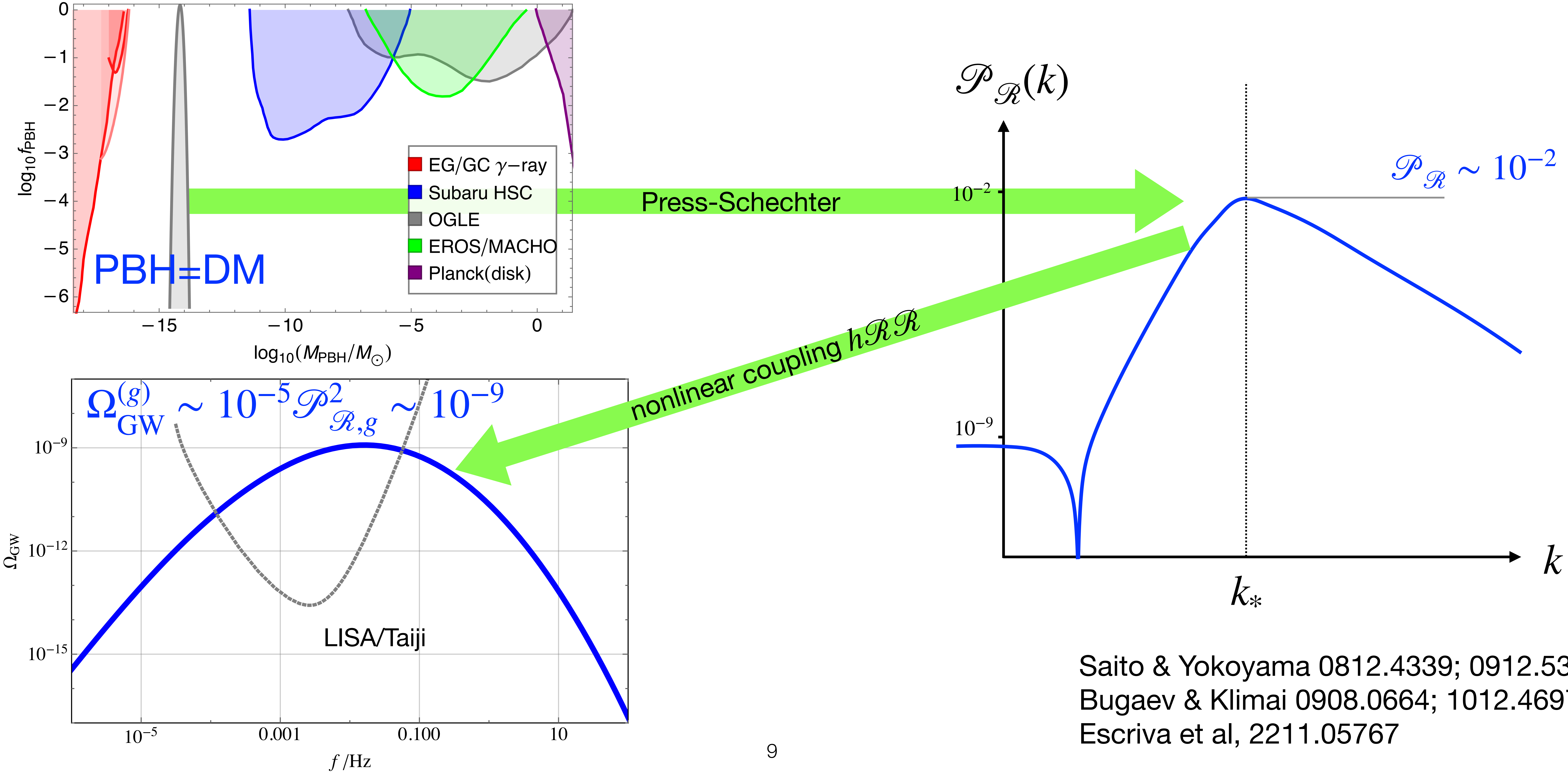
$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

PBH constraints ⟷ crosscheck ⟷ Induced GW predictions





PBH-IGW crosscheck



Saito & Yokoyama 0812.4339; 0912.5317
Bugaev & Klimai 0908.0664; 1012.4697
Escriva et al, 2211.05767

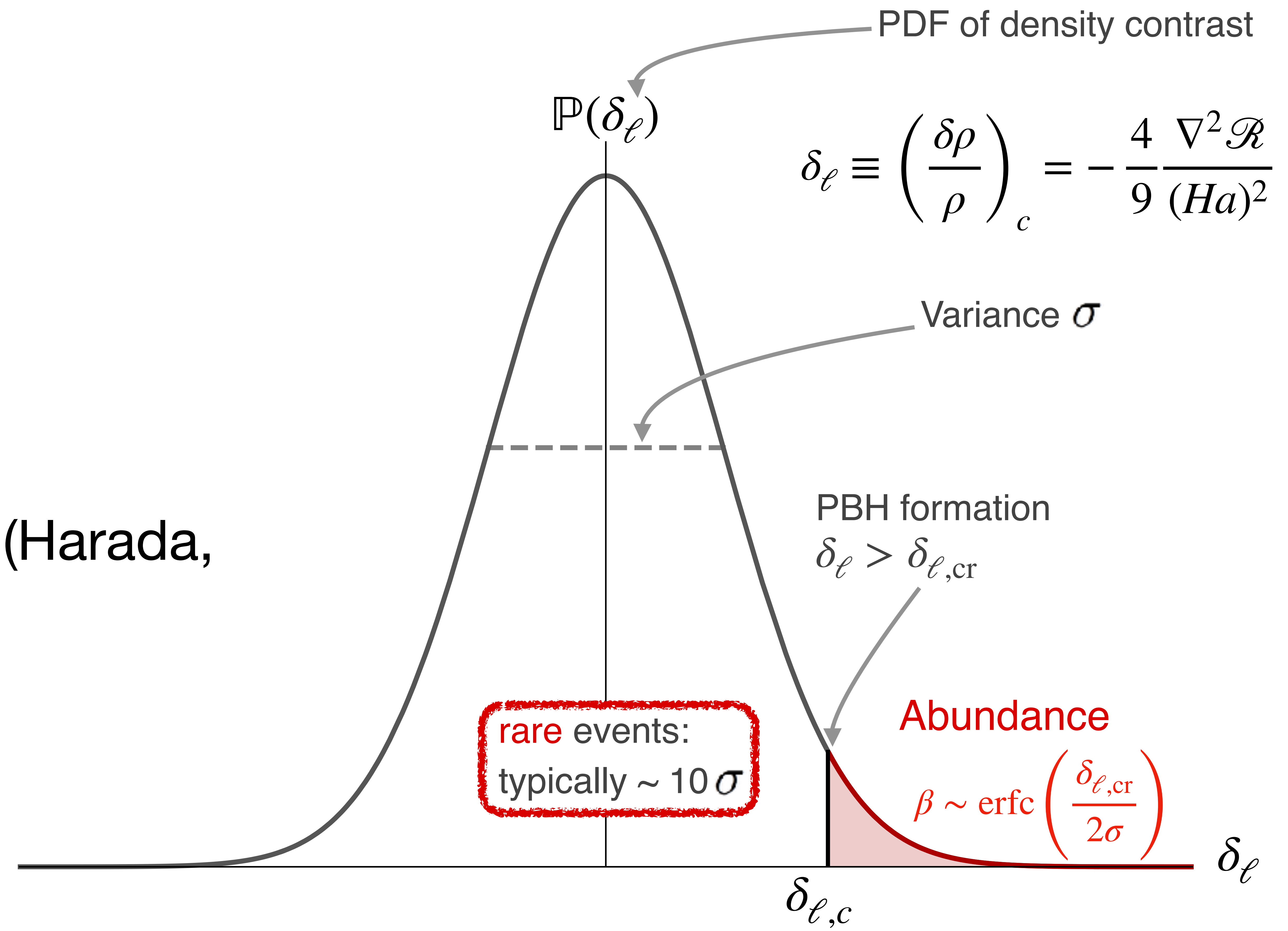
(Simplest) Press-Schechter

$$\left.\begin{array}{c} \mathscr{R} \xrightarrow{(1)} \delta_\ell \\ \mathbb{P}(\mathscr{R}) \xrightarrow{(2)} \mathbb{P}(\delta_\ell) \end{array}\right\} \xrightarrow[\text{(4) Window function}]{\text{(3) given } \delta_{\ell,\mathrm{cr}}} \beta = \int_{\delta_{\ell,\mathrm{cr}}} \mathbb{P}(\delta_\ell)\frac{M(\delta_\ell)}{M_H}\mathrm{d}\delta_\ell$$

Every step is linear/Gaussian:

- (1) Linear Poisson equation.
- (2) Gaussian PDF $\mathbb{P}(\mathscr{R})$ gives Gauss PDF $\mathbb{P}(\delta_\ell)$:

  $\mathbb{P}(\mathscr{R})d\mathscr{R} = \mathbb{P}(\delta_\ell)d\delta_\ell$
- (3) Critical density contrast $\delta_{\ell,\mathrm{cr}}$ given by HYK limit (Harada, Yoo, Kohri, 1309.4201).
- (4) Window function.



(Simplest) Press-Schechter

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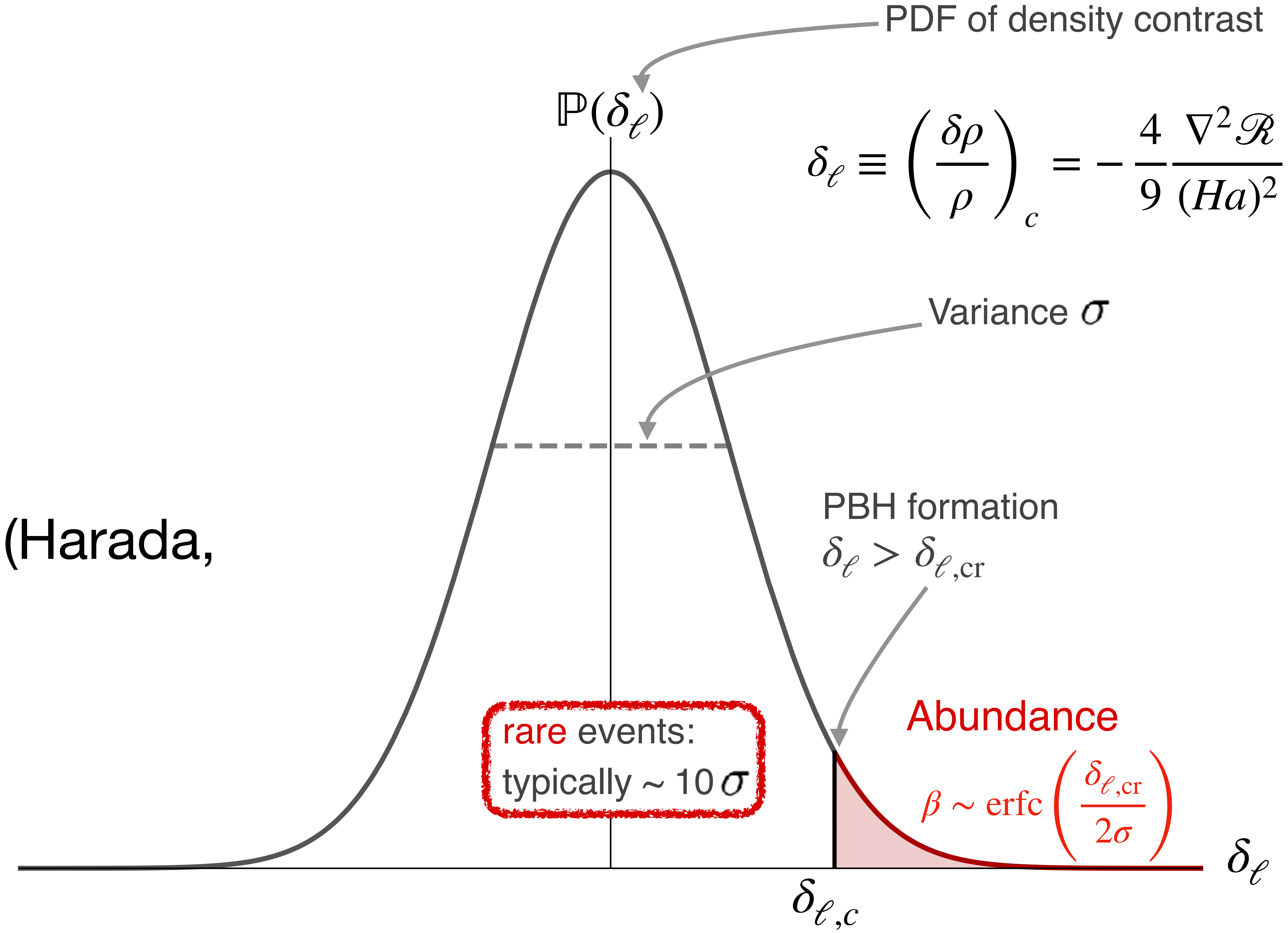
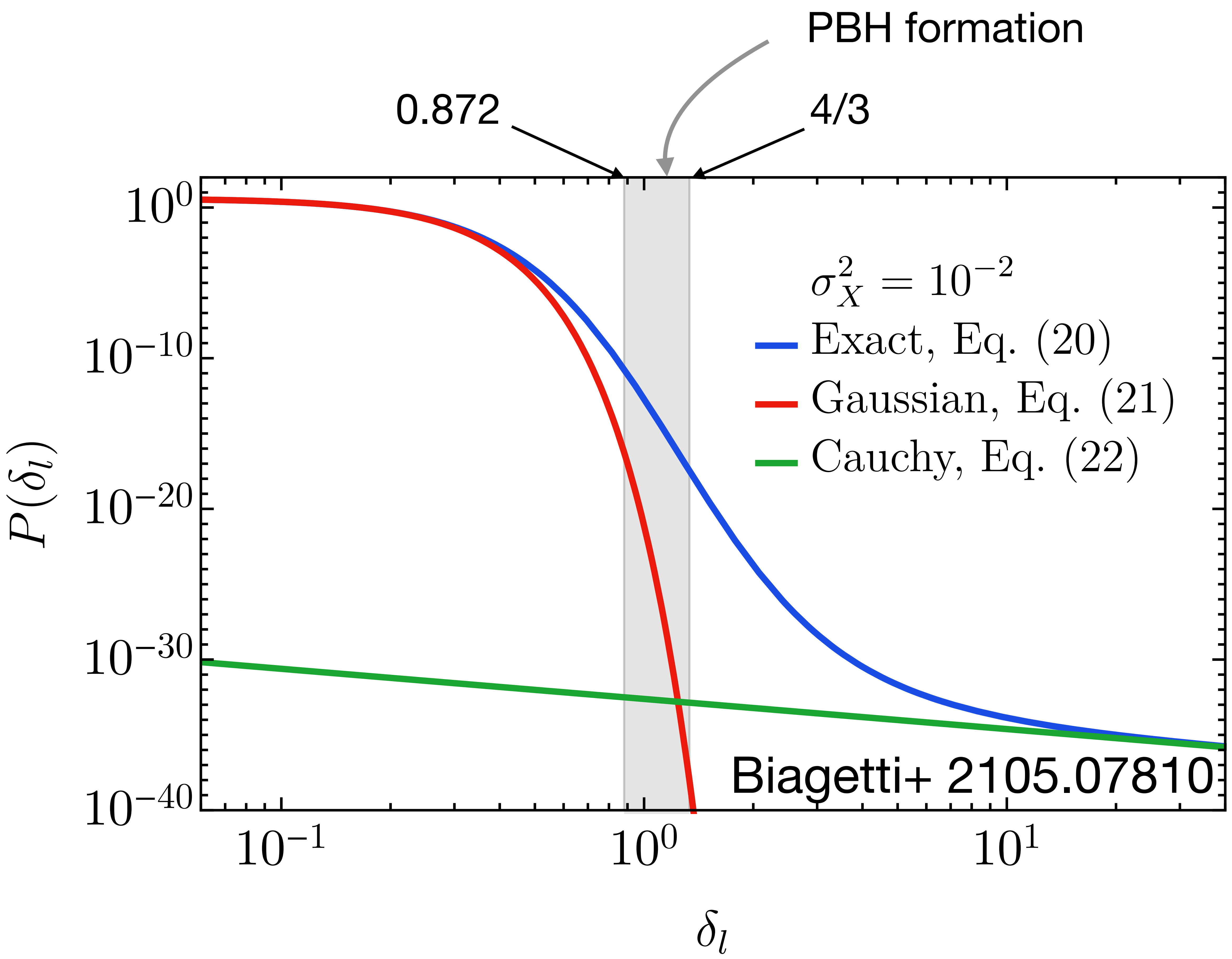


Why non-Gaussianity?

$$\left.\begin{array}{l}\mathscr{R} \xrightarrow{(1)} \mathscr{C}_\ell \\ \mathbb{P}(\mathscr{R}) \xrightarrow{(2)} \mathbb{P}(\mathscr{C}_\ell)\end{array}\right\} \xrightarrow[\text{(4) Window function}]{\text{(3) given } \mathscr{C}_{\rm cr}} \beta = \int_{\mathscr{C}_{\ell,\rm cr}}^{4/3} \mathbb{P}(\mathscr{C}_\ell)\frac{M(\mathscr{C}_\ell)}{M_H}\mathrm{d}\mathscr{C}_\ell$$

Non-Gaussianity must be taken into account：

- (1) Use compaction function $\mathscr{C}$ which nonlinearly depends on $\mathscr{R}$. (Harada et al 1503.03934; De Luca et al 1904.00970.)
- (2) Primordial non-Gaussianity of $\mathscr{R}$.
- (3) $\mathscr{C}_{\rm cr}$ depends on profile. (Musco 1809.02127; Escrivà et al 1907.13311)



Compaction function

- PBH form when $\mathscr{C} \equiv 2G\delta M/(a\bar{r})$ goes to 1 in the sub-horizon evolution. But for analytical calculation, the threshold is chosen on superhorizon scales, of which $\mathscr{C}$ is a constant and determined by inflation models.

Shibata, Sasaki, gr-qc/9905064
Harada et al 1503.03934

- $$ds^2 \simeq \frac{d\bar{r}^2}{1 - K(\bar{r})\bar{r}^2} + \bar{r}^2 d\Omega_2^2\,, \;\; \mathscr{C} = \frac{2G\delta M}{a\bar{r}} \simeq \frac{2}{3}K(\bar{r})\bar{r}^2$$ (comoving slicing)

- $$ds^2 \simeq e^{2\mathscr{R}(r)}\left(dr^2 + r^2 d\Omega_2^2\right), \;\; \mathscr{C} \simeq \mathscr{C}_\ell - \frac{3}{8}\mathscr{C}_\ell^2, \;\; \mathscr{C}_\ell = -\frac{4}{3} r\mathscr{R}'(r)$$

Musco, 1809.02127
Escrivà, Germani, Sheth, 1907.13311

Compaction function

- On superhorizon scales the threshold of $\mathscr{C}$ only depends on its width at the maximum $\bar{r}_m$, characterised by

$$q \equiv -\frac{\bar{r}_m^2 \partial_{\bar{r}}^2 \mathscr{C}(\bar{r}_m)}{4\mathscr{C}(\bar{r}_m)} = \frac{-r_m^2 \partial_r^2 \mathscr{C}(r_m)}{4\mathscr{C}(r_m)\left(1 - \frac{3}{2}\mathscr{C}(r_m)\right)}$$

- A good fitting formula is

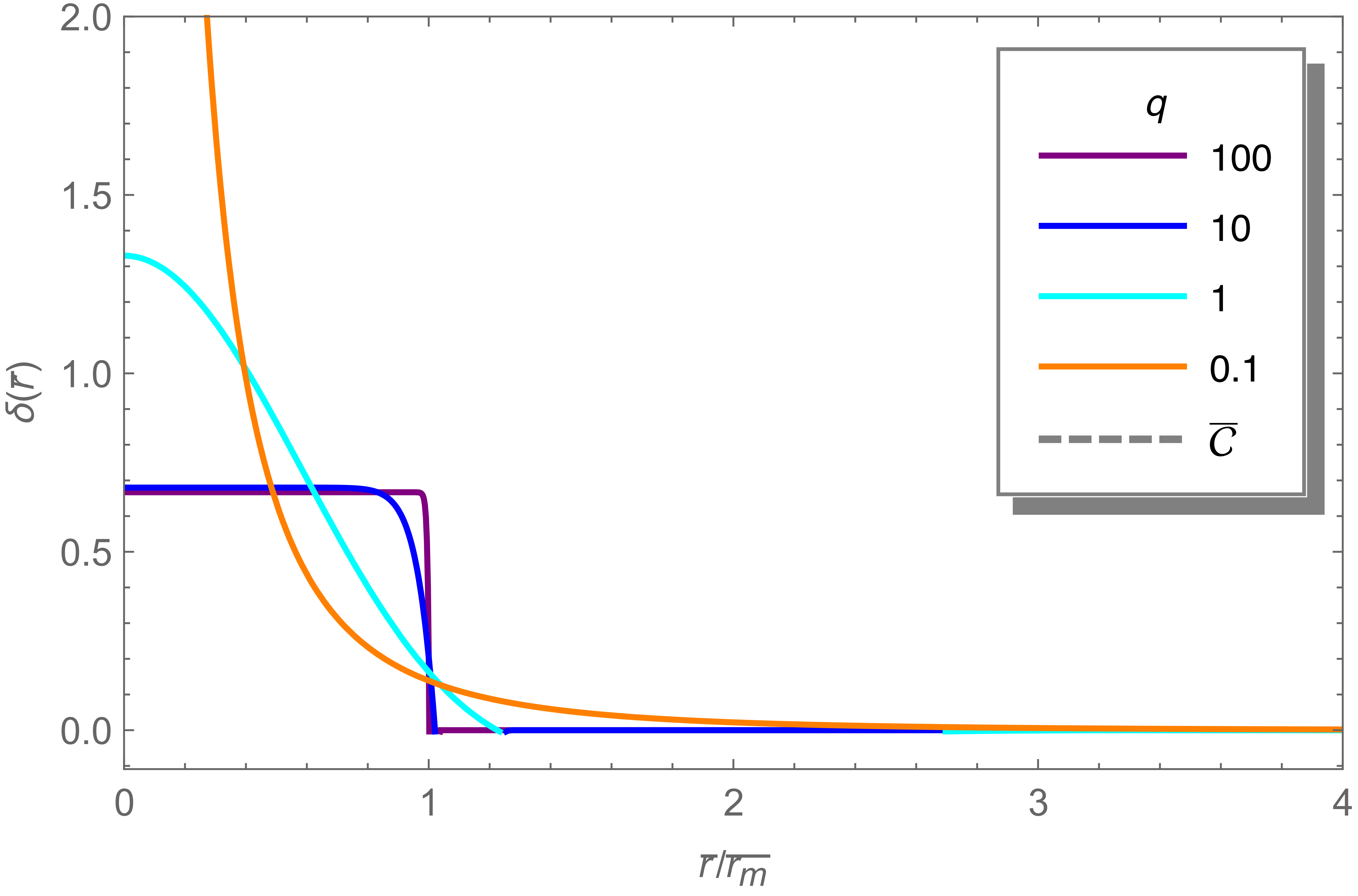
$$\mathscr{C}(\bar{r}) = \mathscr{C}(\bar{r}_m)\frac{\bar{r}^2}{\bar{r}_m^2}\exp\left[\frac{1}{q}\left(1 - (\frac{\bar{r}}{\bar{r}_m})^{2q}\right)\right]$$

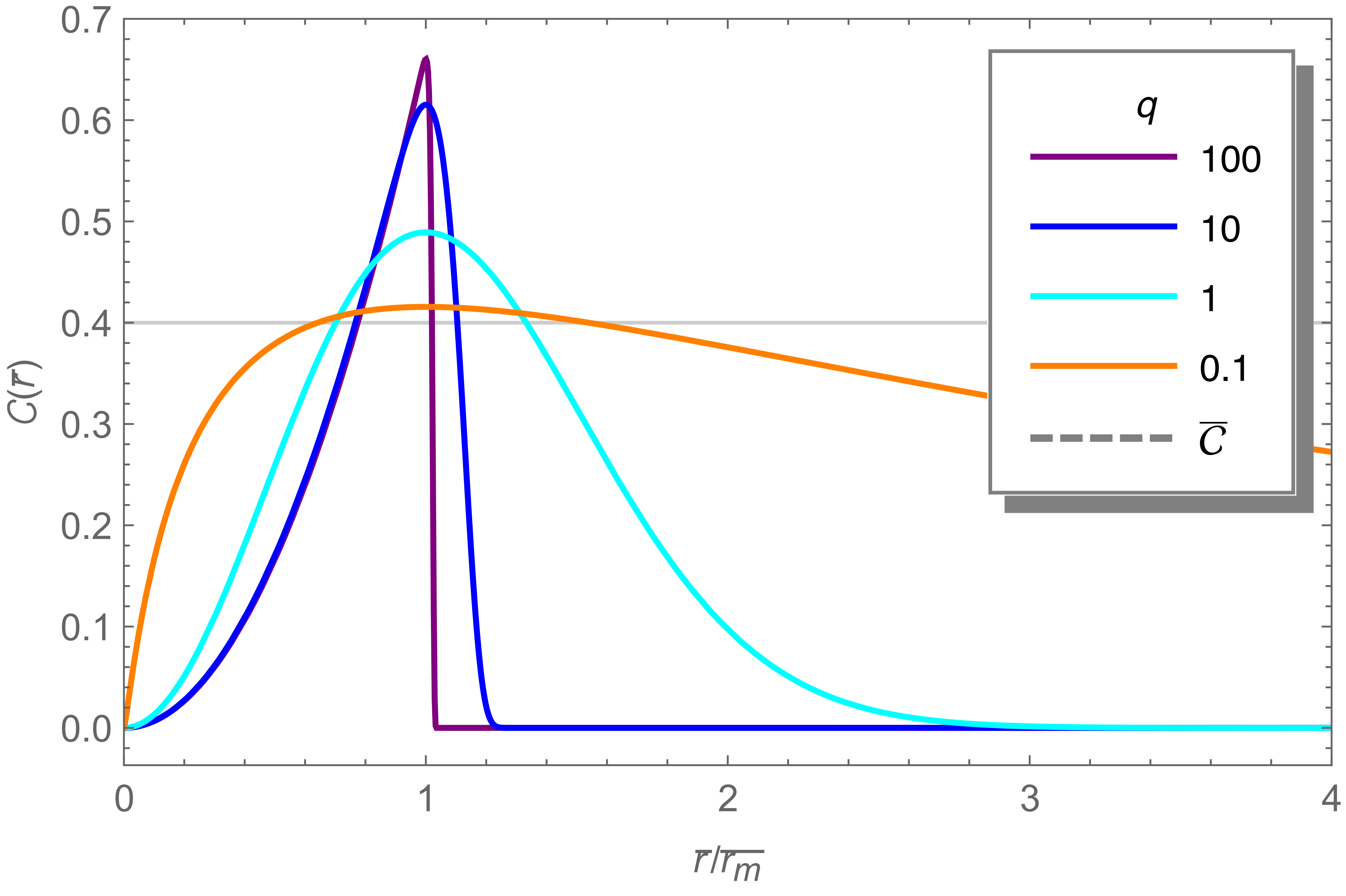
- $q \gg 1$ limit is the top-hat density profile, which has the maximal pressure gradient, $\mathscr{C}_{\rm th}(\bar{r}_m) \to 2/3$

- $q \to 0$ limit density concentrates at the center, and pressure gradient is negligible: Harada-Yoo-Kohri limit $\mathscr{C}_{\rm th}(\bar{r}_m) \approx 0.4$

Musco, 1809.02127
Escrivà, Germani, Sheth, 1907.13311

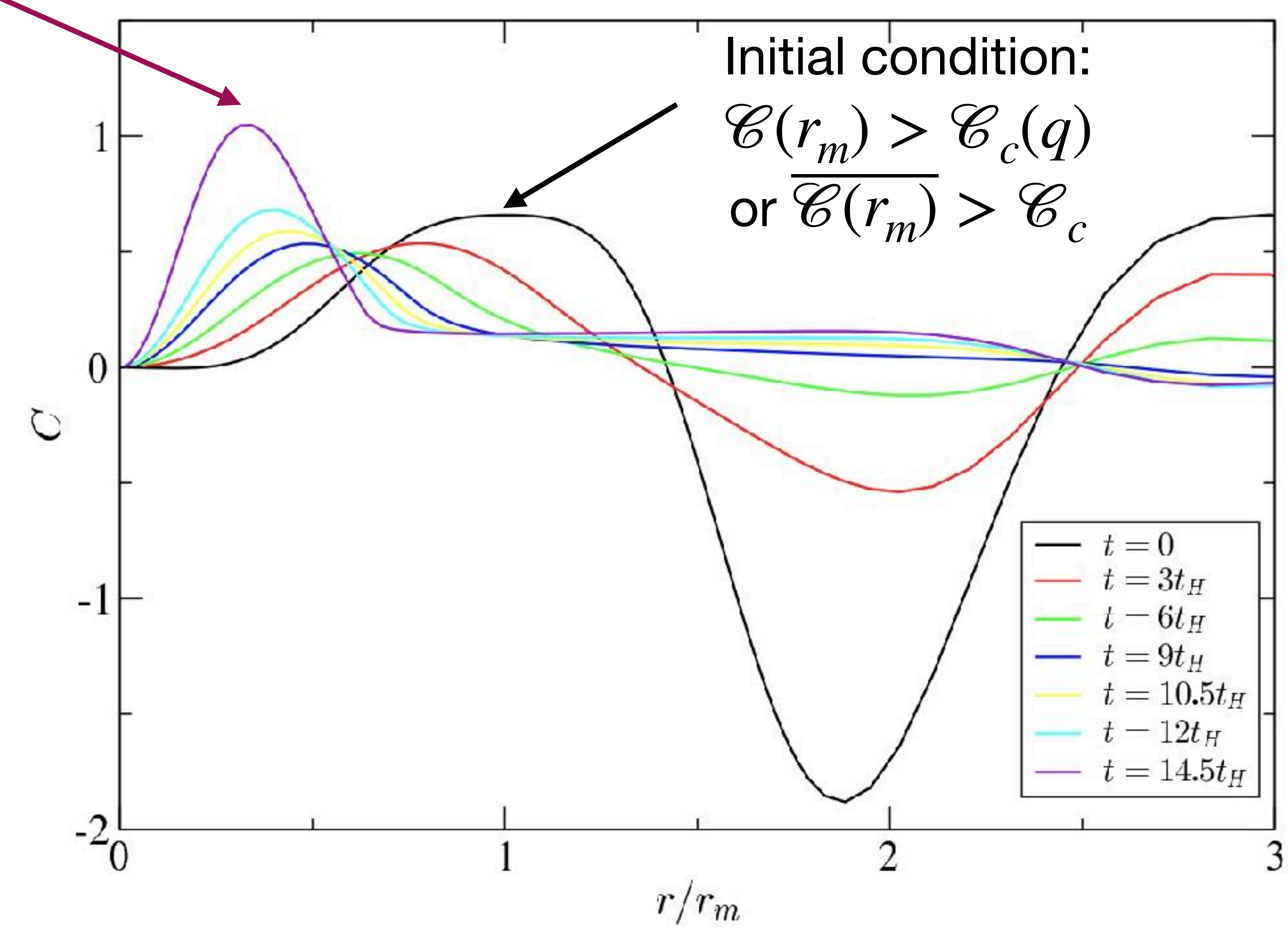
Compaction function





Compaction function

PBH forms:
$\mathscr{C}(r_m, t_f) > 1$



Escriva et al, 2202.01028

Conditions for PBH

- Find the overdressed region

$$\nabla \mathscr{C}(\mathbf{x})\Big|_{\mathbf{x}_0} = 0 \qquad \nabla^2 \mathscr{C}(\mathbf{x})\Big|_{\mathbf{x}_0} < 0$$

- Find the maximum of $\mathscr{C}(\bar{r})$, $\bar{r}_m$, which determines the boundary of the overdensity

$$\frac{\partial}{\partial \bar{r}}\mathscr{C}(\bar{r}, \mathbf{x}_0)\Big|_{\bar{r}_m} = 0 \qquad \frac{\partial^2}{\partial^2 \bar{r}}\mathscr{C}(\bar{r}, \mathbf{x}_0)\Big|_{\bar{r}_m} < 0$$

- $\mathscr{C}(\bar{r}_m, \mathbf{x}_0)$ (or its average) must exceed the threshold

$$\mathscr{C}(r_m, \mathbf{x}_0) > \mathscr{C}_c(q) \quad \text{or} \quad \overline{\mathscr{C}}(r, \mathbf{x}_0) > \overline{\mathscr{C}}_c = 2/5$$

Germani, Sheth, 2308.02971
Escriva, Germani, Sheth, 1907.13311

Linear perturbation

- Define linear compaction function $\mathscr{C}_\ell \equiv -\frac{4}{3} r\partial_r \mathscr{R}(r)$, such that $\mathscr{C} \simeq \mathscr{C}_\ell - \frac{3}{8}\mathscr{C}_\ell^2$
- The threshold $\mathscr{C}_{\rm th}$ can be converted to the threshold of $\mathscr{C}_{\ell,{\rm th}} = \frac{4}{3}\left(1 - \sqrt{1 - \frac{3}{2}\mathscr{C}_{\rm th}}\right)$
- The maximum of $\mathscr{C}(r, \mathbf{x})$ can be either the maximum or the minimum of $\mathscr{C}_\ell(r, \mathbf{x})$, as $\nabla^2 \mathscr{C}(r, \mathbf{x}) = \left(1 - \frac{3}{4}\mathscr{C}_\ell(\mathbf{x})\right) \nabla^2 \mathscr{C}_\ell(r, \mathbf{x})$
- $\mathscr{C}_\ell < 4/3$ is consistent with small perturbation, which is called Type I PBH
- $\mathscr{C}_\ell > 4/3$ is rarer, called Type II PBH

Germani, Sheth, 2308.02971
Escriva, Germani, Sheth, 1907.13311

Press-Schechter-type formalism

- For Type I PBH there is an upper bound

$$\frac{4}{3}\left(1-\sqrt{1-\frac{3}{2}\times\mathscr{C}_{\rm th}(q)}\right)\equiv\mathscr{C}_{\ell,\rm th}<\mathscr{C}_{\ell}<4/3$$

- PDF $\mathbb{P}(\mathscr{C}_\ell)$ is given by the PDF of $\mathbb{P}(\mathscr{R})$, which depends on inflation model

$$\mathbb{P}(\mathscr{C}_\ell)=\mathbb{P}(\mathscr{R})\left|\frac{\partial\mathscr{R}}{\partial\mathscr{C}_\ell}\right|=\mathbb{P}(\delta\varphi)\left|\frac{\partial\delta\varphi}{\partial\mathscr{R}}\frac{\partial\mathscr{R}}{\partial\mathscr{C}_\ell}\right|$$

- The PBH mass function is

$$\beta=\int_{\mathscr{C}_{\ell,\rm cr}}^{4/3}\mathbb{P}(\mathscr{C}_\ell)\kappa\left[\left(\mathscr{C}_\ell-\frac{3}{8}\mathscr{C}_\ell^2\right)-\mathscr{C}_{\rm th}\right)\Big]^{\gamma}{\rm d}\mathscr{C}_\ell$$

Biagetti et al, 2105.07810
Gow et al, 2211.08348

Press-Schechter-type formalism

Profile can not be calculated in Press-Schechter formalism

- For Type I PBH there is an upper bound

$$\frac{4}{3}\left(1-\sqrt{1-\frac{3}{2}\times\mathscr{C}_{\rm th}(q)}\right)\equiv\mathscr{C}_{\ell,\rm th}<\mathscr{C}_\ell<4/3$$

- PDF $\mathbb{P}(\mathscr{C}_\ell)$ is given by the PDF of $\mathbb{P}(\mathscr{R})$, which depends on inflation model

$$\mathbb{P}(\mathscr{C}_\ell)=\mathbb{P}(\mathscr{R})\left|\frac{\partial\mathscr{R}}{\partial\mathscr{C}_\ell}\right|=\mathbb{P}(\delta\varphi)\left|\frac{\partial\delta\varphi}{\partial\mathscr{R}}\frac{\partial\mathscr{R}}{\partial\mathscr{C}_\ell}\right|$$

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$$\beta=\int_{\mathscr{C}_{\ell,\rm cr}}^{4/3}\mathbb{P}(\mathscr{C}_\ell)\kappa\left[\left(\mathscr{C}_\ell-\frac{3}{8}\mathscr{C}_\ell^2\right)-\mathscr{C}_{\rm th}\right)\Big]^\gamma{\rm d}\mathscr{C}_\ell$$

Biagetti et al, 2105.07810
Gow et al, 2211.08348

Peak Theory

- Instead, in peak theory, BBKS gives the profile of a local peak, from which the critical value of $\overline{\mathscr{C}}_c$ can be calculated analytically. Then we transfer it to the critical value of the Laplacian of the curvature perturbation ($\nabla^2\mathscr{R}$), μ_2.

- The statistic quantities are μ_2 and its dispersion, μ_4.

- The PBH mass function is then

$$\beta(M) = \int_{\mu_2 \geq \mu_{2,th}} d\mu_2 d\mu_4 \cdot n_{\text{peak}}(\mu_2(M, \mu_4), \mu_4) \left| \frac{d \ln M}{d\mu_2} \right|^{-1} M(\mu_2, \mu_4)$$

Kitajima et al, 2109.00791
SP, Sasaki, Takhistov, Jianing Wang, in prep

Why non-Gaussianity?

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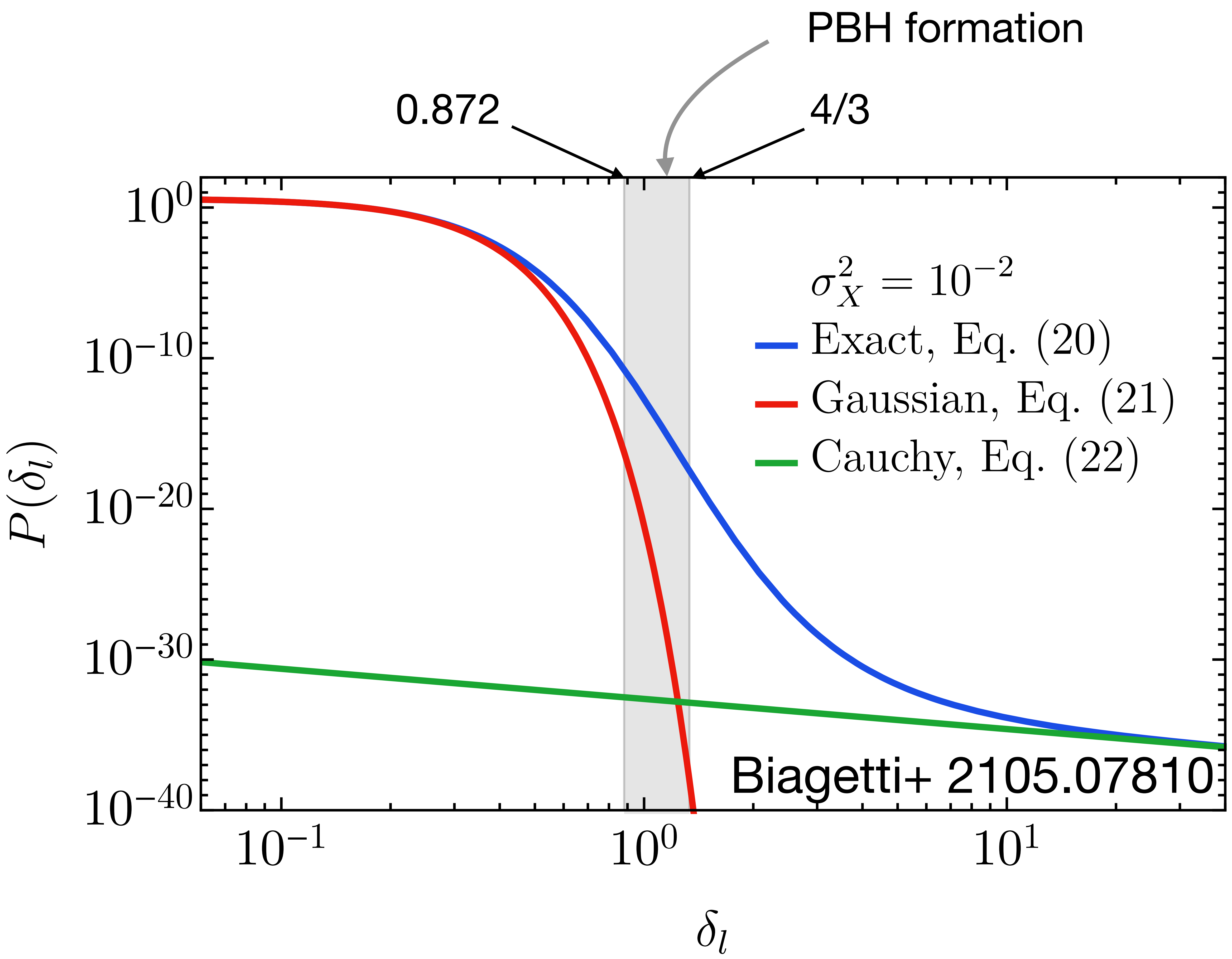
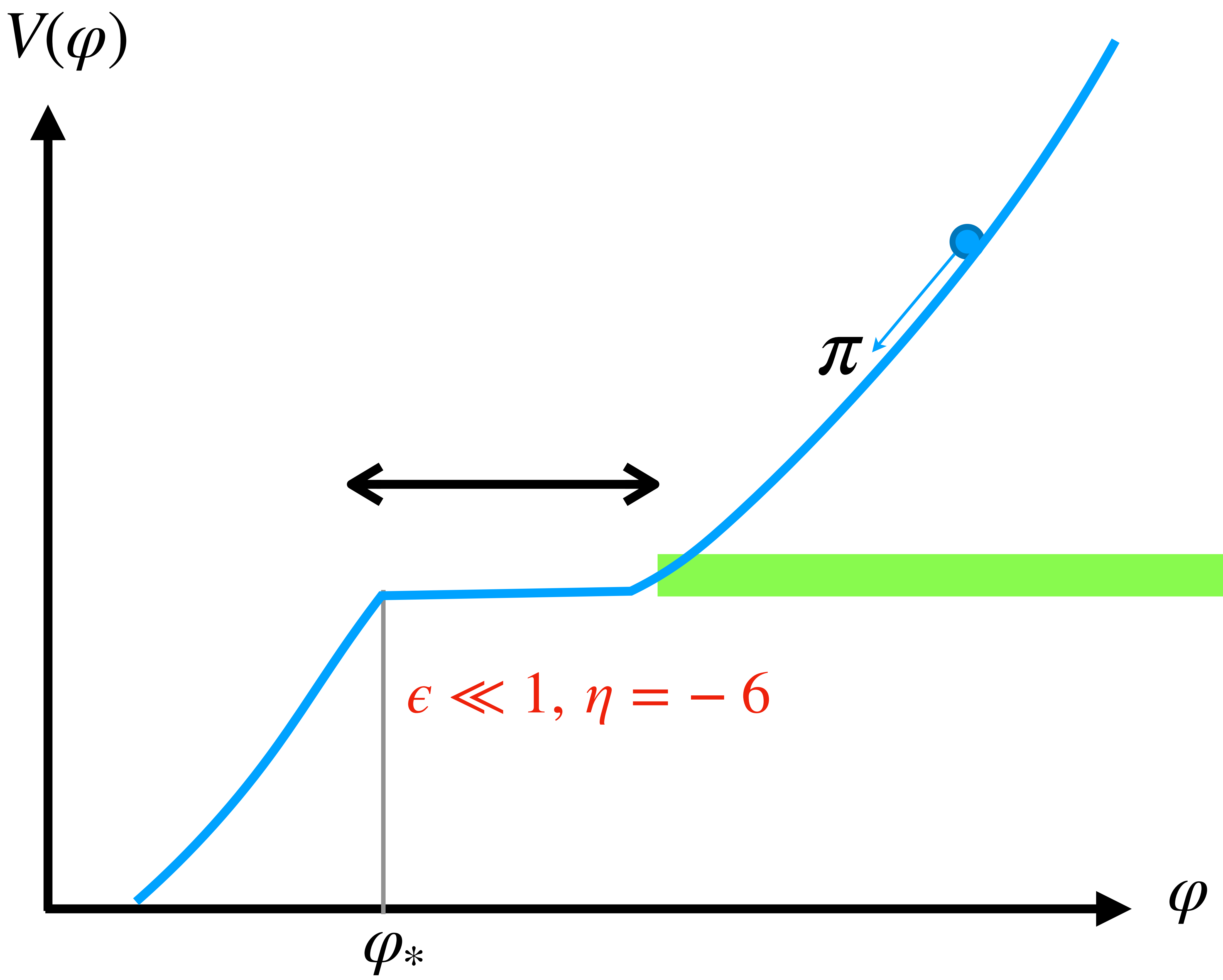
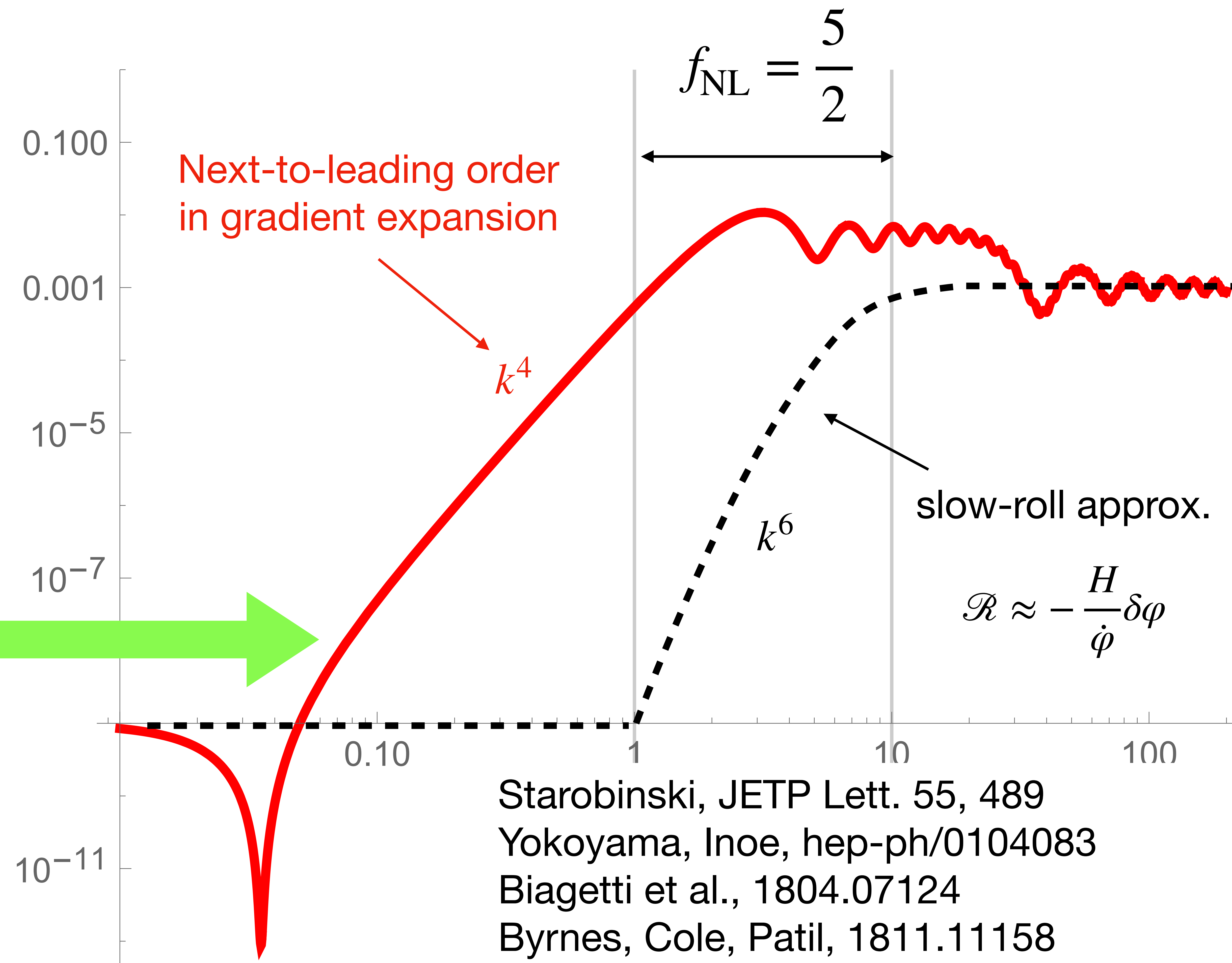


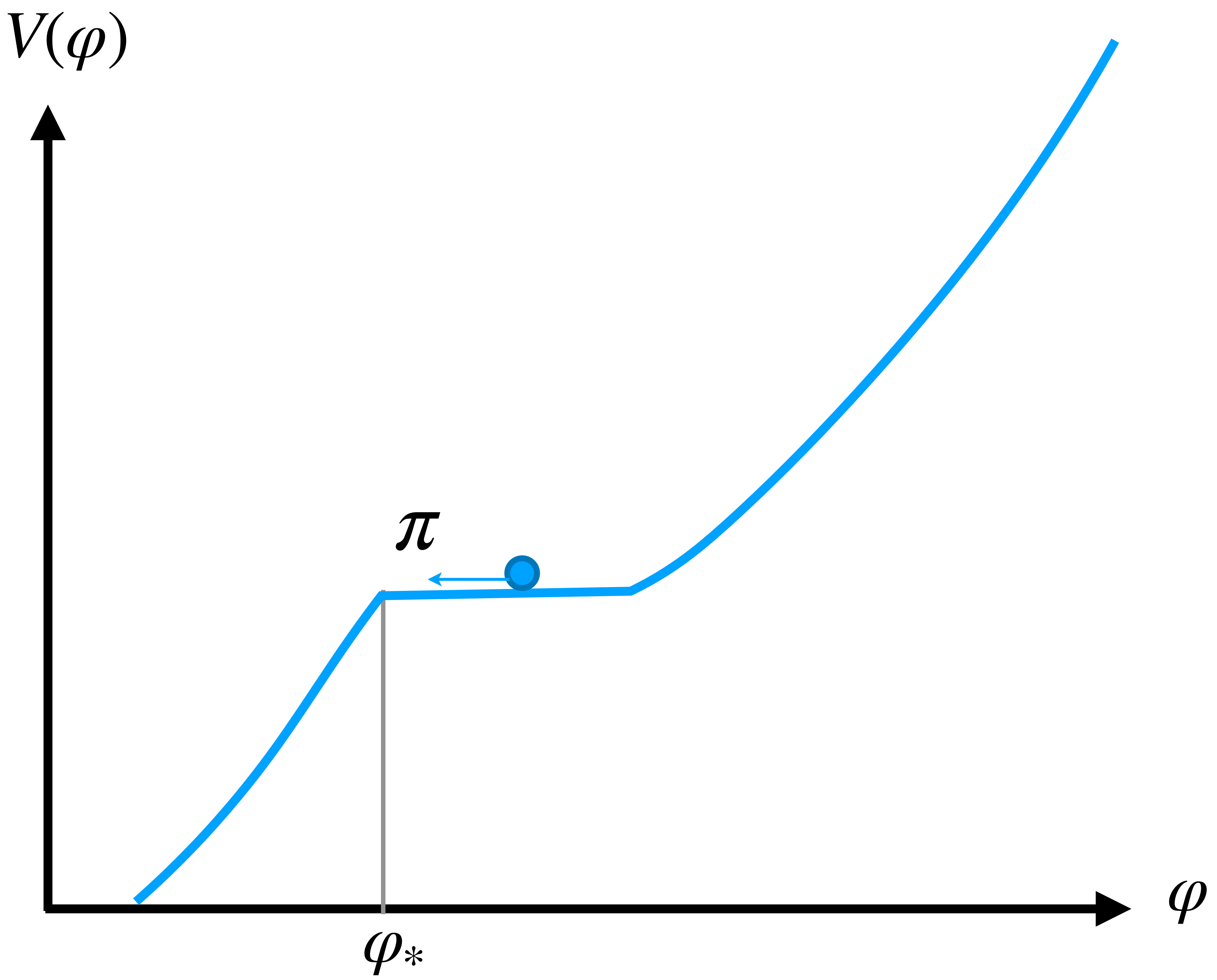
Ultra-slow-roll inflation





Starobinski, JETP Lett. 55, 489
Yokoyama, Inoe, hep-ph/0104083
Biagetti et al., 1804.07124
Byrnes, Cole, Patil, 1811.11158
Cole, Gow, Byrnes, Patil, 2204.07573
SP, Jianing Wang, 2209.14183
Domenech, Vargas, Vargas, 2309.05750

Ultra-slow-roll inflation



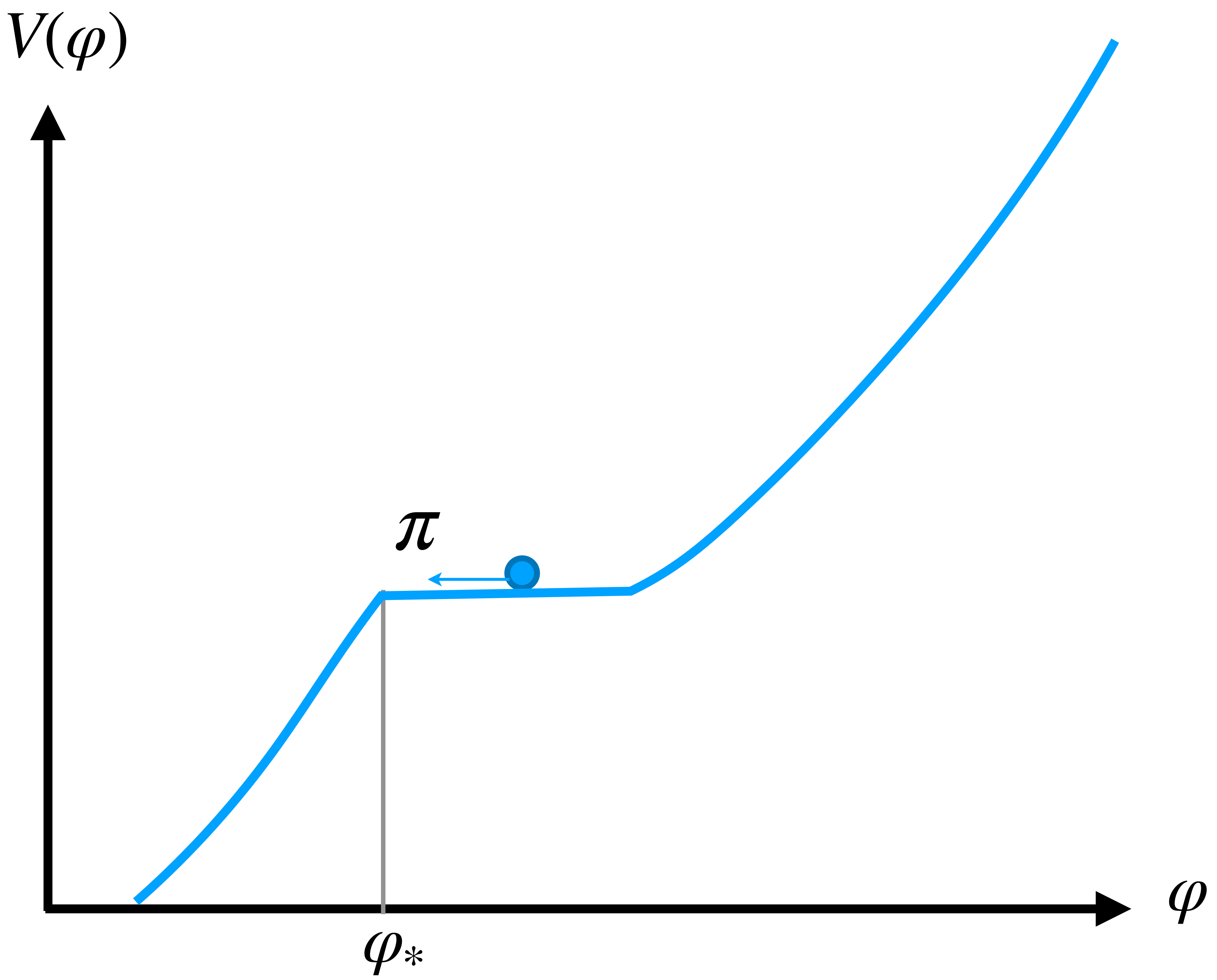
$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}N^2} - 3\frac{\mathrm{d}\varphi}{\mathrm{d}N} = 0 \qquad N = \int_{t_*}^{t} H\mathrm{d}t$$

$$\varphi(N) = \varphi_* + \frac{\pi_*}{3}\left(1 - e^{3N}\right)$$

$$\pi(N) \equiv -\frac{\mathrm{d}\varphi}{\mathrm{d}N} = \pi_* e^{3N}$$

$$N = -\frac{1}{3}\ln\frac{\pi_*}{\pi}$$

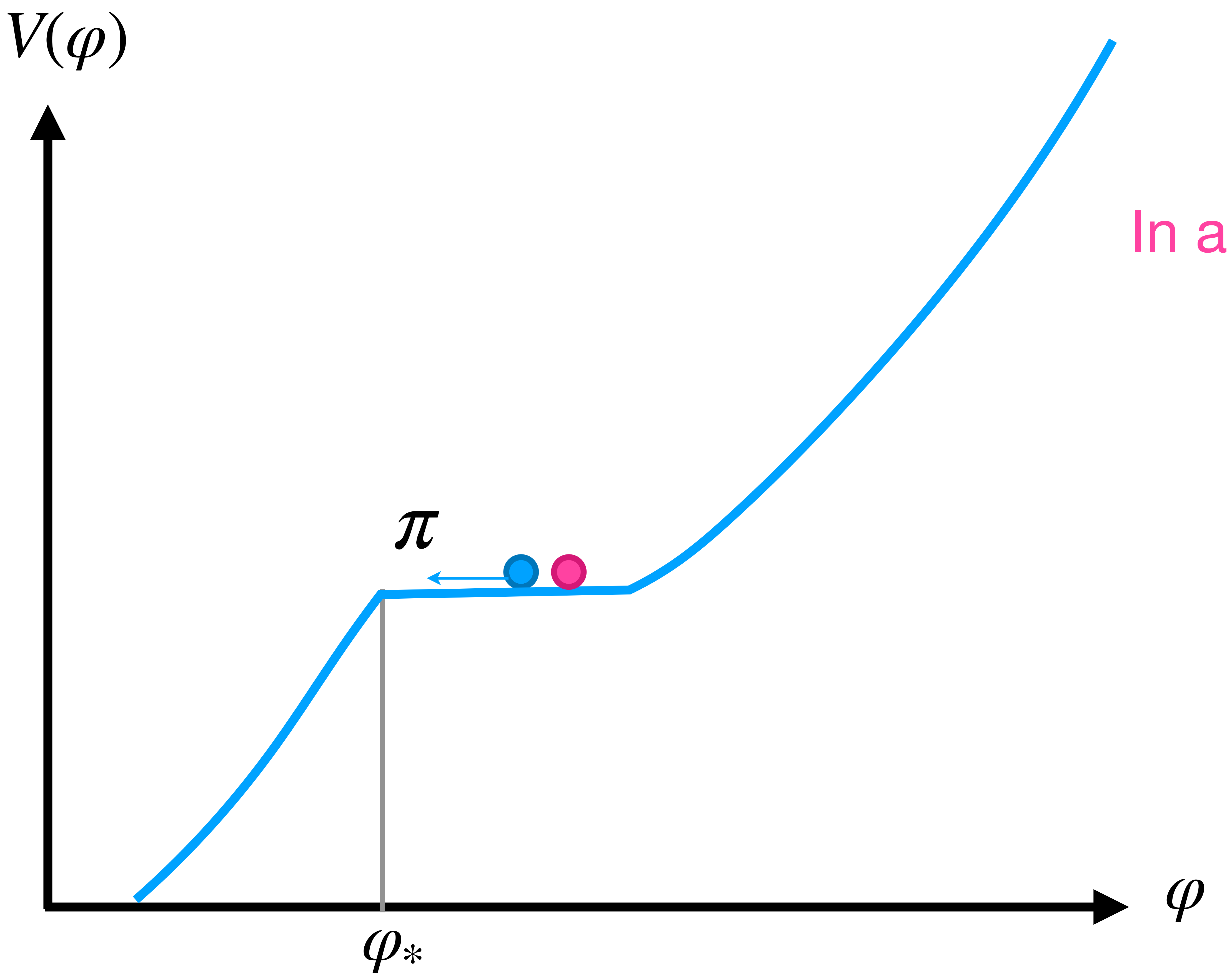
Ultra-slow-roll inflation



In the "fiducial" patch

$$N = -\frac{1}{3}\ln\frac{\pi_*}{\pi}$$

Ultra-slow-roll inflation



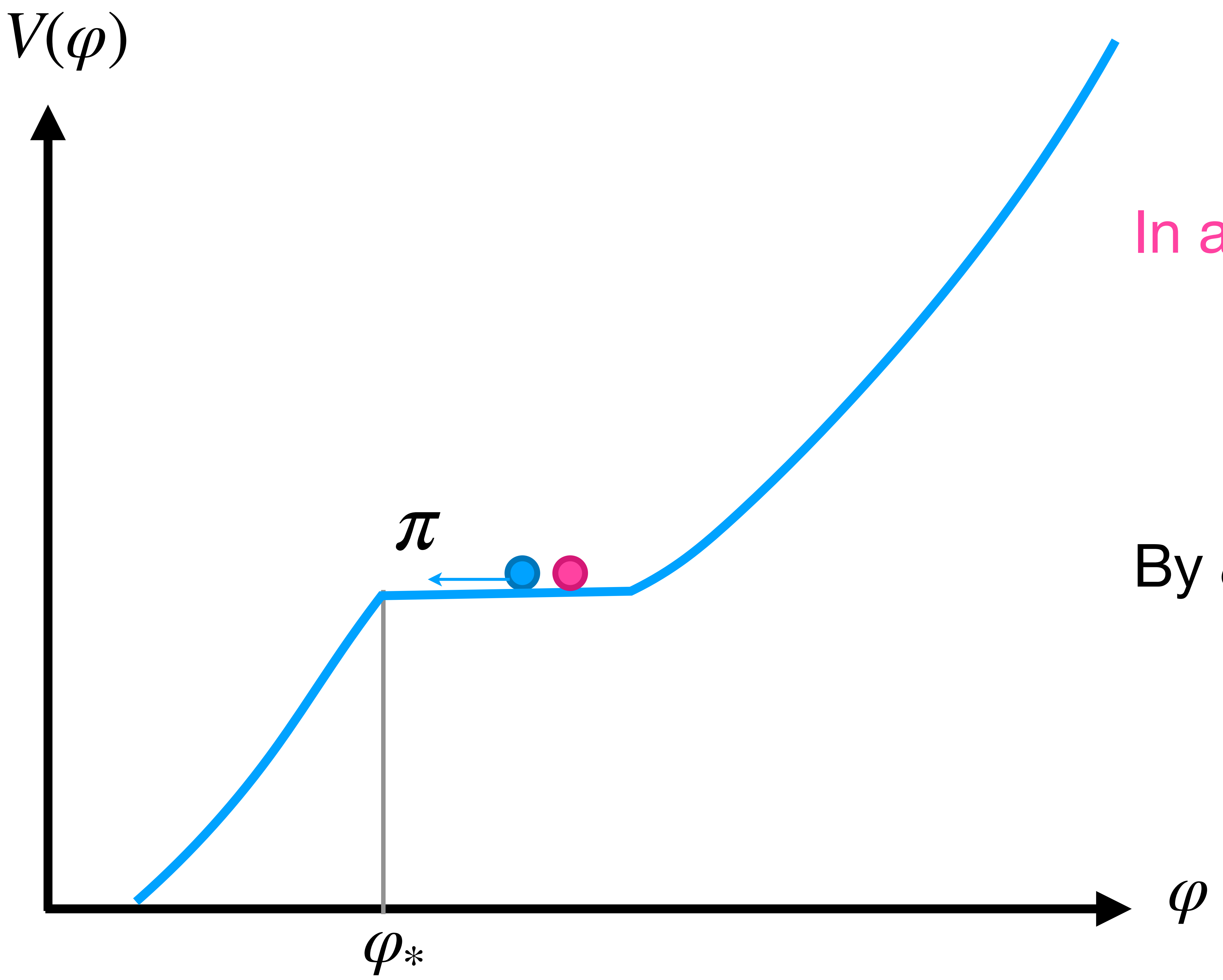
In the “fiducial” patch

$$N = -\frac{1}{3}\ln\frac{\pi_*}{\pi}$$

In a perturbed patch

$$\tilde{N} = -\frac{1}{3}\ln\frac{\tilde{\pi}_*}{\tilde{\pi}}$$

Ultra-slow-roll inflation



In the “fiducial” patch

$$N = -\frac{1}{3}\ln\frac{\pi_*}{\pi}$$

In a perturbed patch

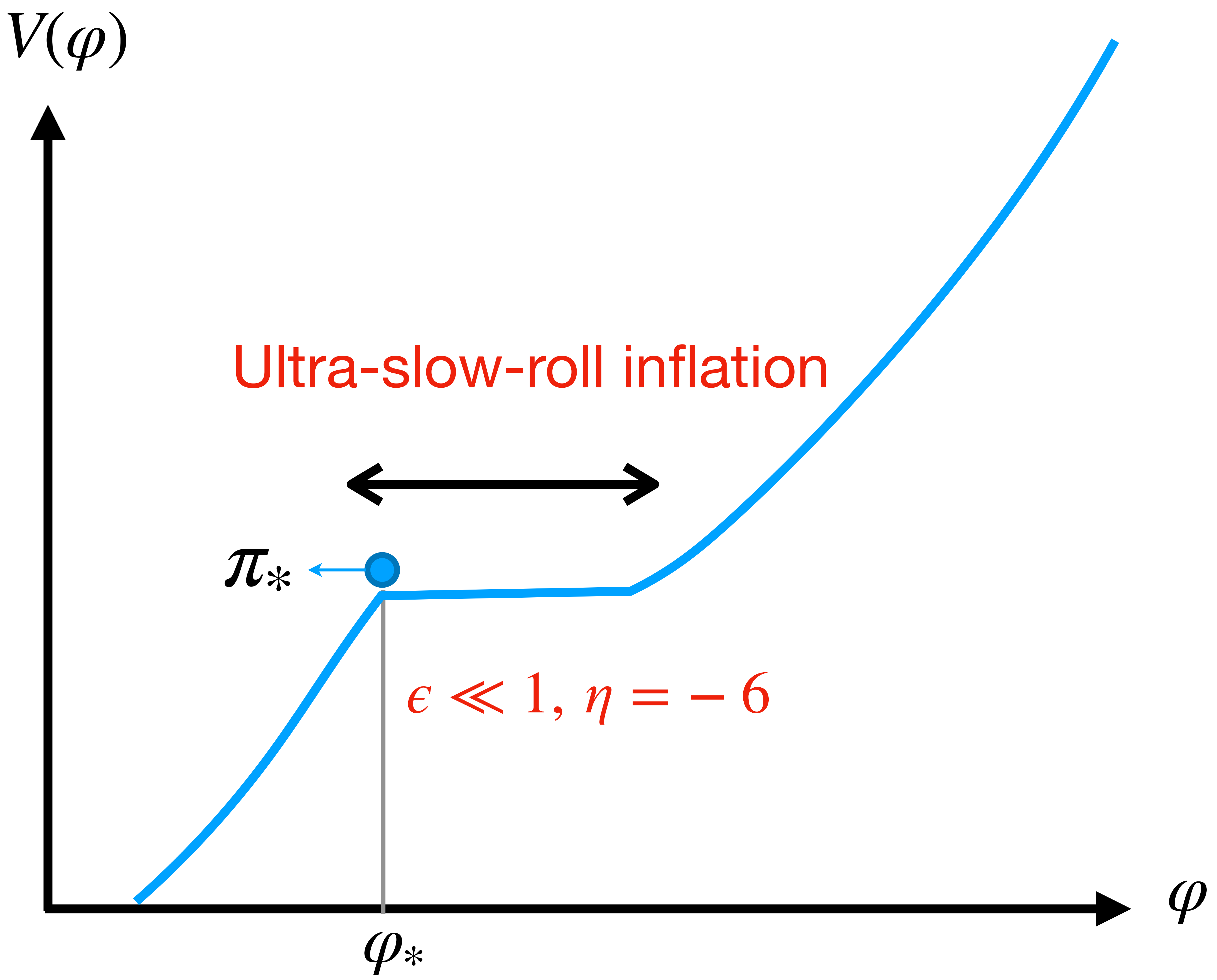
$$\tilde{N} = -\frac{1}{3}\ln\frac{\tilde{\pi}_*}{\tilde{\pi}}$$

By δN formalism, the curvature perturbation is

$$\mathcal{R} = \delta N = \tilde{N} - N = \frac{1}{3}\ln\frac{\tilde{\pi}}{\pi}\frac{\pi_*}{\tilde{\pi}_*}$$

$$= \frac{1}{3}\ln\left(1 - \frac{\delta\pi}{\pi}\right) - \frac{1}{3}\ln\left(1 - \frac{\delta\pi_*}{\pi_*}\right)$$

Ultra-slow-roll inflation

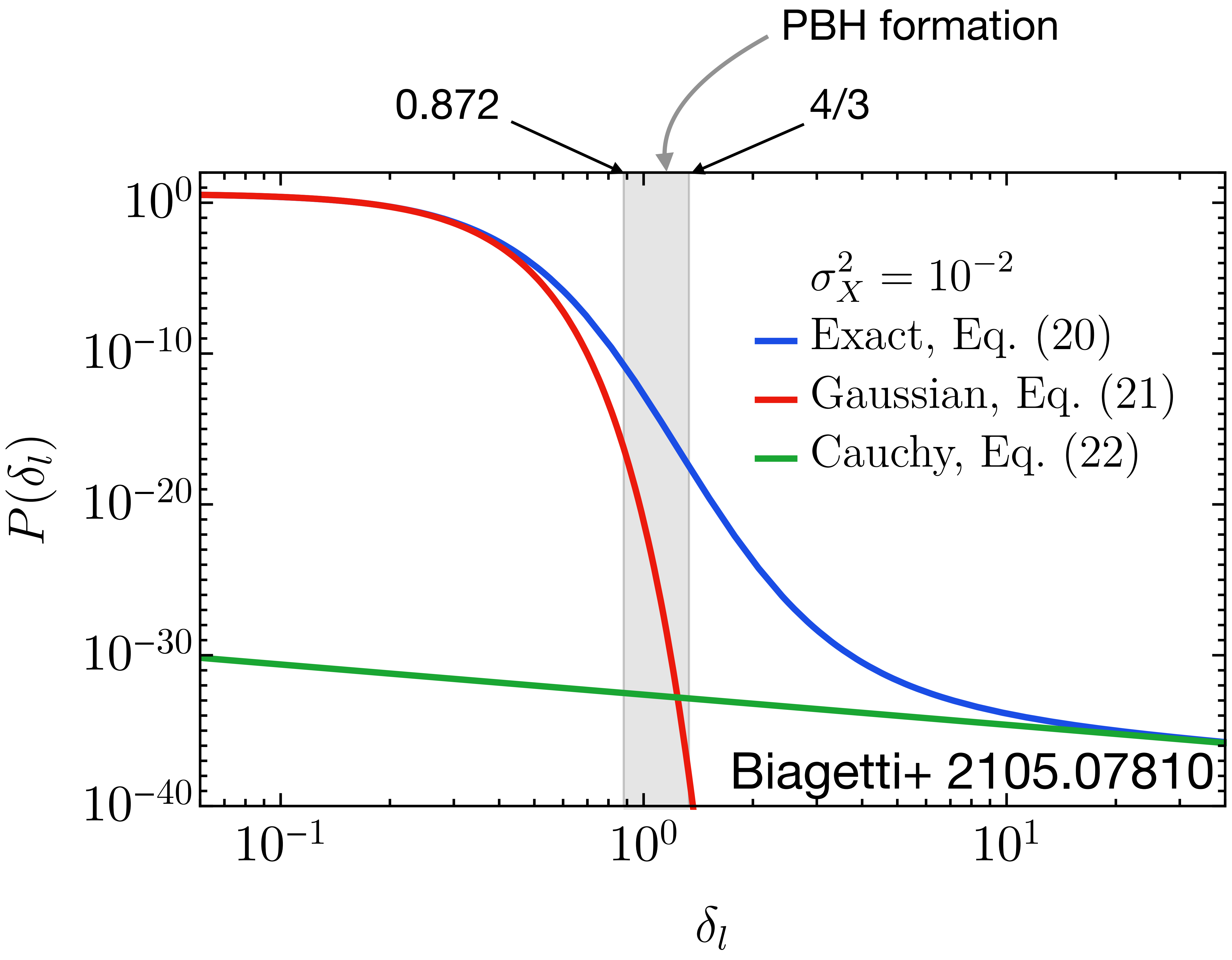


$$\mathcal{R} \approx -\frac{1}{3}\ln\left(1 - \frac{\delta\pi_*}{\pi_*}\right)$$

$$\left(f_{\rm NL} = \frac{5}{2}, \quad g_{\rm NL} = -\frac{25}{3}, \quad \cdots\right)$$

Namjoo, Firouzjahi, Sasaki, 1210.3692
Chen, Firouzjahi, Komatsu, Namjoo, Sasaki, 1308.5341
Cai, Chen, Namjoo, Sasaki, Wang, Wang, 1712.09998
Biagetti, Franciolini, Kehagias, Riotto, 1804.07124
Passaglia, Hu, Motohashi, 1812.08243
SP and Sasaki, 2211.13932
SP, 2404.06151
Also verified by stochastic approach, see e.g.
Pattison et al 2101.05741

Ultra-slow-roll inflation



$$\mathscr{R} = -\frac{1}{3}\ln\left(1 - \frac{\delta\pi_*}{\pi_*}\right)$$

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Namjoo, Firouzjahi, Sasaki, 1210.3692
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SP and Sasaki, 2211.13932
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Also verified by stochastic approach, see e.g.
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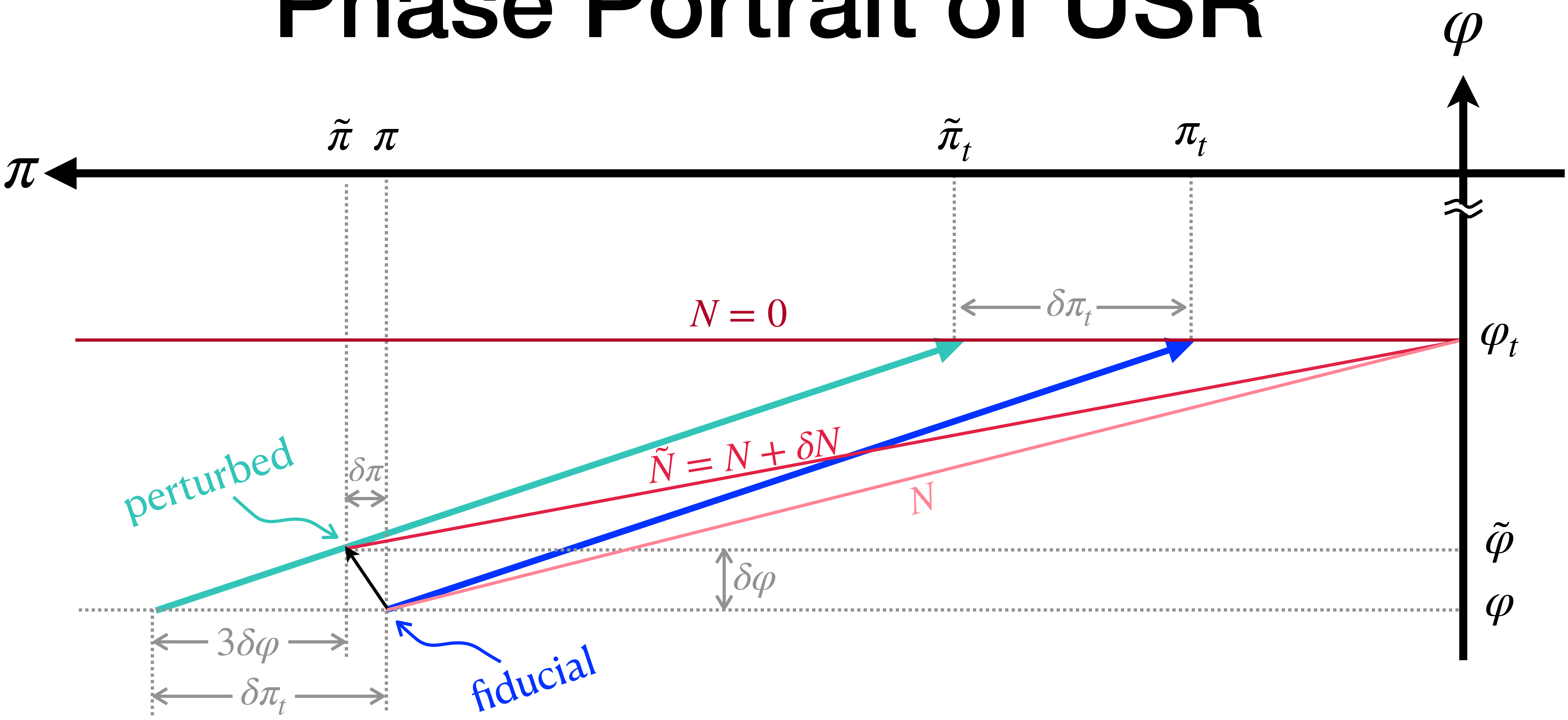
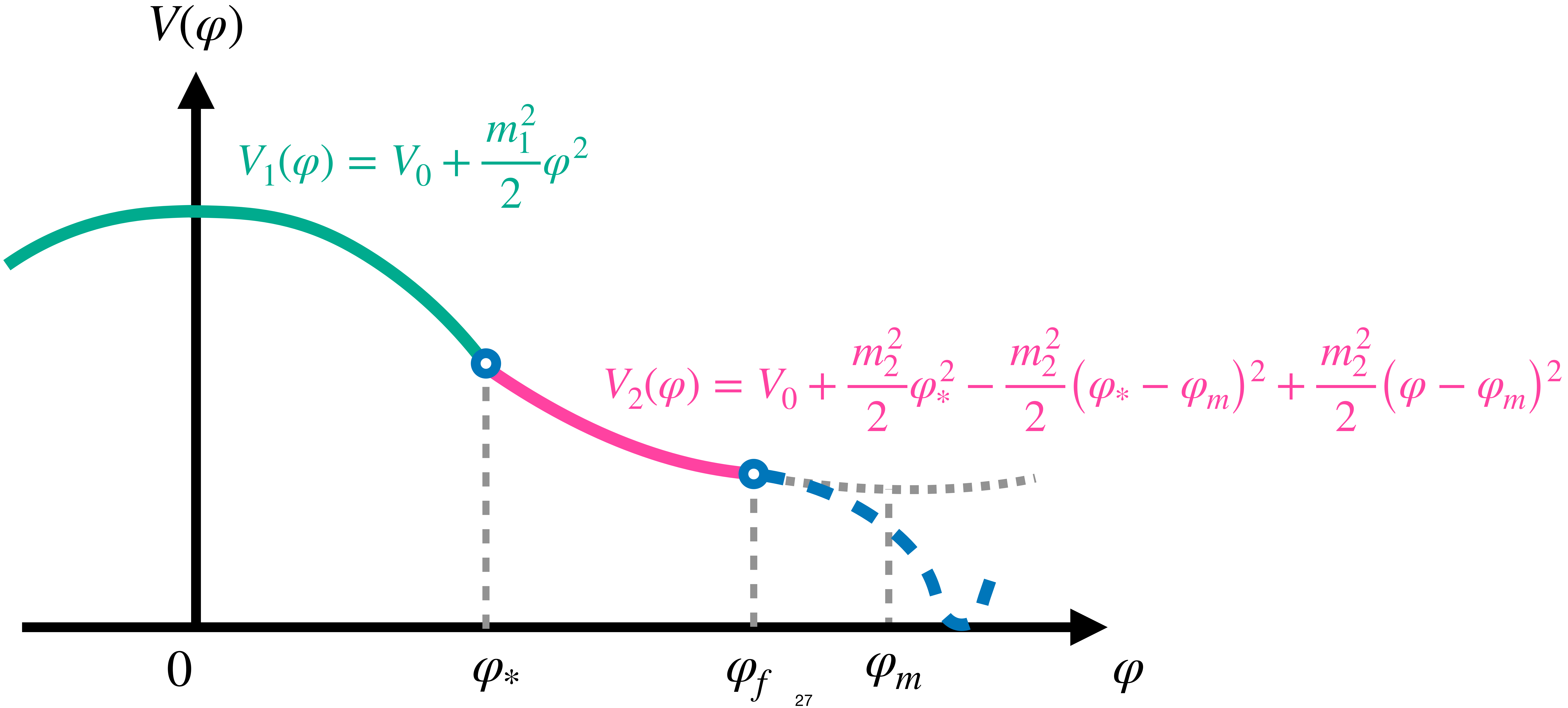
Phase Portrait of USR

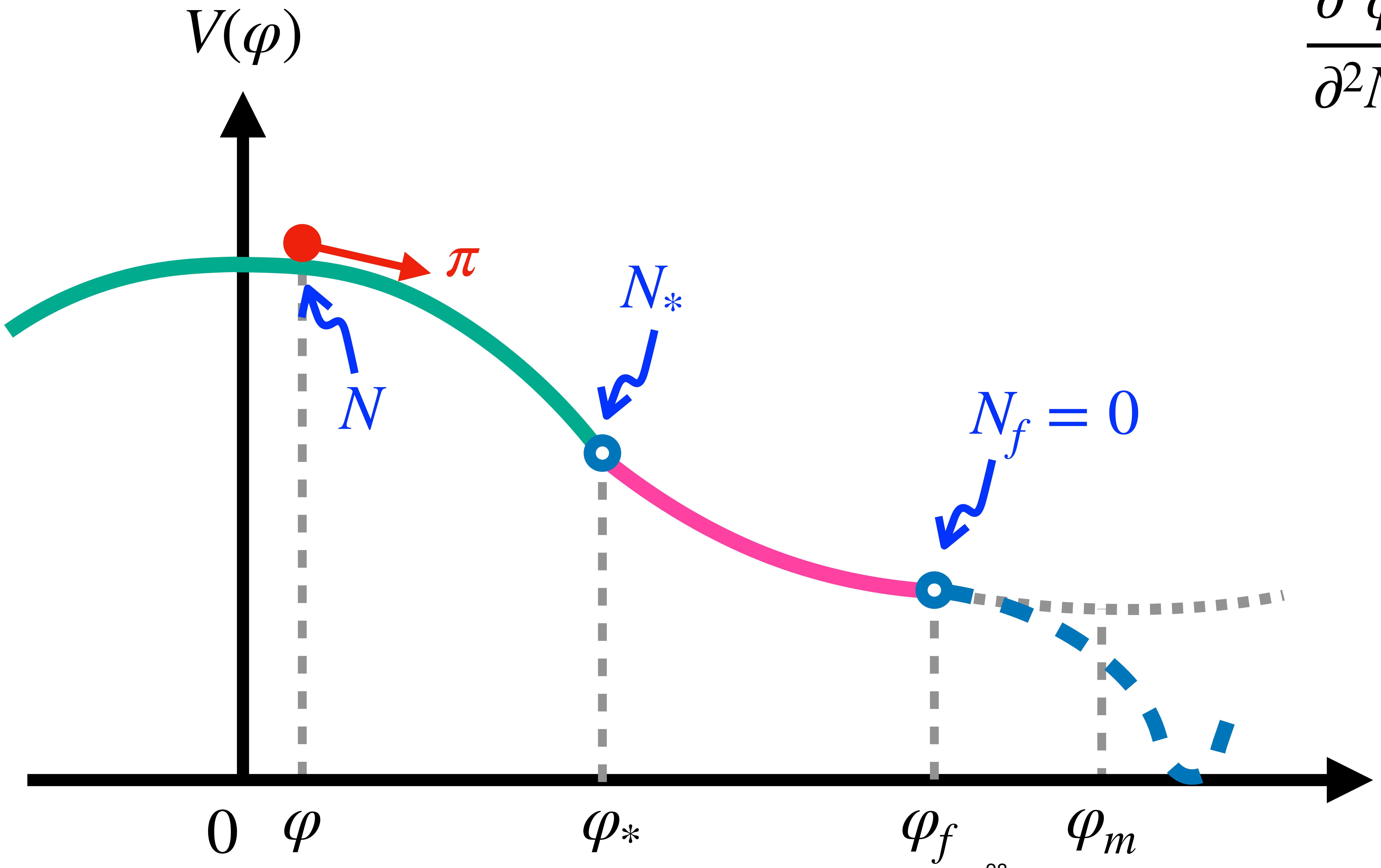


SP and Sasaki, 2211.13932
SP, 2404.06151

piecewise quadratic potential



piecewise quadratic potential



$$\frac{\partial^2\varphi}{\partial^2 N} - 3\frac{\partial\varphi}{\partial N} + 3\eta_V\varphi = 0$$

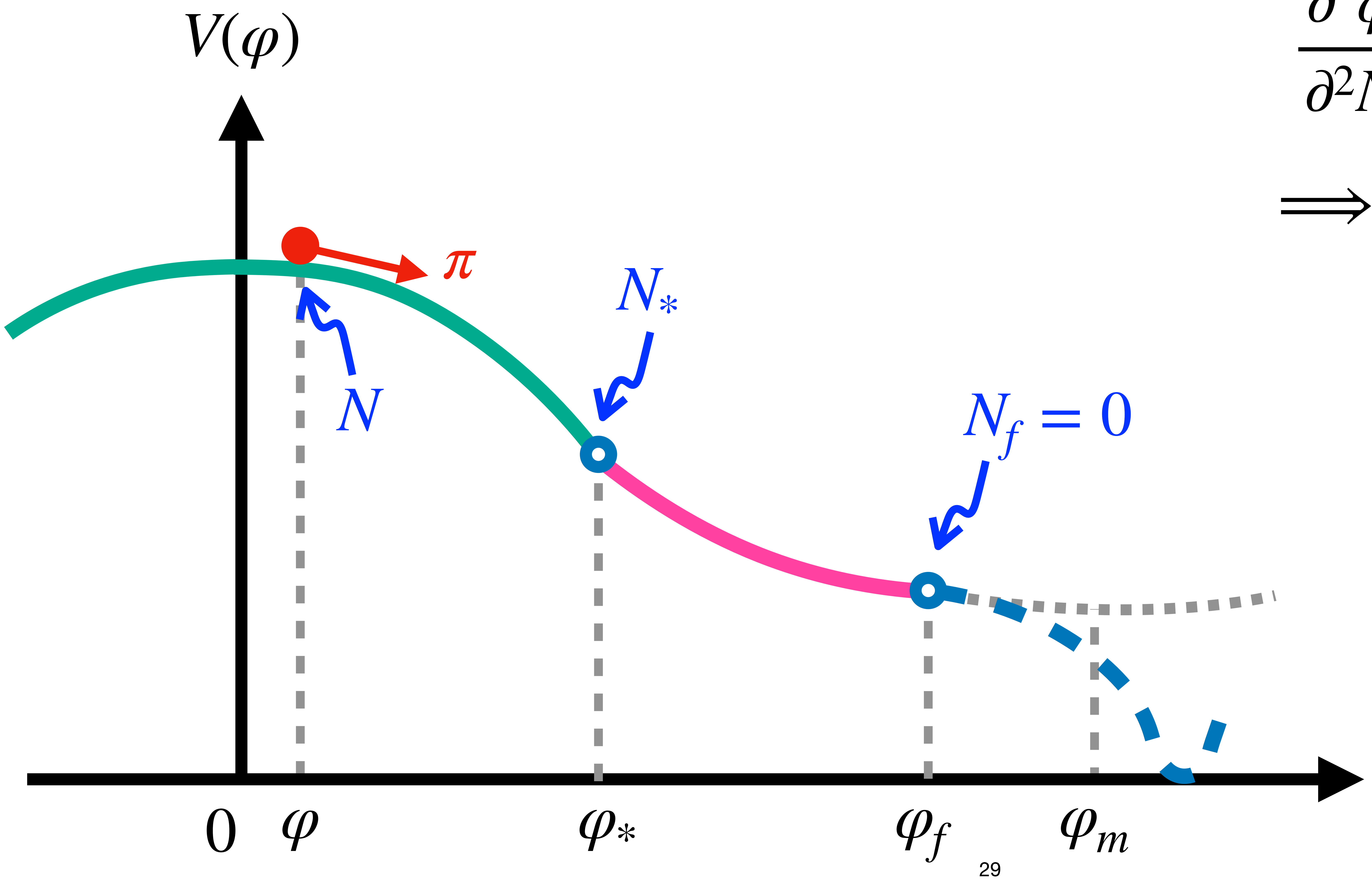
$$\eta_V = \frac{m_1^2}{3H^2}$$

$$N = \int_t^{t_f} H dt$$

$$V_1(\varphi) = V_0 + \frac{m_1^2}{2}\varphi^2$$

$$V_2(\varphi) = V_0 + \frac{m_2^2}{2}\varphi_*^2 - \frac{m_2^2}{2}(\varphi_* - \varphi_m)^2 + \frac{m_2^2}{2}(\varphi - \varphi_m)^2$$

background solution



$$\frac{\partial^2\varphi}{\partial^2 N} - 3\frac{\partial\varphi}{\partial N} + 3\eta_V\varphi = 0$$

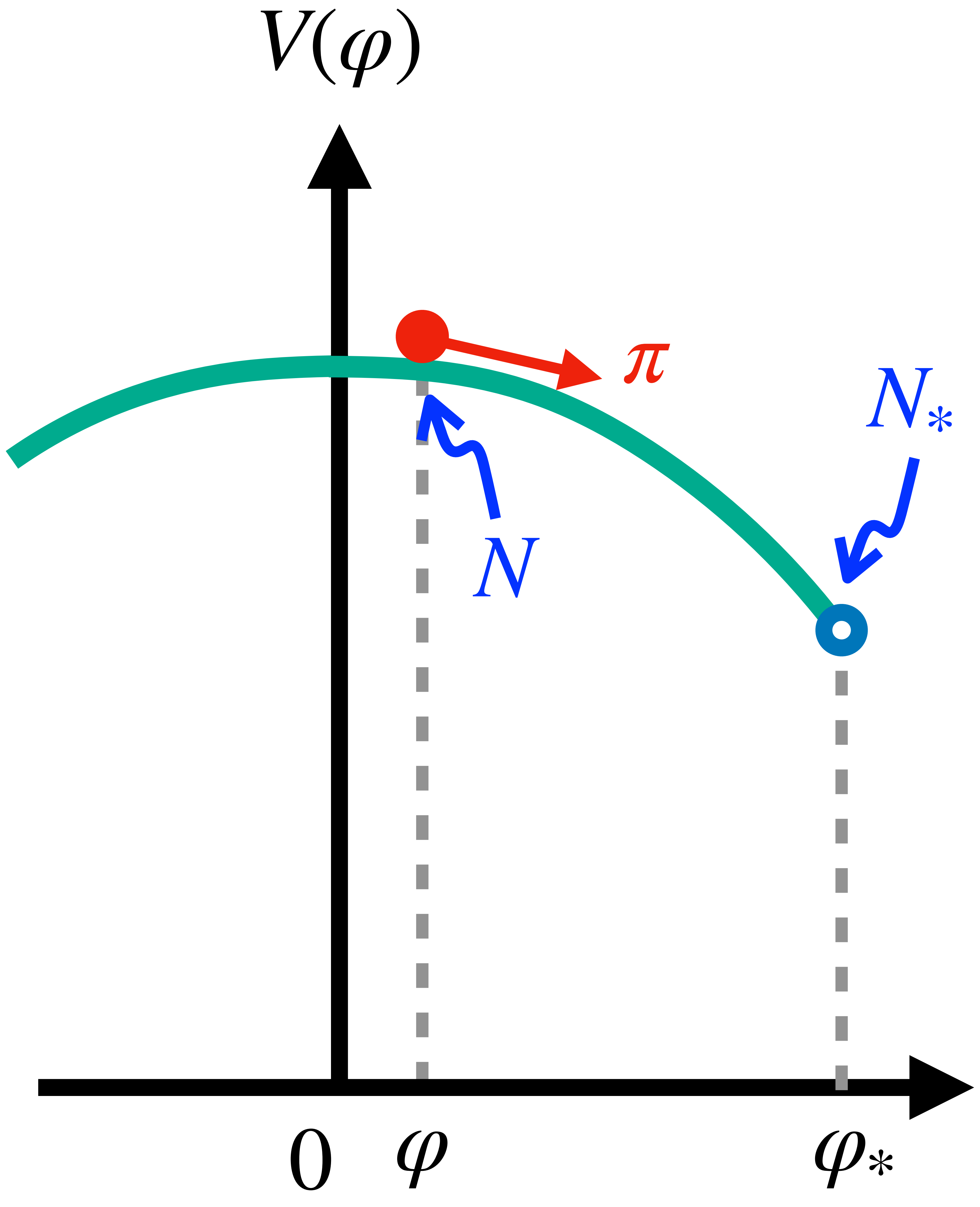
$$\Longrightarrow \quad \varphi = c_+ e^{\lambda_+ N} + c_- e^{\lambda_- N}$$

$$\lambda_\pm = \frac{3 \pm \sqrt{9 - 12\eta_V}}{2}$$

$$V_1(\varphi) = V_0 + \frac{m_1^2}{2}\varphi^2$$

$$V_2(\varphi) = V_0 + \frac{m_2^2}{2}\varphi_*^2 - \frac{m_2^2}{2}(\varphi_* - \varphi_m)^2 + \frac{m_2^2}{2}(\varphi - \varphi_m)^2$$

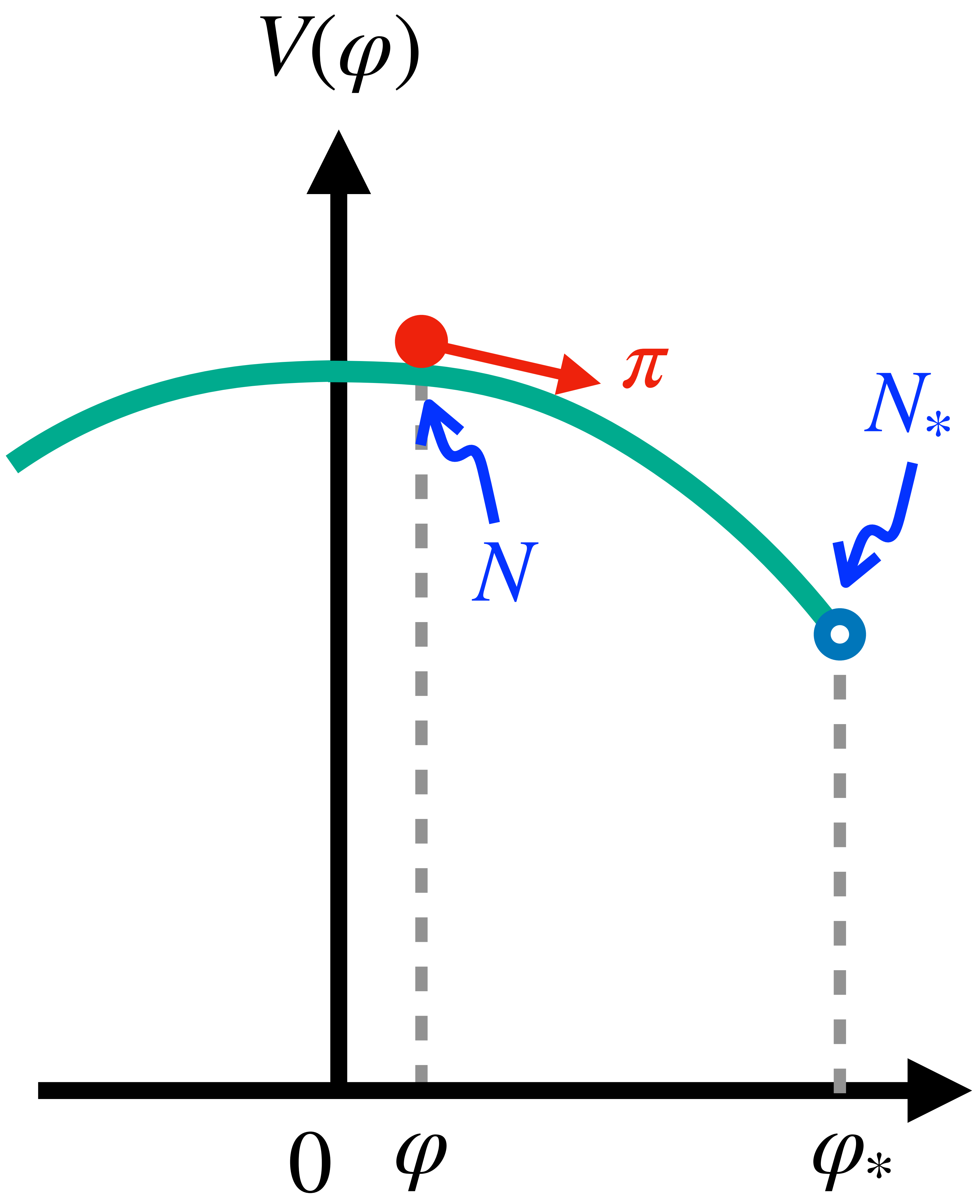
background solution



$$\varphi(N) = c_+ e^{\lambda_+(N-N_*)} + c_- e^{\lambda_-(N-N_*)}$$

$$-\pi(N) \equiv \frac{\partial \varphi}{\partial N} = \lambda_+ c_+ e^{\lambda_+(N-N_*)} + \lambda_- c_- e^{\lambda_-(N-N_*)}$$

background solution



$$\varphi(N) = c_+ e^{\lambda_+(N-N_*)} + c_- e^{\lambda_-(N-N_*)}$$

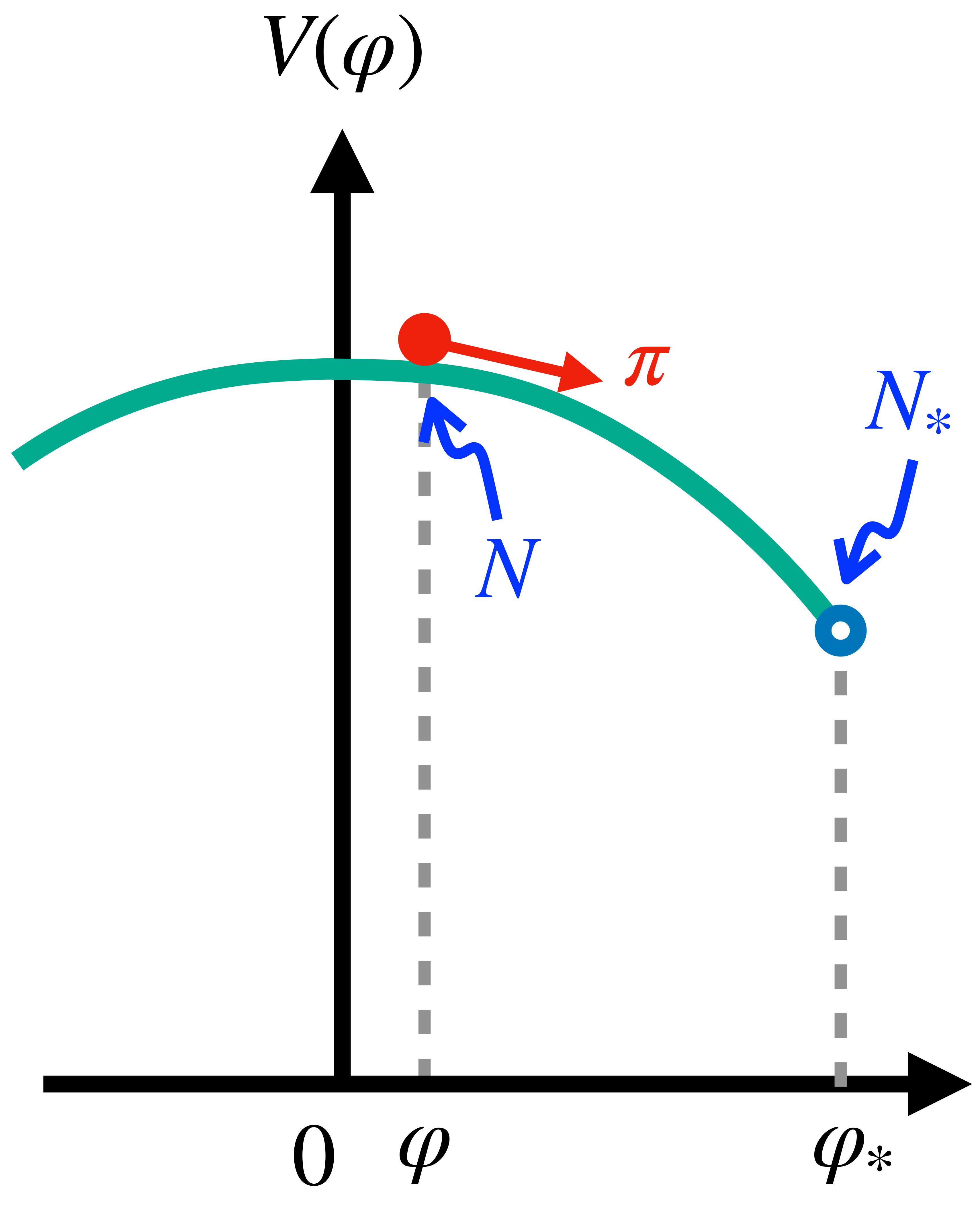
$$-\pi(N) \equiv \frac{\partial\varphi}{\partial N} = \lambda_+ c_+ e^{\lambda_+(N-N_*)} + \lambda_- c_- e^{\lambda_-(N-N_*)}$$

$$\varphi(N_*) \equiv \varphi_* = c_+ + c_-$$

$$-\pi(N_*) \equiv \pi_* = \lambda_+ c_+ + \lambda_- c_-$$

$$\Longrightarrow \quad c_\pm = \mp \frac{\pi_* + \lambda_\mp \varphi_*}{\lambda_+ - \lambda_-}$$

background solution



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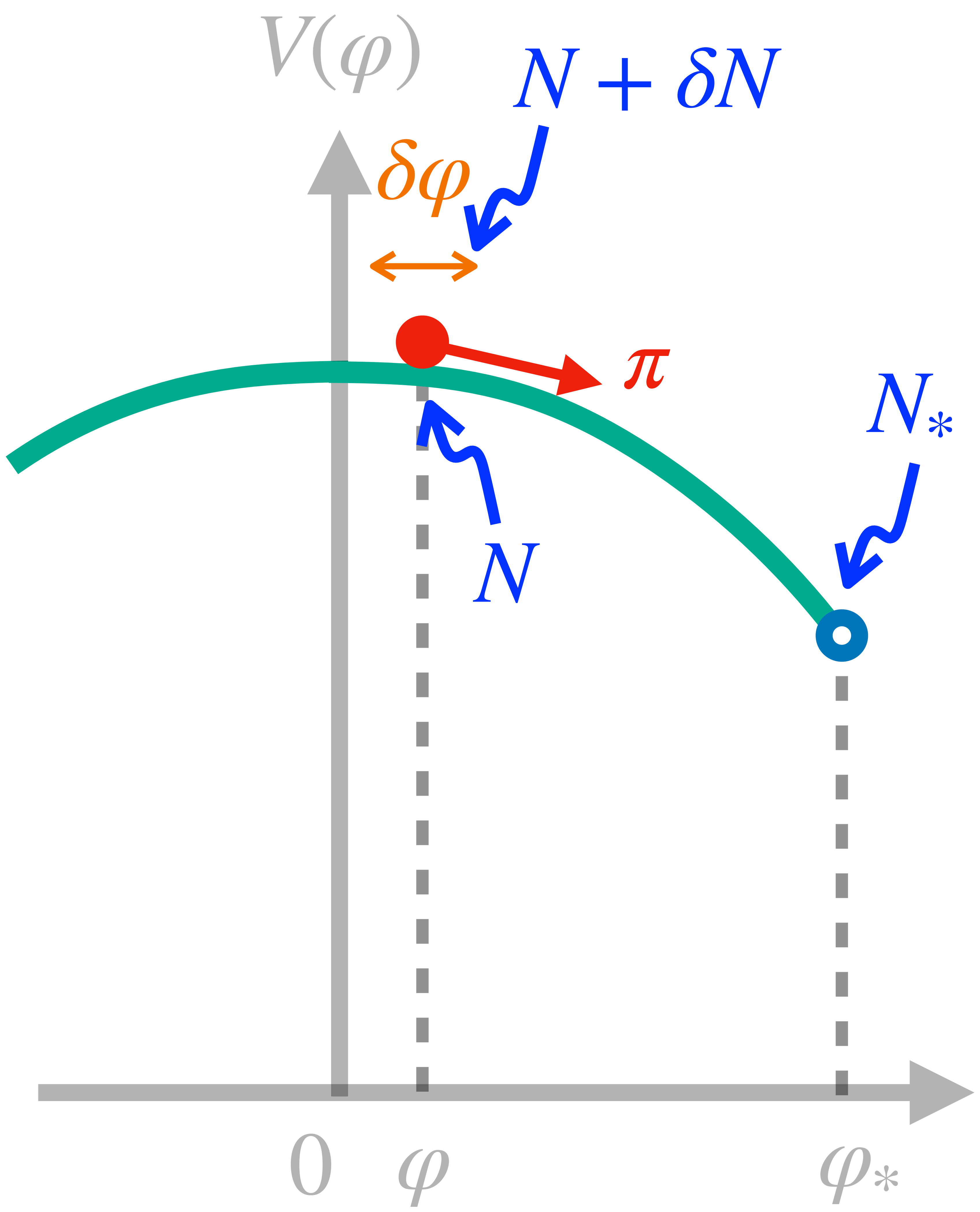
$$\Longrightarrow \quad \boxed{c_\pm = \mp \frac{\pi_* + \lambda_\mp \varphi_*}{\lambda_+ - \lambda_-}}$$

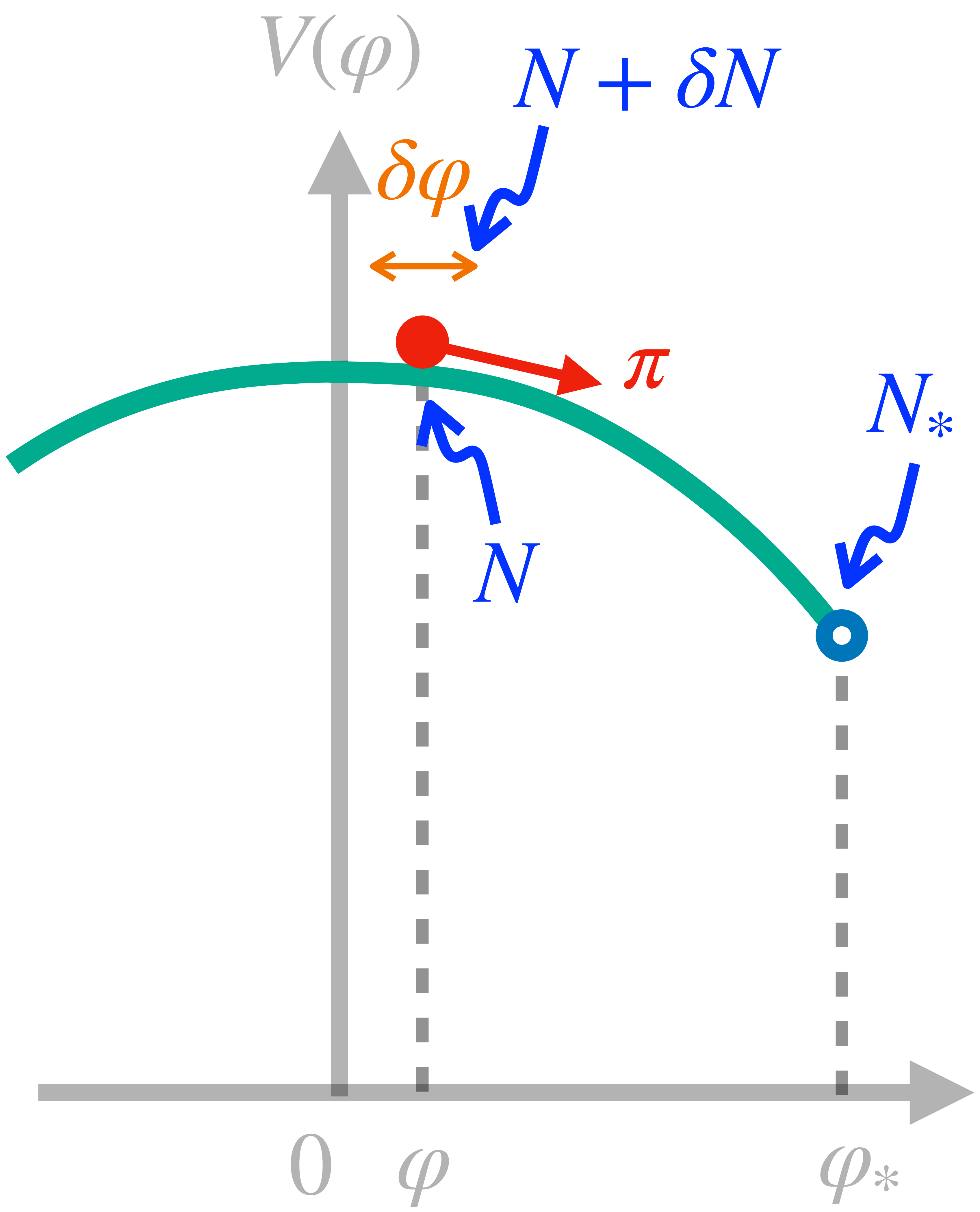
Logarithmic Duality



The (fiducial) e-folding number can be expressed by (φ, π) and their values on the boundary (φ_*, π_*).

$$\left.\begin{aligned} \frac{\pi + \lambda_+\varphi}{\pi_* + \lambda_+\varphi_*} &= e^{\lambda_+(N-N_*)} \\ \frac{\pi + \lambda_-\varphi}{\pi_* + \lambda_-\varphi_*} &= e^{\lambda_-(N-N_*)} \end{aligned}\right\} \Longrightarrow N - N_* = \frac{1}{\lambda_\pm} \ln \frac{\pi + \lambda_\mp\varphi}{\pi_* + \lambda_\mp\varphi_*}$$

Logarithmic Duality



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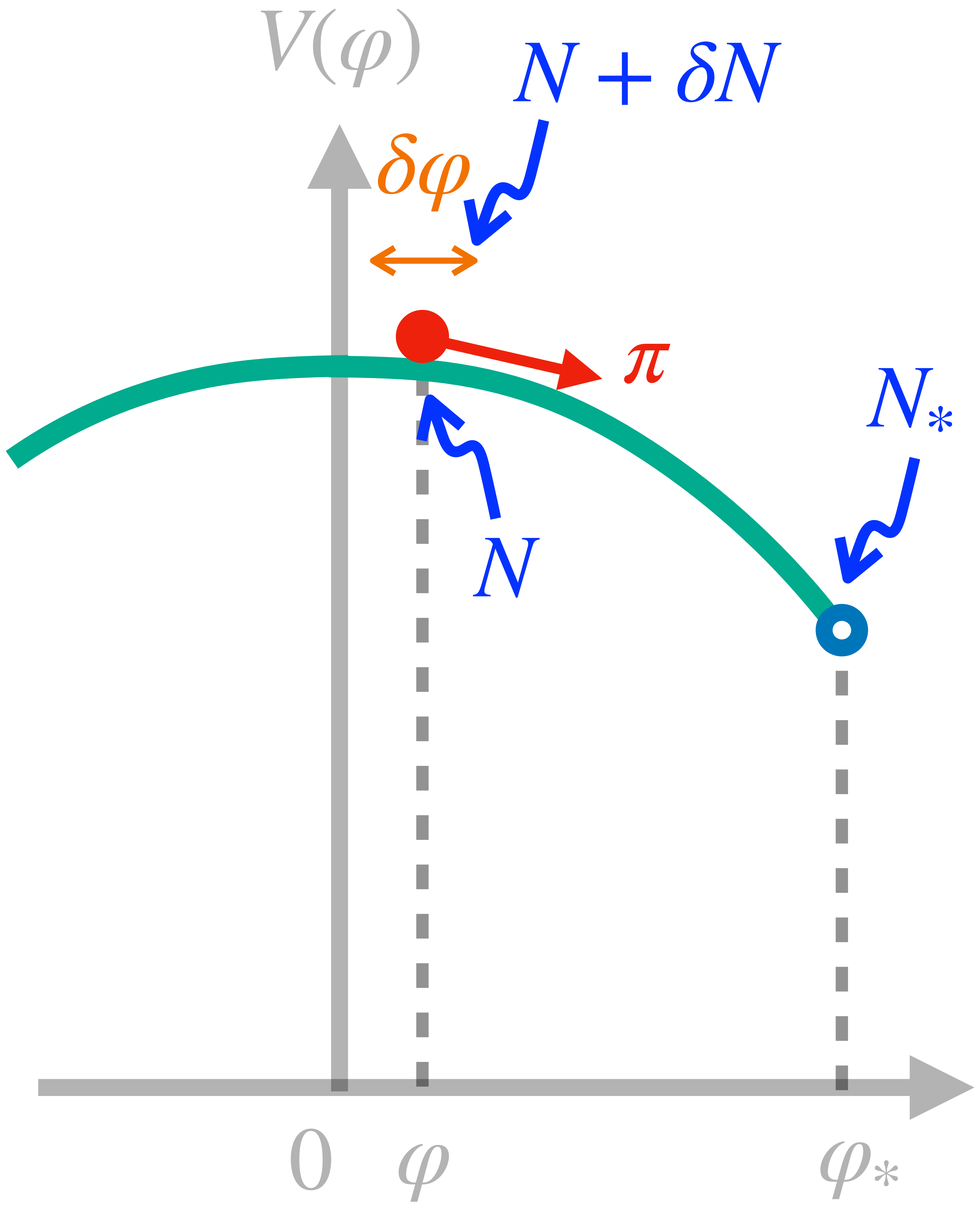
For another trajectory, we take the perturbation as

$$\left.\begin{aligned} N &\to N + \delta N \\ \varphi &\to \varphi + \delta\varphi \\ \pi &\to \pi + \delta\pi \\ \pi_* &\to \pi_* + \delta\pi_* \end{aligned}\right\} \quad N - N_* + \delta(N - N_*) = \frac{1}{\lambda_\pm} \ln \frac{\pi + \delta\pi + \lambda_\mp(\varphi + \delta\varphi)}{\pi_* + \delta\pi_* + \lambda_\mp\varphi_*}$$

And then subtract the fiducial N from $N + \delta N$:

Logarithmic Duality



$$\left.\begin{aligned}\frac{\pi + \lambda_+\varphi}{\pi_* + \lambda_+\varphi_*} &= e^{\lambda_+(N-N_*)}\\ \frac{\pi + \lambda_-\varphi}{\pi_* + \lambda_-\varphi_*} &= e^{\lambda_-(N-N_*)}\end{aligned}\right\} \Longrightarrow N - N_* = \frac{1}{\lambda_\pm}\ln\frac{\pi + \lambda_\mp\varphi}{\pi_* + \lambda_\mp\varphi_*}$$

$$\Longrightarrow \mathscr{R} = \delta(N - N_*)$$

$$= \frac{1}{\lambda_\pm}\ln\left(1 + \frac{\delta\pi + \lambda_\mp\delta\varphi}{\pi + \lambda_\mp\varphi}\right) - \frac{1}{\lambda_\pm}\ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_\mp\varphi_*}\right)$$

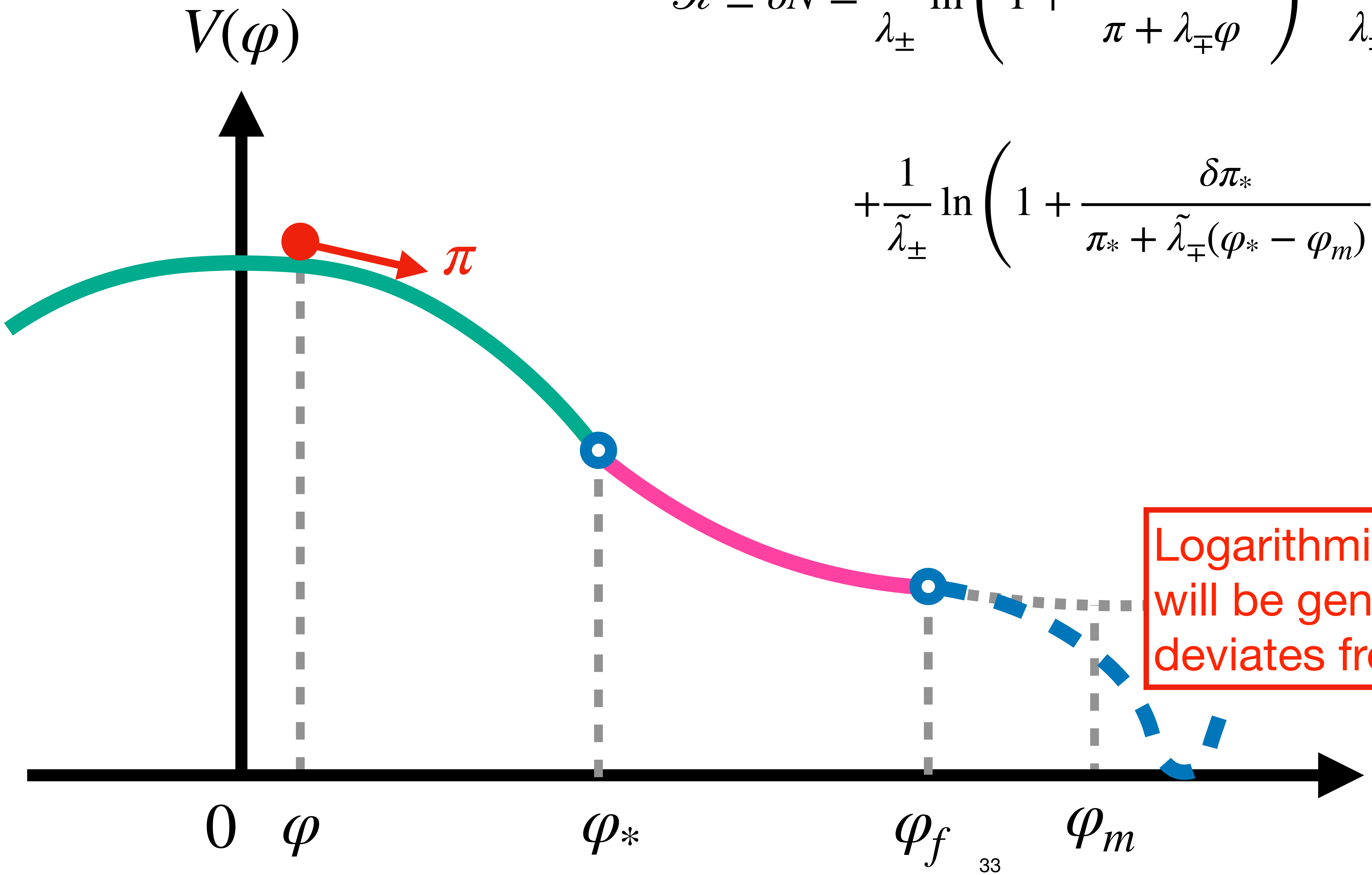
Logarithmic duality of the curvature perturbation

SP and Sasaki, 2211.13932

Logarithmic Duality

$$\mathscr{R} \equiv \delta N = \frac{1}{\lambda_\pm}\ln\left(1+\frac{\delta\pi+\lambda_\mp\delta\varphi}{\pi+\lambda_\mp\varphi}\right) - \frac{1}{\lambda_\pm}\ln\left(1+\frac{\delta\pi_*}{\pi_*+\lambda_\mp\varphi_*}\right)$$

$$+\frac{1}{\tilde{\lambda}_\pm}\ln\left(1+\frac{\delta\pi_*}{\pi_*+\tilde{\lambda}_\mp(\varphi_*-\varphi_m)}\right) - \frac{1}{\tilde{\lambda}_\pm}\ln\left(1+\frac{\delta\pi_f}{\pi_f+\tilde{\lambda}_\mp(\varphi_f-\varphi_m)}\right)$$



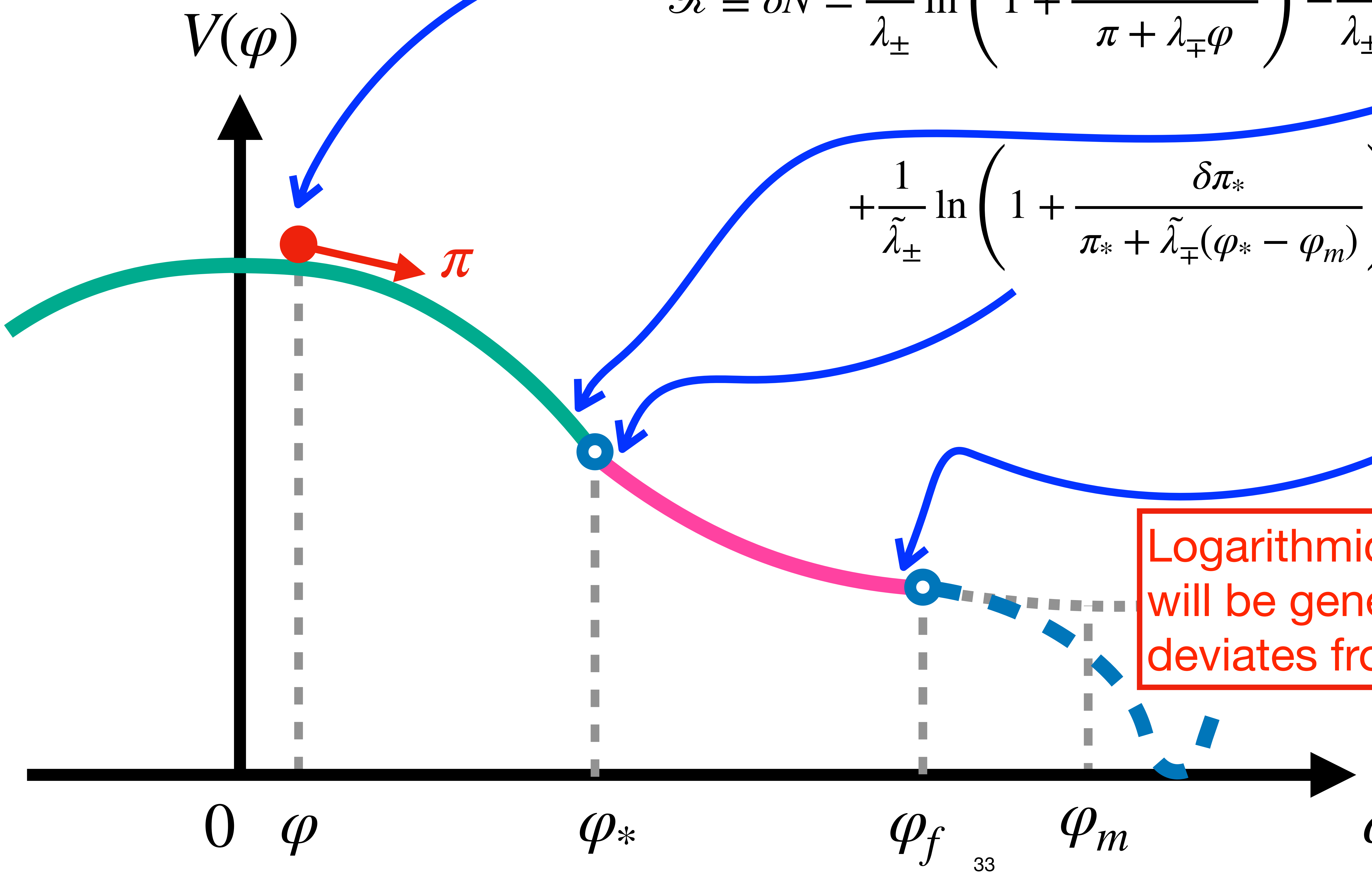
Logarithmic relation (i.e. non-Gaussianity) will be generated when the evolution deviates from attractor.

SP and Sasaki, 2211.13932
SP, 2404.06151

Logarithmic Duality

$$\mathscr{R} \equiv \delta N = \frac{1}{\lambda_\pm}\ln\left(1+\frac{\delta\pi+\lambda_\mp\delta\varphi}{\pi+\lambda_\mp\varphi}\right) - \frac{1}{\lambda_\pm}\ln\left(1+\frac{\delta\pi_*}{\pi_*+\lambda_\mp\varphi_*}\right)$$

$$+\frac{1}{\tilde\lambda_\pm}\ln\left(1+\frac{\delta\pi_*}{\pi_*+\tilde\lambda_\mp(\varphi_*-\varphi_m)}\right) - \frac{1}{\tilde\lambda_\pm}\ln\left(1+\frac{\delta\pi_f}{\pi_f+\tilde\lambda_\mp(\varphi_f-\varphi_m)}\right)$$



Logarithmic relation (i.e. non-Gaussianity) will be generated when the evolution deviates from attractor.

SP and Sasaki, 2211.13932
SP, 2404.06151

$$\mathscr{R}(\delta\varphi, \delta\pi)$$

SP and Sasaki, 2211.13932

$$\mathscr{R} = \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi + \lambda_\mp \delta\varphi}{\pi + \lambda_\mp \varphi}\right) - \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_\mp \varphi_*}\right) + \cdots$$

$\lambda_- \ll 1$

$$(f_{NL} = -\frac{5}{6}\lambda_-)$$

$$\mathscr{R} = -H\frac{\delta\varphi}{\dot\varphi} + \frac{3}{5} f_{\rm NL}\left(-H\frac{\delta\varphi}{\dot\varphi}\right)^2$$

Slow-roll inflation
Stewart and Sasaki, 1995
Lyth and Roquigez, 2005

$\lambda_- = -1/\mu$

$$\mathscr{R} = -\mu \ln\left(1 - \frac{\mathscr{R}_g}{\mu}\right)$$

Constant-roll
Atal, Garriga, Marcos-Caballero, 1905.13202
Atal, Cid, Escrivà, Garriga, 1908.11357
Escrivà, Atal, Garriga, 2306.09990

$\lambda_- = 0$

$$\mathscr{R} = -\frac{1}{3} \ln\left(1 + \frac{\delta\pi_*}{\pi_*}\right)$$

Ultra-slow-roll
Namjoo, Firouzjahi, Sasaki, 1210.3692
Cai, Chen, et al 1712.09998
Biagetti et al 1804.07124
Passaglia et al 1812.08243

$\lambda_- = 3/2$

$$\mathscr{R} = \frac{2}{3} \ln(1 + \delta)$$

Curvaton scenario,
SP and Sasaki, 2112.12680
Ferrante et al, 2211.01728
Hooper et al. 2308.00756

$\lambda_- = -6$

$$\mathscr{R} = -\frac{1}{6} \ln\left(1 - 6\mathscr{R}_{\rm G}\right)$$

Modulated reheating,
SP and Yokoyama, in prep.

Probability Distribution Function

For the simplest single-logarithm case: $$\mathscr{R} \equiv \delta N = \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi + \lambda_+ \delta\varphi}{\pi + \lambda_+ \varphi}\right)$$

$$\Downarrow \quad P(\mathscr{R})d\mathscr{R} = P(\delta\varphi)d\delta\varphi$$

$P(\delta\varphi)$ → Gaussian PDF with variance $\sigma_{\delta\varphi}^2$

$$P(\mathscr{R}) = \frac{e^{\lambda_- \mathscr{R}}}{\sqrt{2\pi}\sigma_{\delta\varphi}} \left|\lambda_-\right| \varphi \exp\left[-\frac{\varphi^2}{2\sigma_{\delta\varphi}^2}\left(e^{\lambda_- \mathscr{R}} - 1\right)^2\right]$$

SP and Sasaki, 2211.13932

Probability Distribution Function

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$$P(\mathscr{R}) = \frac{e^{\lambda_- \mathscr{R}}}{\sqrt{2\pi}\sigma_{\delta\varphi}} \left|\lambda_-\right| \varphi \exp\left[-\frac{\varphi^2}{2\sigma_{\delta\varphi}^2}\left(e^{\lambda_- \mathscr{R}} - 1\right)^2\right]$$

$\lambda_- < 0$

$$P(\mathscr{R}) \sim e^{\lambda_- \mathscr{R}}$$

exponential tail

$\lambda_- > 0$

$$P(\mathscr{R}) \sim \exp\left(-c^2 e^{2\lambda_- \mathscr{R}}\right)$$

Gumbel-distribution-like tail

SP and Sasaki, 2211.13932

PDF of $\mathscr{C}_\ell$

For the simplest single-logarithm case: $\mathscr{R} \equiv \delta N = \frac{1}{\lambda_-} \ln\left(1 + \lambda_- \mathscr{R}_g\right)$

$X \equiv -\frac{4}{3} r \mathscr{R}' \qquad Y \equiv 1 + \lambda_- \mathscr{R}$

$\Downarrow$

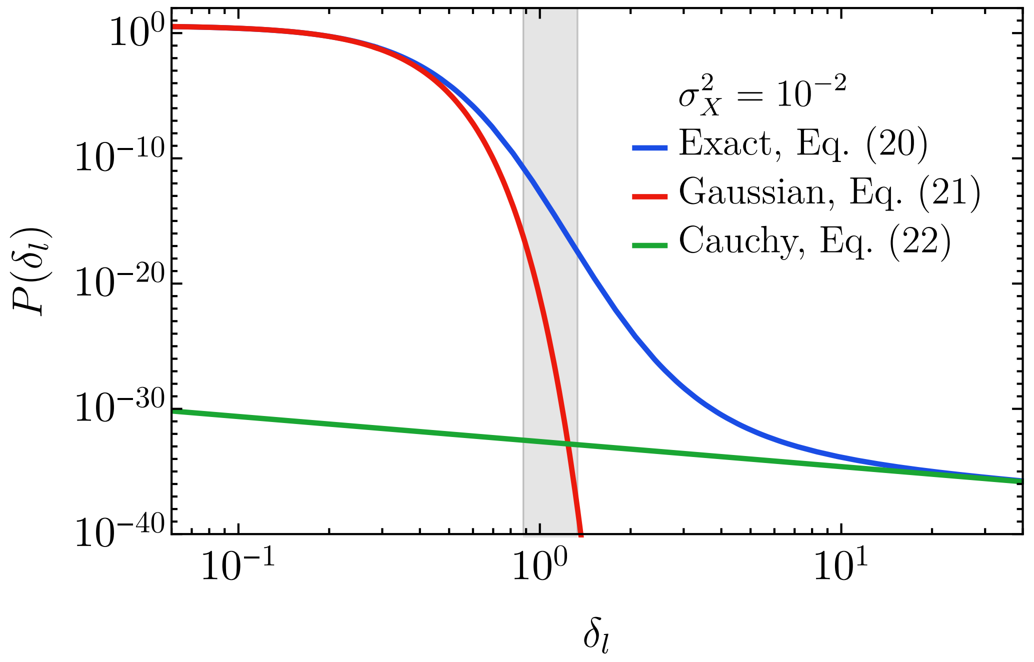
$$\mathbb{P}(\mathscr{C}_\ell) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \left(\frac{\sigma_X}{\sigma_X - \mathrm{sgn}(\lambda_-)\sigma_Y \mathscr{C}_\ell} \right)^2 \exp\left[-\frac{1}{2} \left(\frac{\mathscr{C}_\ell}{\sigma_X - \mathrm{sgn}(\lambda_-)\sigma_Y \mathscr{C}_\ell} \right)^2 \right]$$

$\lambda_- < 0$

Gaussian

exponential tail $P(\mathscr{R}) \sim e^{\lambda_- \mathscr{R}}$

Cauchy (unphysical)



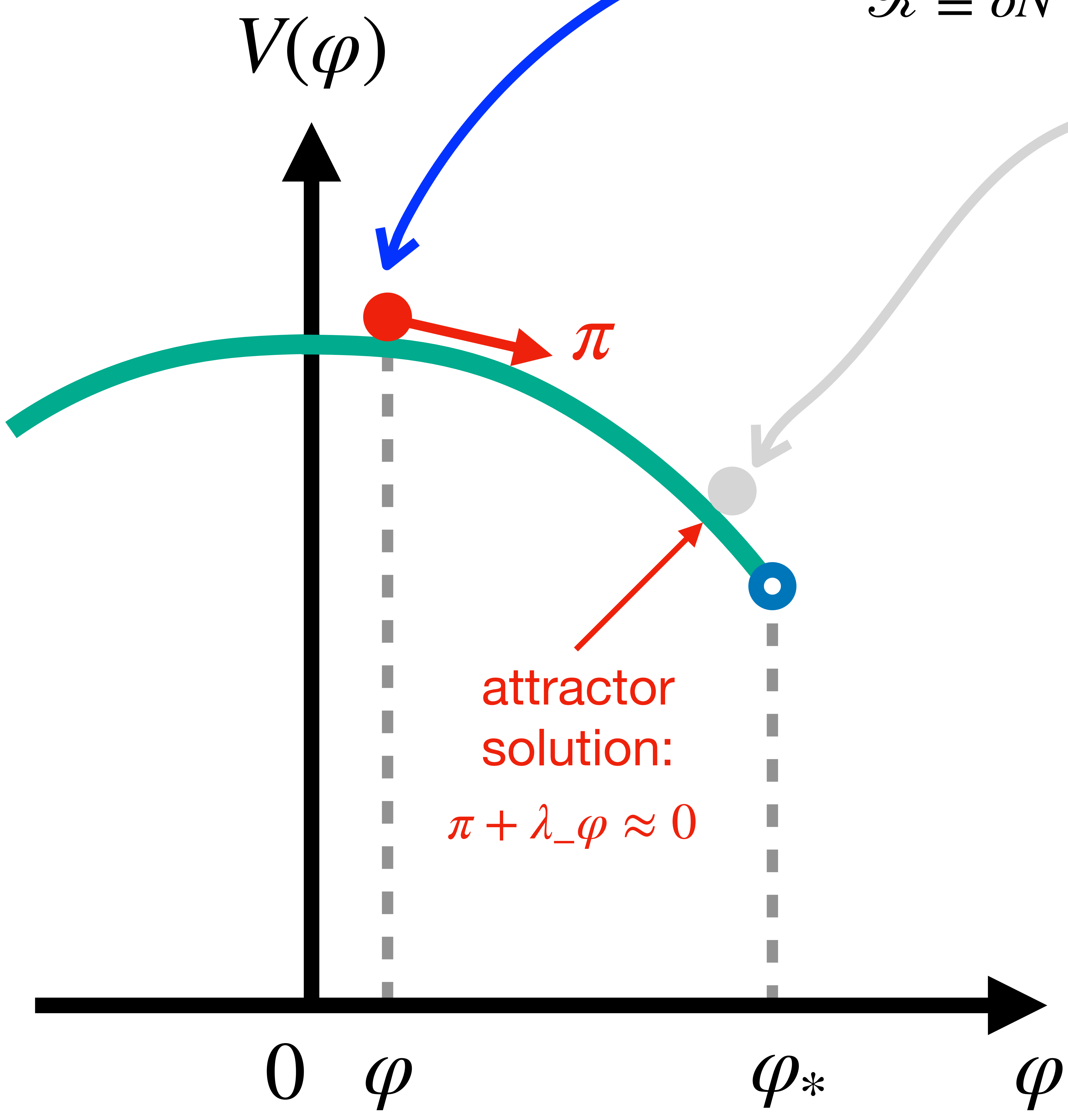
$\lambda_- > 0$

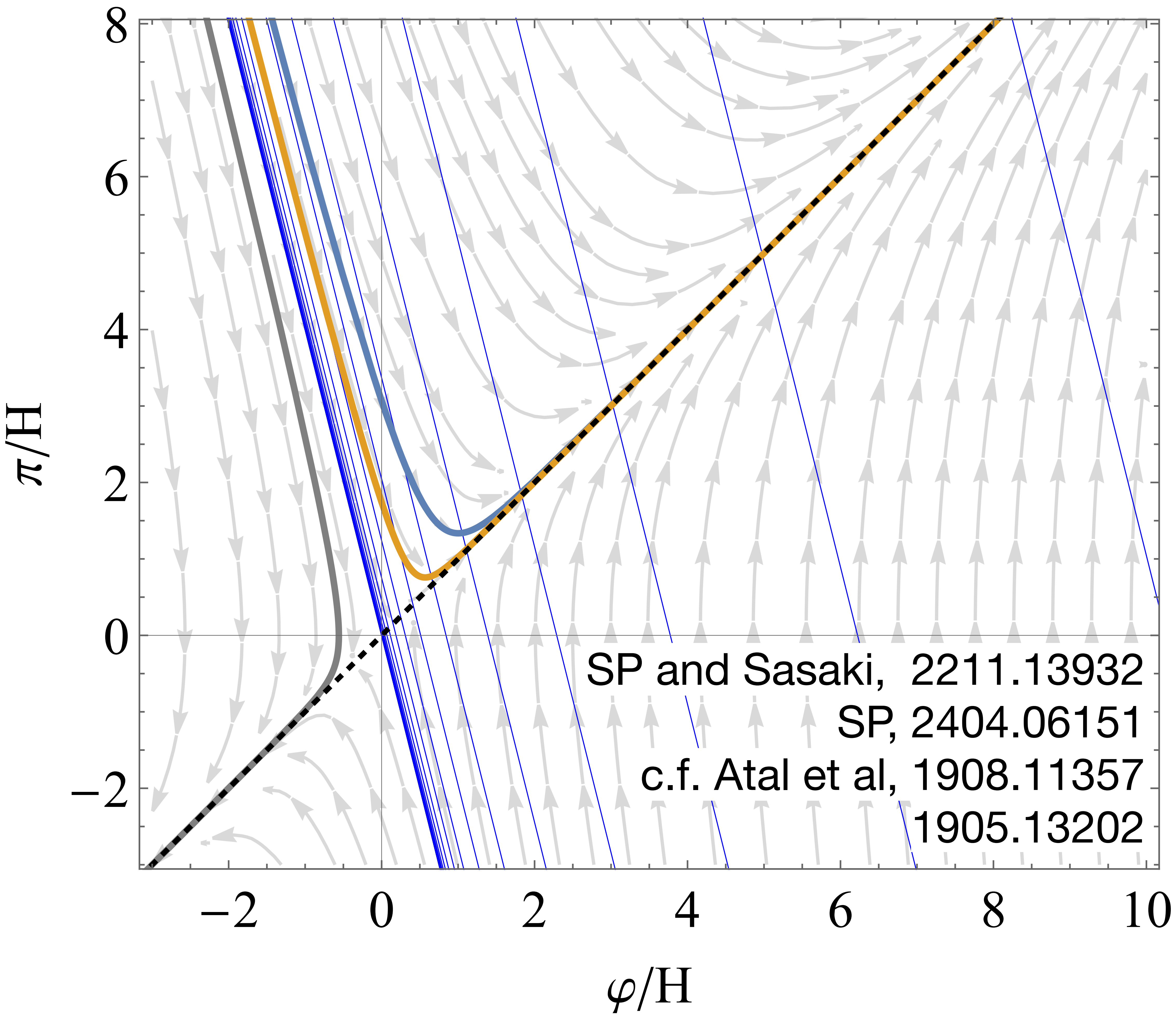
Highly suppressed

SP, 2404.06151

Case1: Constant-roll

$$\mathcal{R} \equiv \delta N = \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi + \lambda_+ \delta\varphi}{\pi + \lambda_+ \varphi}\right) - \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_+ \varphi_*}\right)$$



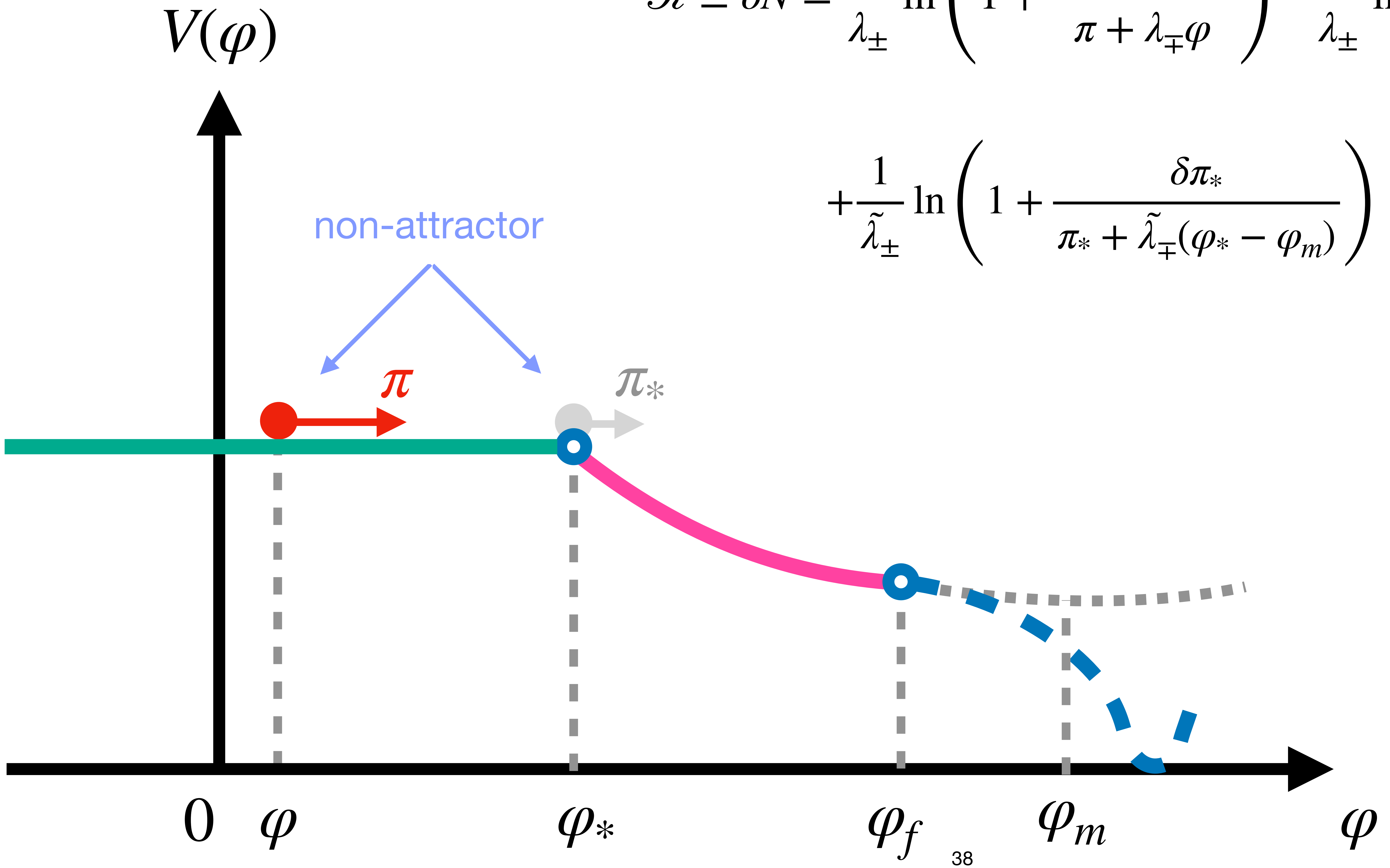


Case 2: USR

$(\lambda_- = 0, \quad \lambda_+ = 3)$
$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$

$$\mathscr{R} \equiv \delta N = \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi + \lambda_\mp \delta\varphi}{\pi + \lambda_\mp \varphi}\right) - \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_\pm \varphi_*}\right)$$

$$+ \frac{1}{\tilde{\lambda}_\pm} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \tilde{\lambda}_\mp(\varphi_* - \varphi_m)}\right) - \frac{1}{\tilde{\lambda}_\pm} \ln\left(1 + \frac{\delta\pi_f}{\pi_f + \tilde{\lambda}_\pm(\varphi_f - \varphi_m)}\right)$$



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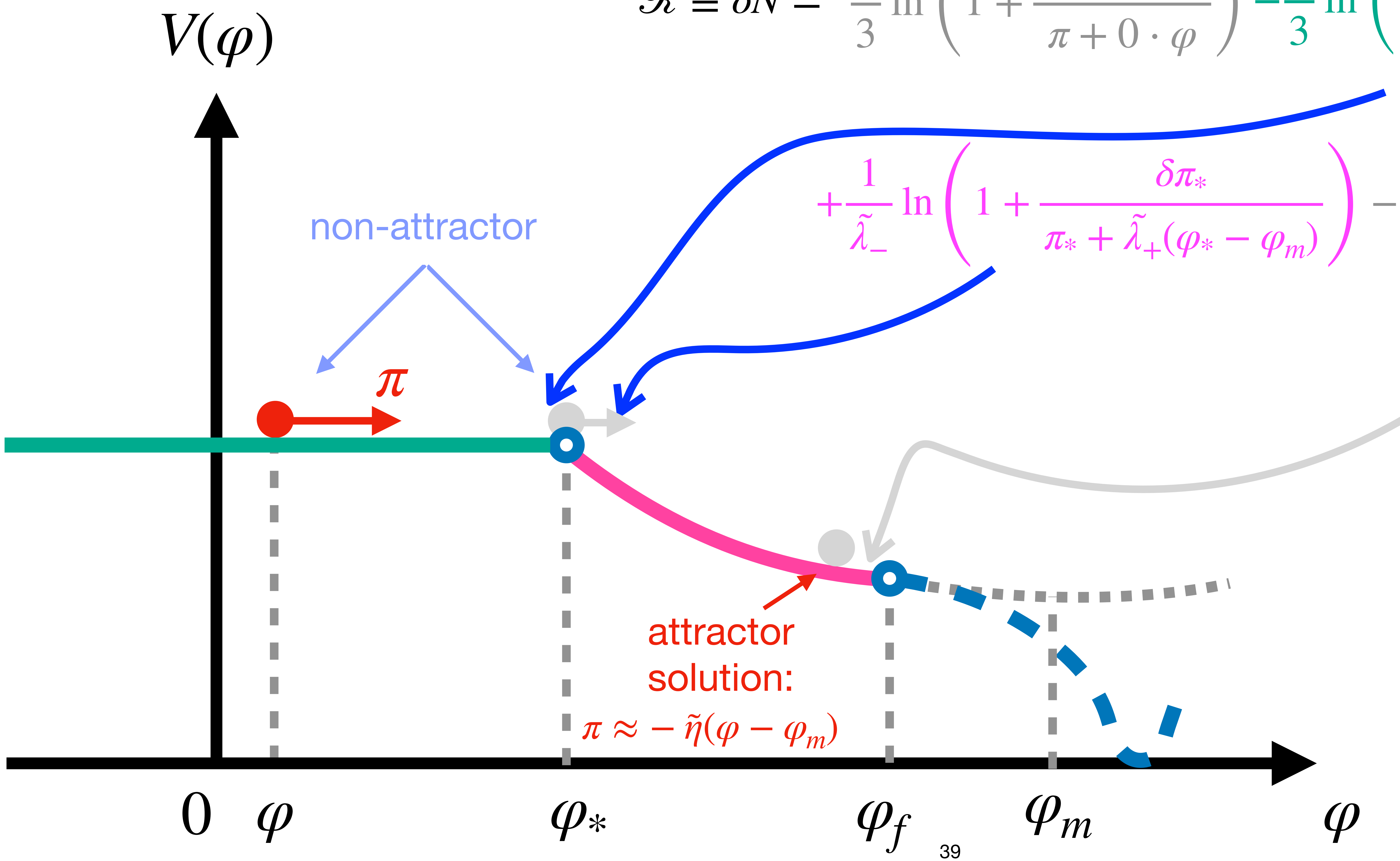
$(\lambda_- = 0, \quad \lambda_+ = 3)$
$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$

$$\mathscr{R} \equiv \delta N = \frac{1}{3}\ln\left(1 + \frac{0 + 0\cdot\delta\varphi}{\pi + 0\cdot\varphi}\right) - \frac{1}{3}\ln\left(1 + \frac{\delta\pi_*}{\pi_*}\right)$$

$$+\frac{1}{\tilde{\lambda}_-}\ln\left(1 + \frac{\delta\pi_*}{\pi_* + \tilde{\lambda}_+(\varphi_* - \varphi_m)}\right) - \frac{1}{\tilde{\lambda}_-}\ln\left(1 + \frac{\delta\pi_f}{\pi_f + \tilde{\lambda}_+(\varphi_f - \varphi_m)}\right)$$



Case 2: USR

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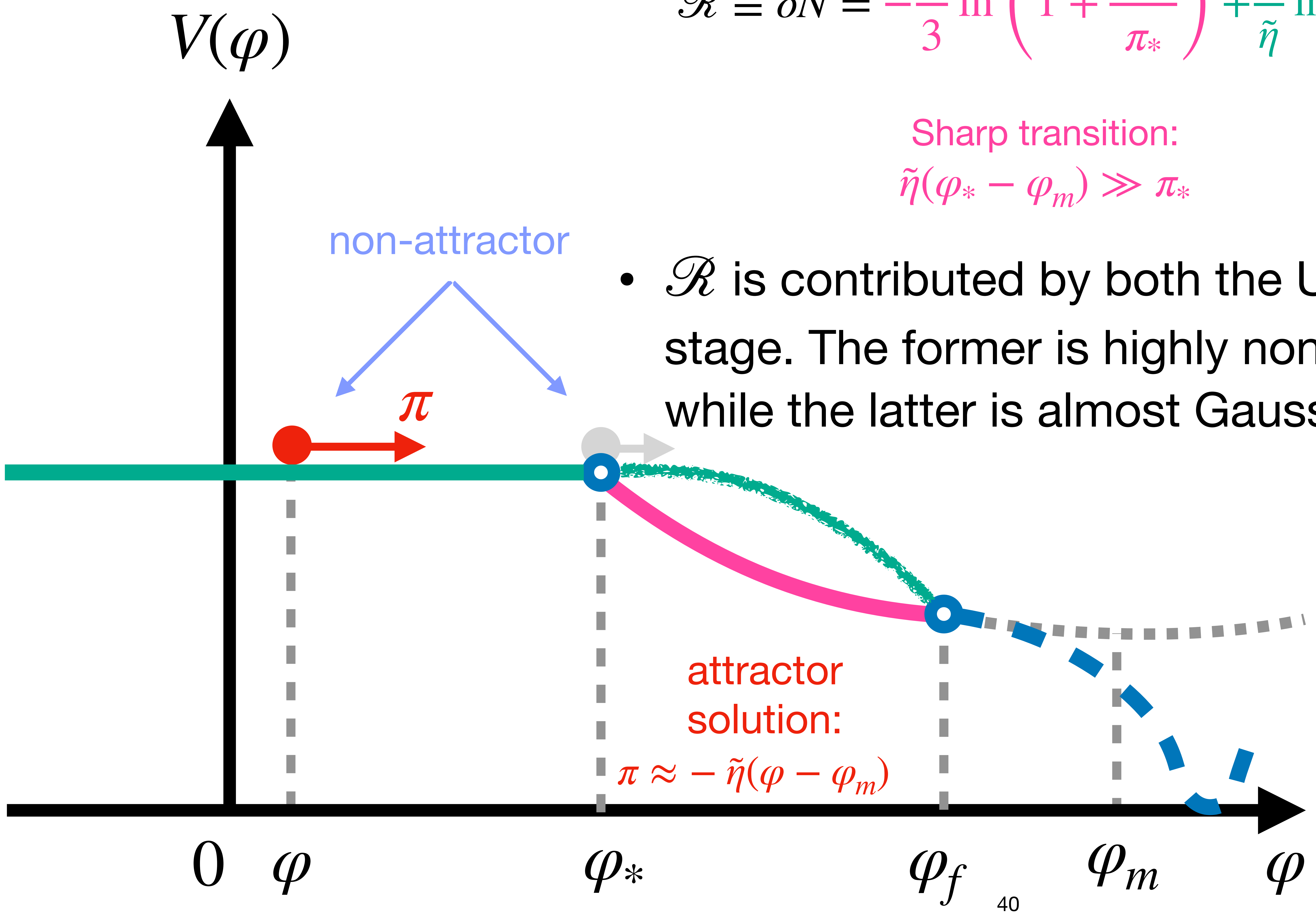
$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$

$$\mathscr{R} \equiv \delta N = -\frac{1}{3}\ln\left(1 + \frac{\delta\pi_*}{\pi_*}\right) + \frac{1}{\tilde{\eta}}\ln\left(1 + \frac{\delta\pi_*}{\pi_* + (3-\tilde{\eta})(\varphi_* - \varphi_m)}\right)$$

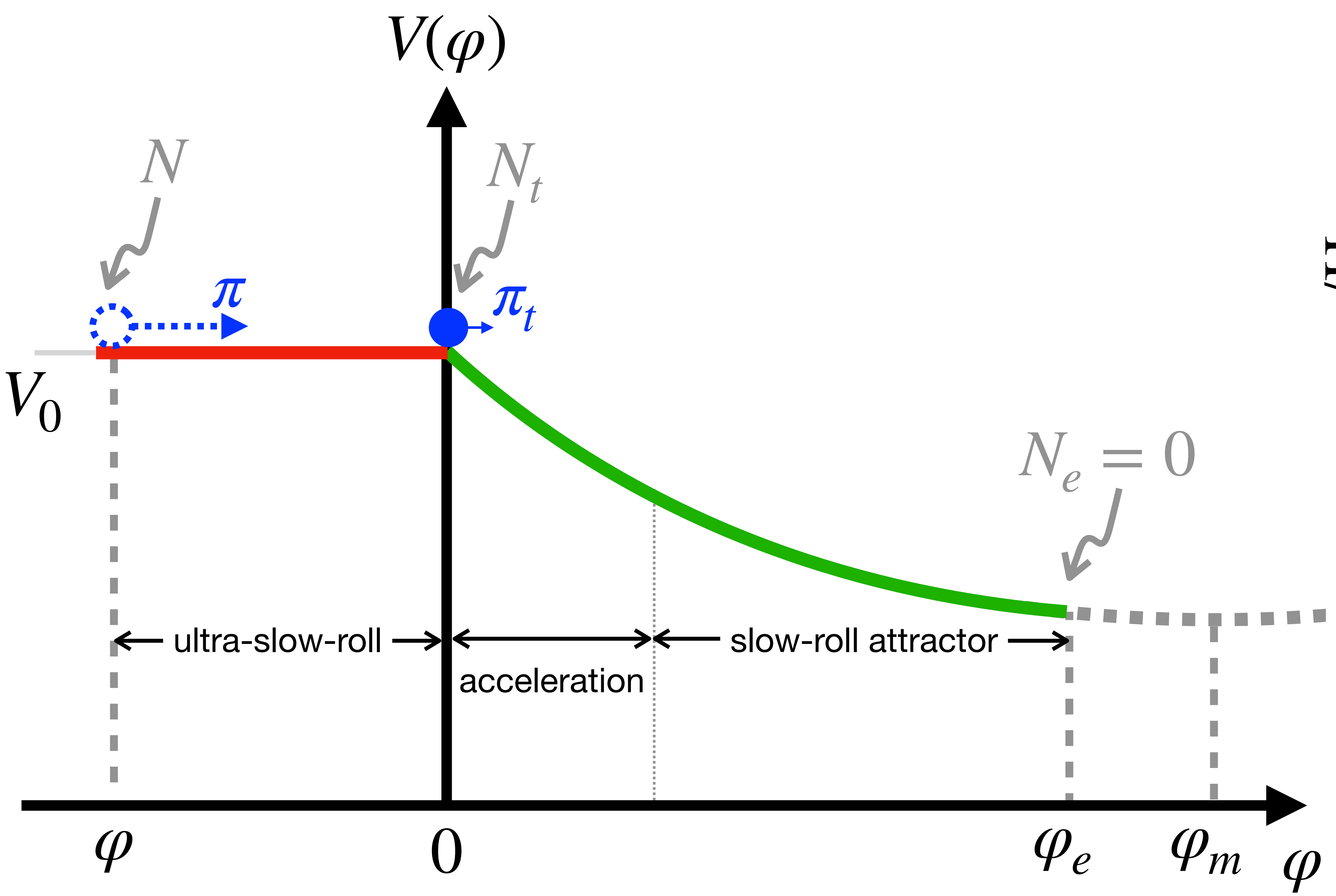
Sharp transition: $\tilde{\eta}(\varphi_* - \varphi_m) \gg \pi_*$

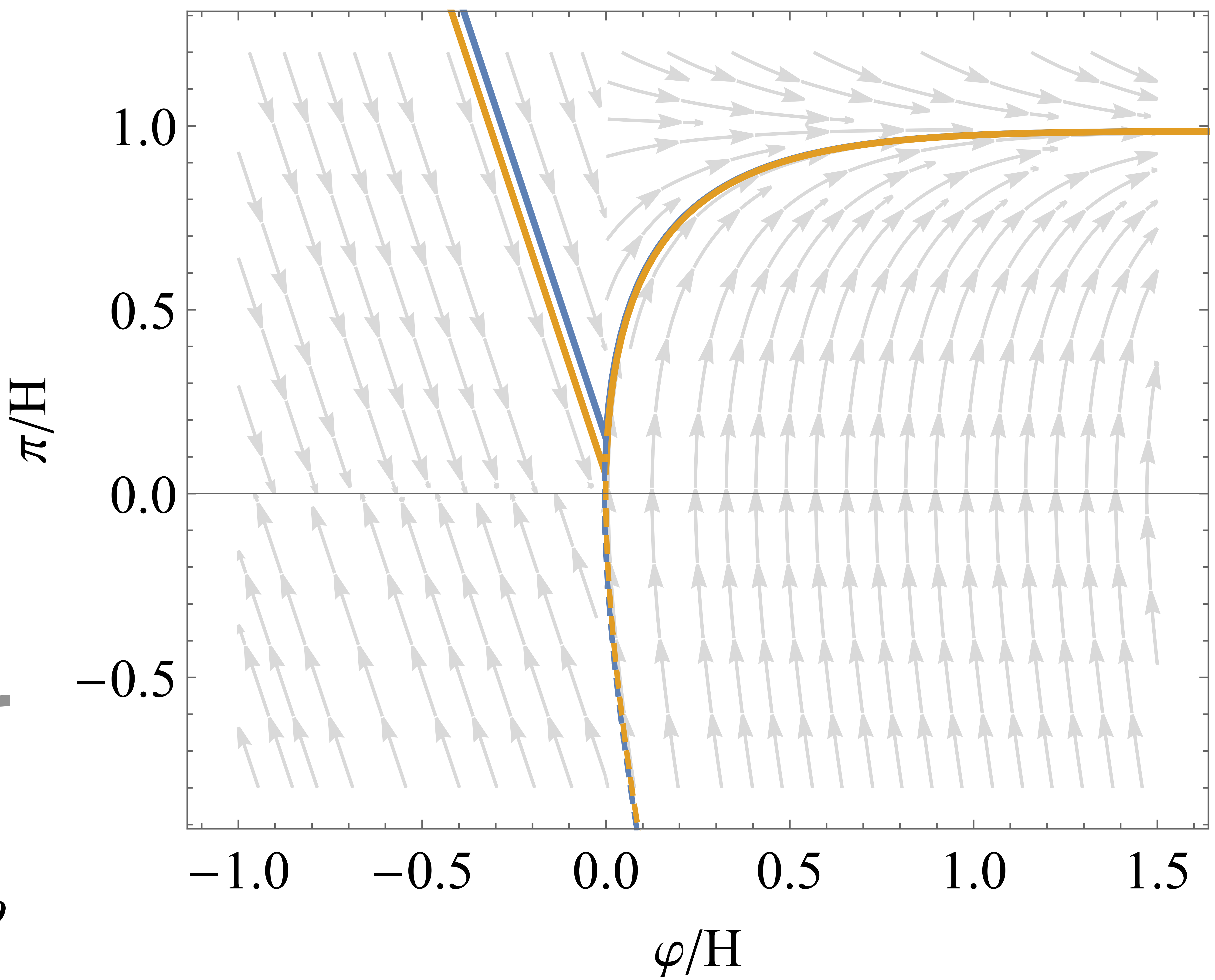
Smooth transition $\tilde{\eta}(\varphi_* - \varphi_m) \ll \pi_*$



- $\mathscr{R}$ is contributed by both the USR stage and the later slow-roll stage. The former is highly non-Gaussian (i.e. exp tail, $f_{\rm NL} = 5/2$), while the latter is almost Gaussian.
- The h factor ($h \equiv -6\sqrt{\epsilon_V/\epsilon_*}$) defined in Cai et al 2017 is the ratio between the slow-roll velocity and the end-of-USR velocity.
- When $h \sim \mathcal{O}(1)$, both of the logarithms are of the same order.

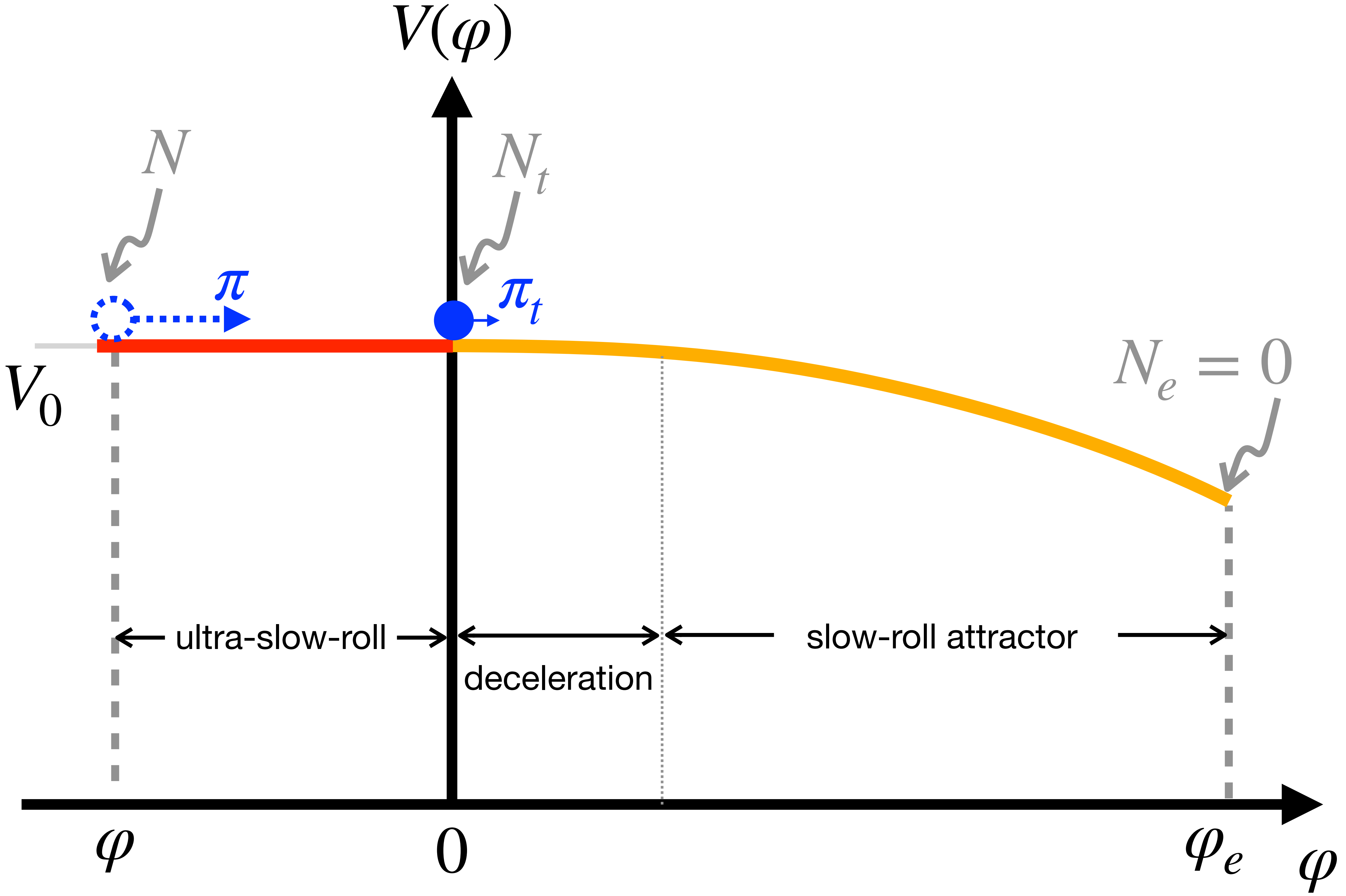
USR: Sharp end

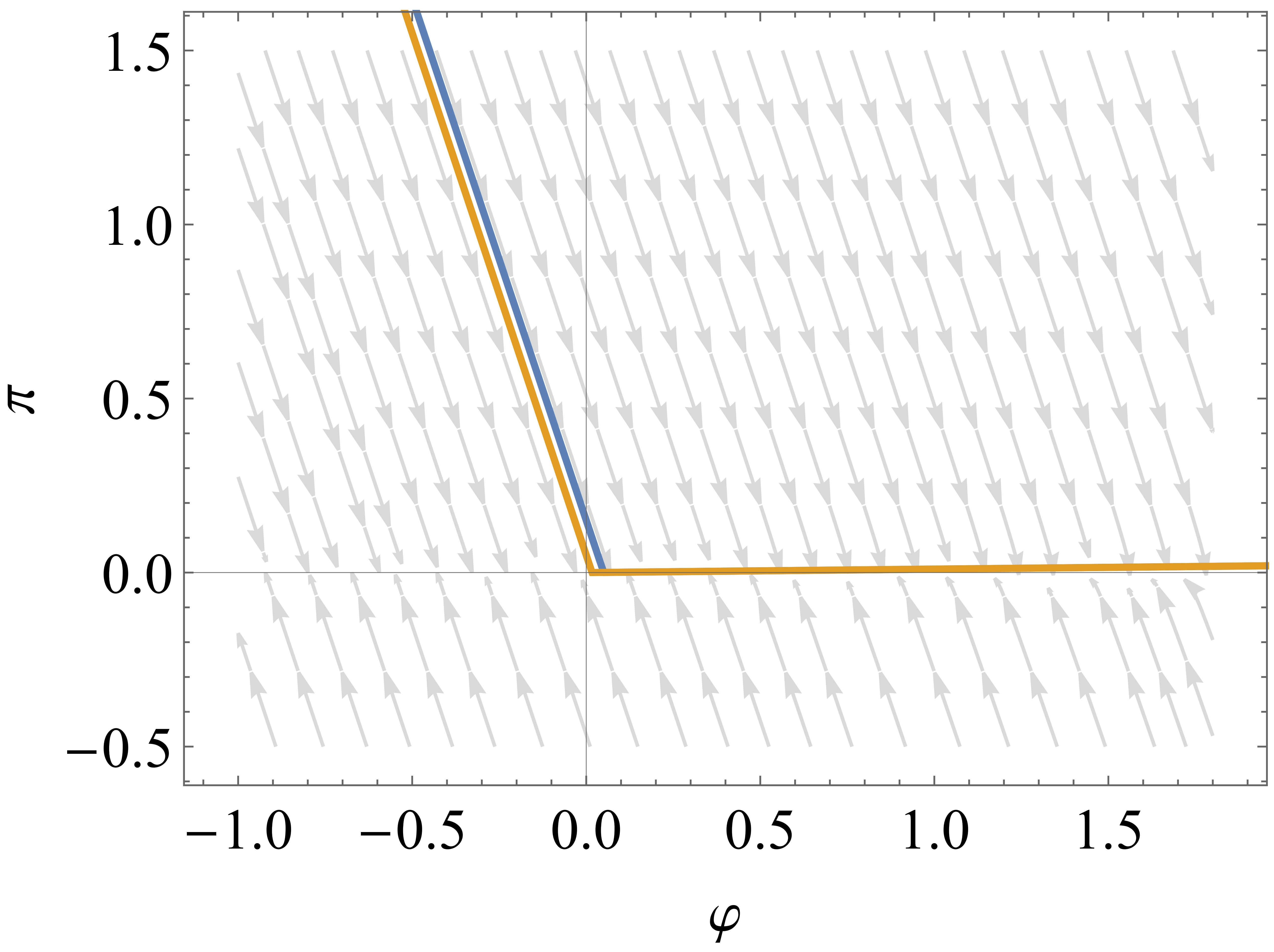




SP and Sasaki, 2211.13932
SP, 2404.06151
c.f. Cai et al, 1712.09998

USR: Smooth end





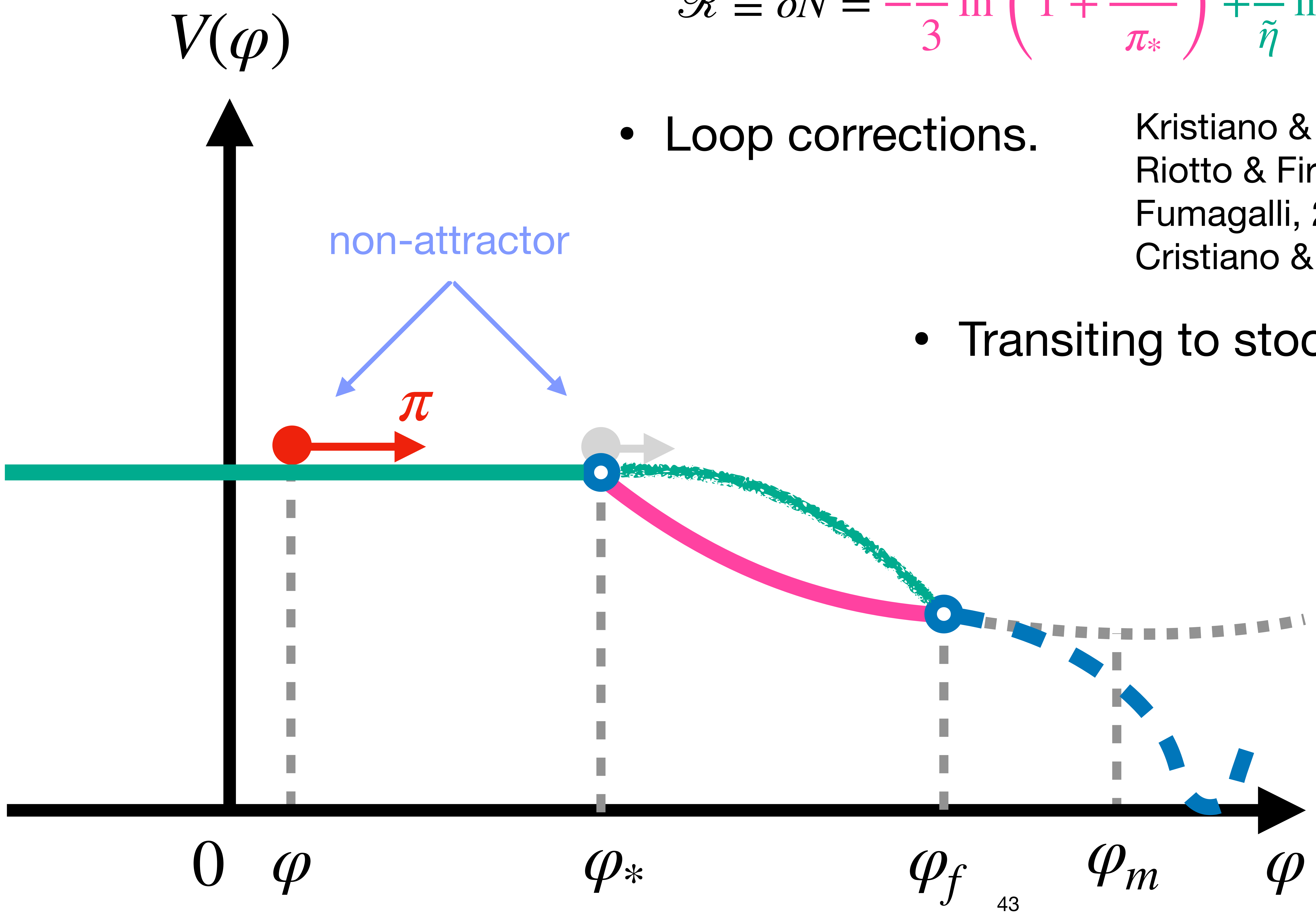
SP and Sasaki, 2211.13932
SP, 2404.06151
c.f. Cai et al, 1712.09998

USR

$(\lambda_- = 0, \quad \lambda_+ = 3)$

$(\tilde{\lambda}_- = \tilde{\eta}, \quad \tilde{\lambda}_+ = 3 - \tilde{\eta})$

$$\mathcal{R} \equiv \delta N = -\frac{1}{3}\ln\left(1 + \frac{\delta\pi_*}{\pi_*}\right) + \frac{1}{\tilde{\eta}}\ln\left(1 + \frac{\delta\pi_*}{\pi_* + (3-\tilde{\eta})(\varphi_* - \varphi_m)}\right)$$



- Loop corrections.

Kristiano & Yokoyama, 2211.03395
Riotto & Firouzjahi, 2304.07801
Fumagalli, 2305.19263
Cristiano & Yokoyama, 2405.12145

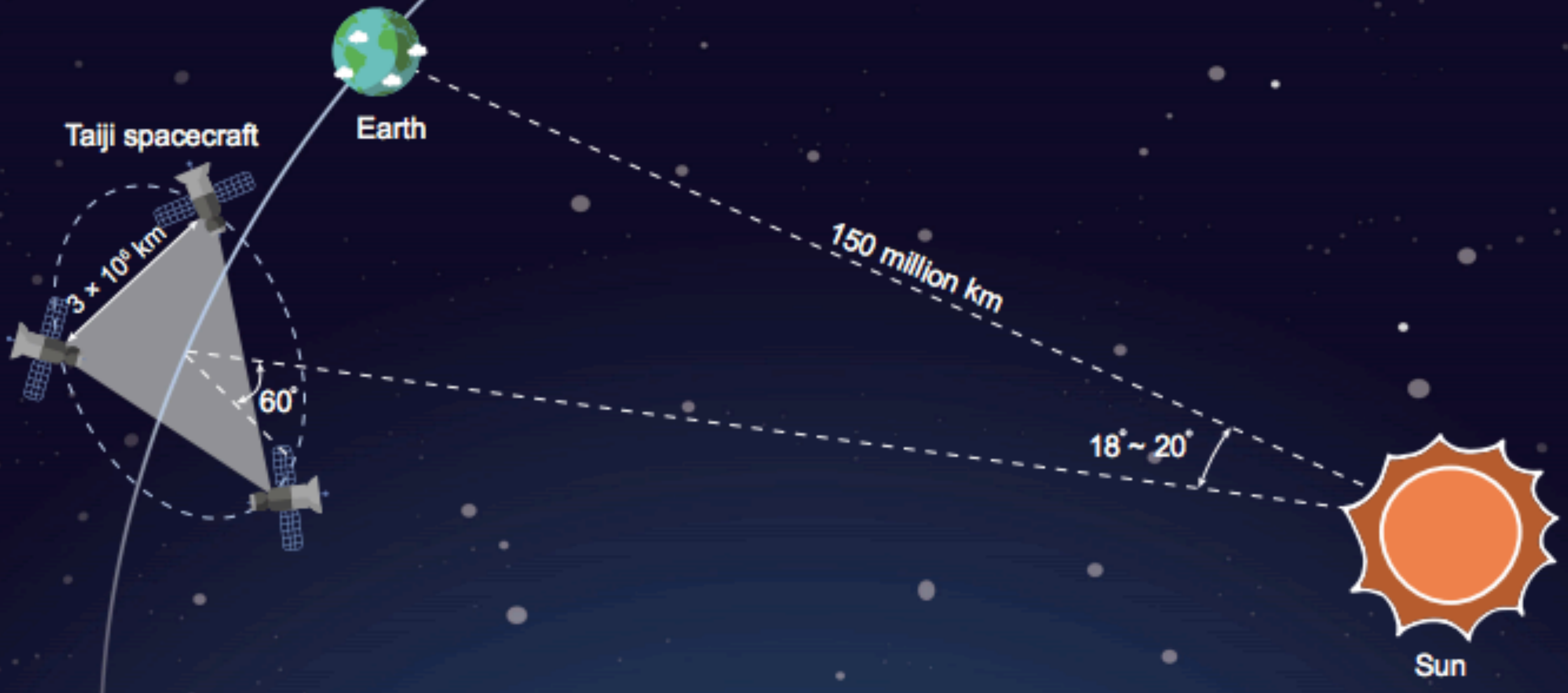
- Transiting to stochastic approach

Pattison et al., 2101.05741
Ballesteros et al 2406.02417
Cruces, SP, Sasaki, in prep.

- Sharp transition will make the separate universe approach (thus δN formalism) invalid transiently.

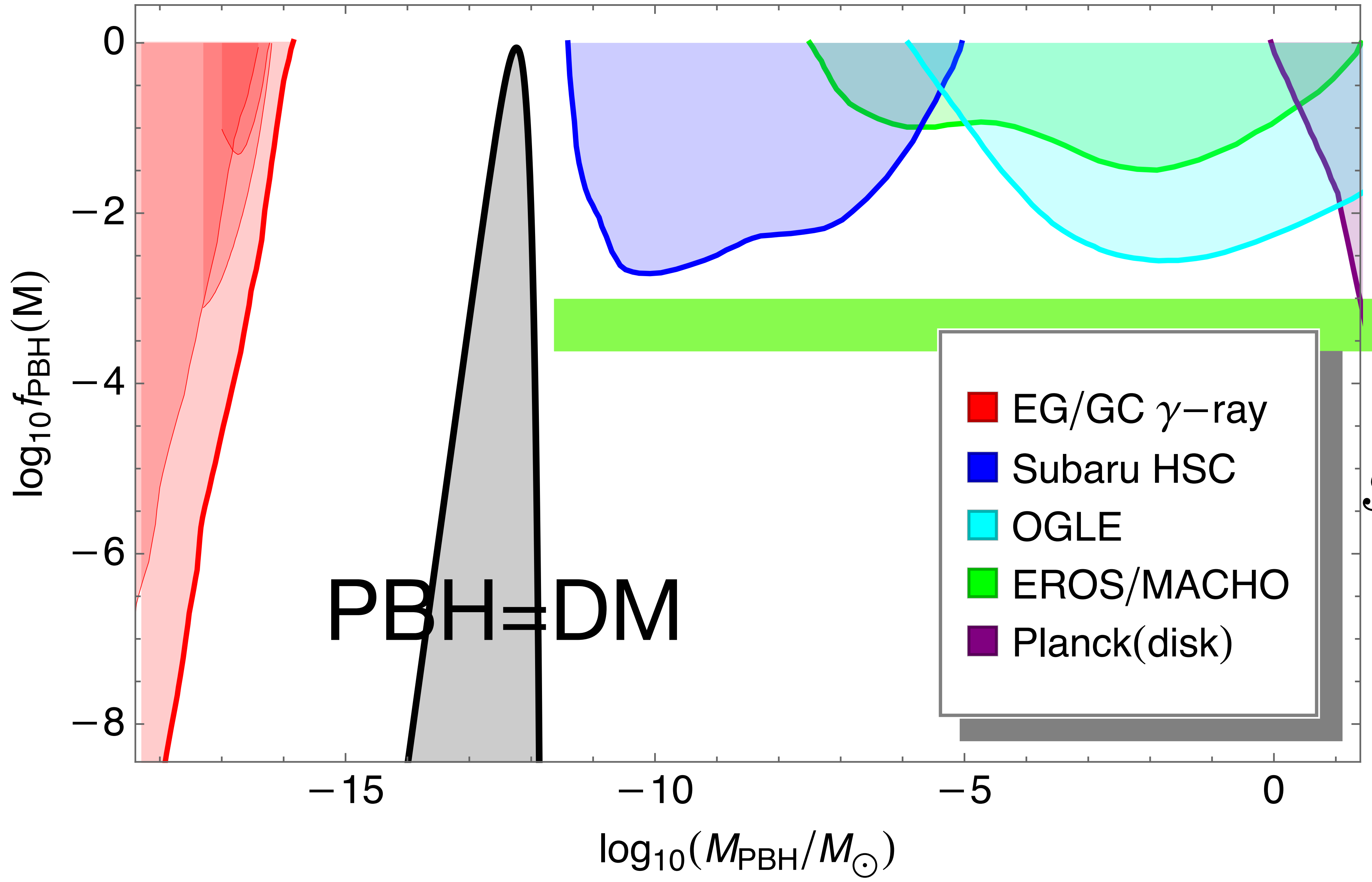
Domenech et al., 2309.05750
Jackson et al., 2311.03281
Artigas, SP, Tanaka, in prep.

Predictions on mHz and nHz GWs



Credit: Natl.Sci.Rev. 4, 5, 685-686

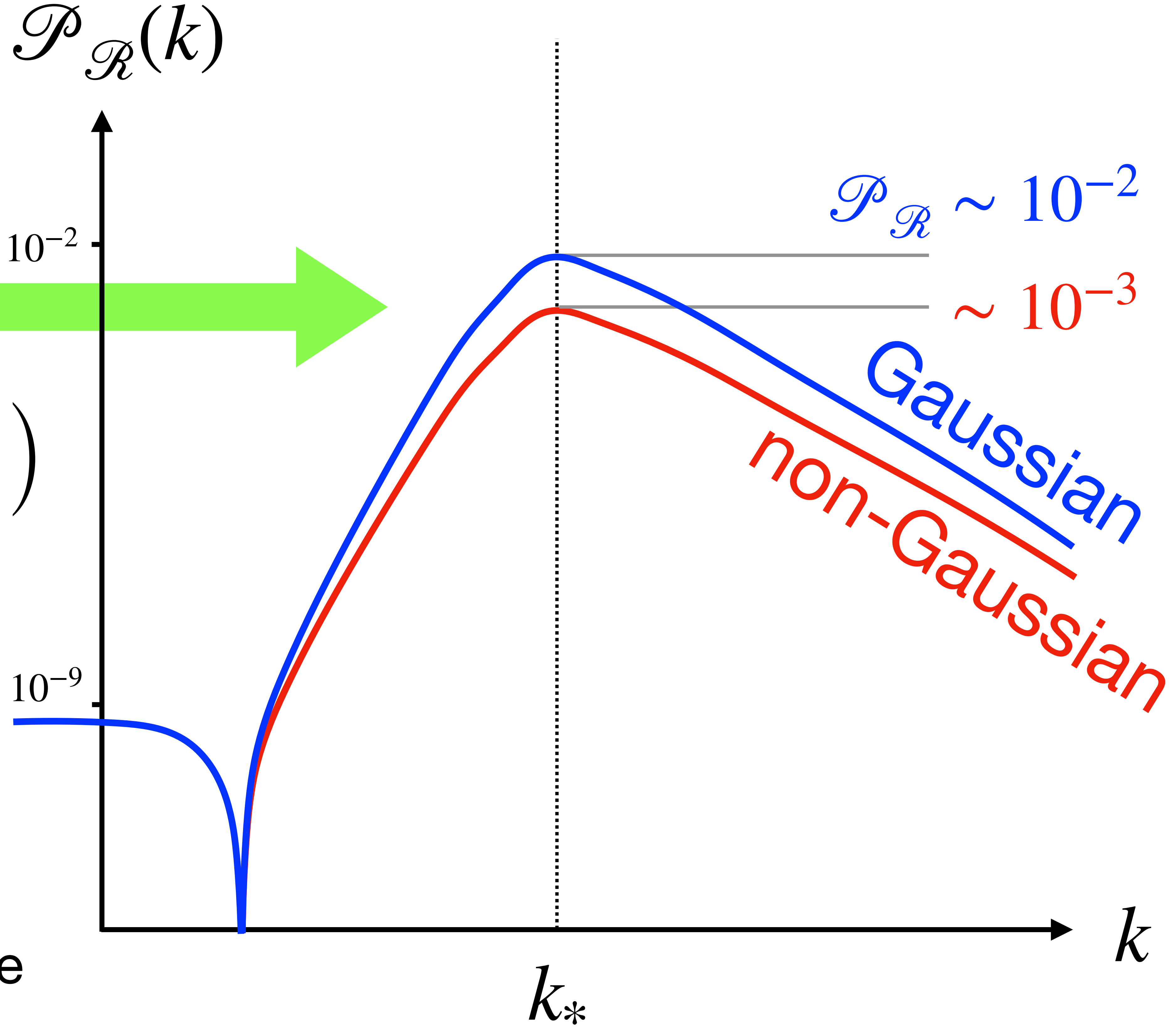
PBH as DM



Press-Schechter-type w/. NG

$$\mathcal{R} = \mathcal{R}_g + \frac{3}{5} f_{\rm NL}\left(\mathcal{R}_g^2 - \langle \mathcal{R}_g^2 \rangle\right)$$

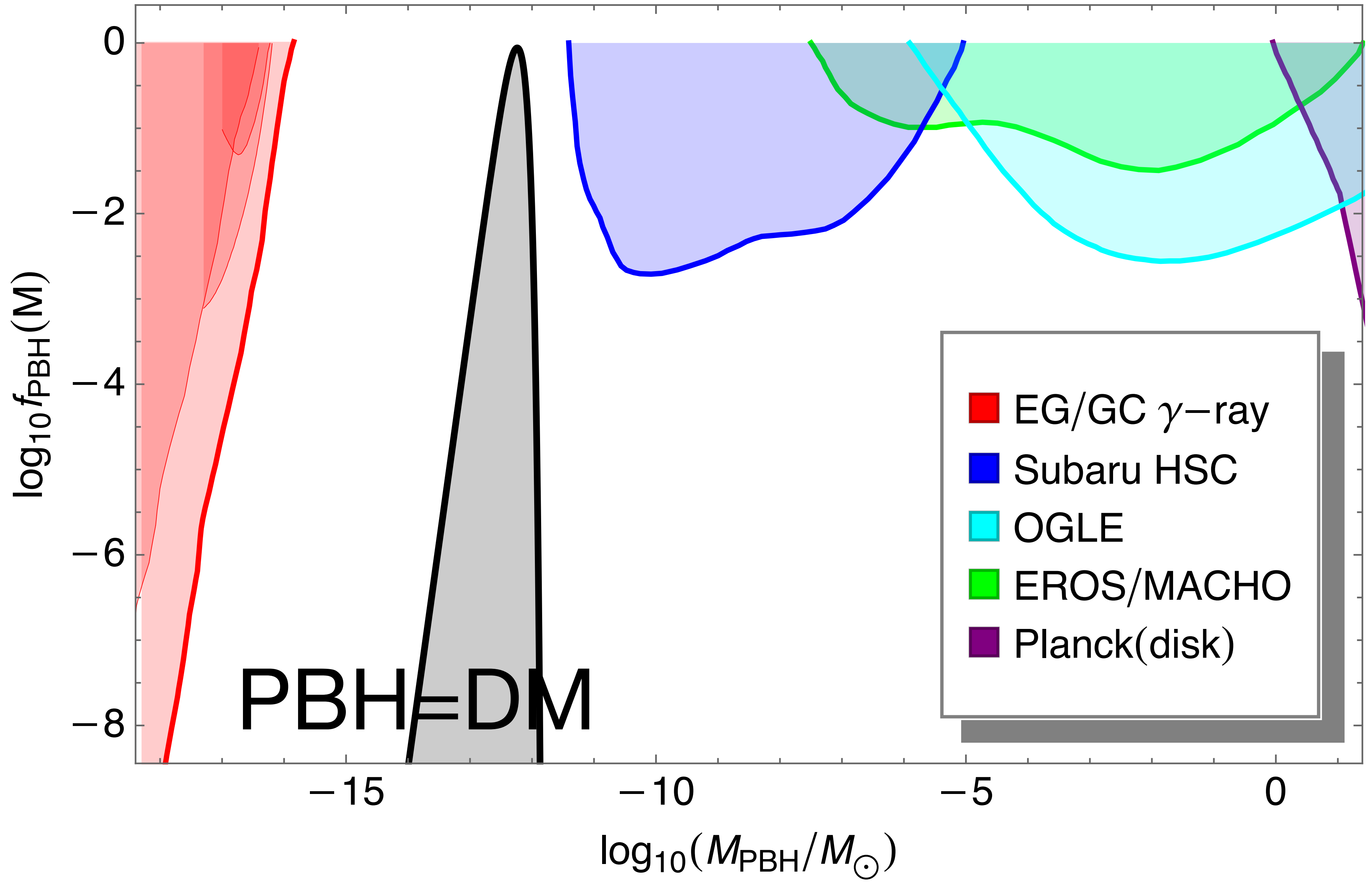
$$\mathcal{R} = \frac{1}{\lambda} \ln\left(1 + \lambda \mathcal{R}_g\right)$$



- Press-Schechter: Young & Byrnes 1307.4995; Young et al 1405.7023
- Press-Schechter-type: Biagetti et al 2105.07810; Gow et al 2211.08348; Ferrante et al 2211.01728
- Peak theory: De Luca et al 1904.00970; Atal et al 1905.13202; Yoo et al 2008.02425; Kitajima et al 2109.00791; Escrivà et al 2202.01028; Germani & Sheth 1912.07072; Jianing Wang, SP, et al in prep.
- Primordial NG must be taken into account when calculating PBH abundance

Cai, SP and Sasaki, 1810.11000
SP, 2404.06151

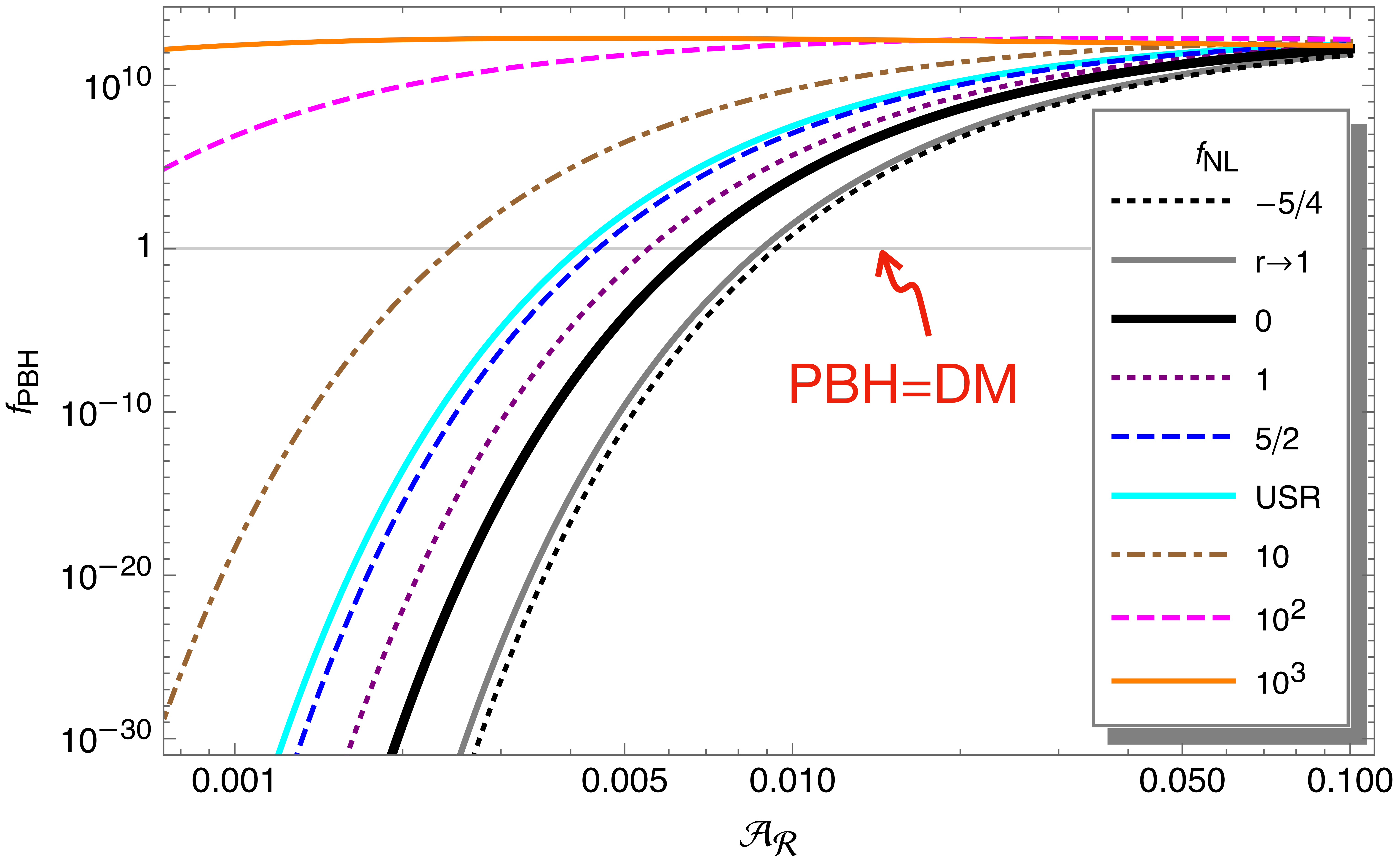
PBH as DM



$$\mathcal{R} = \mathcal{R}_g + \frac{3}{5} f_{\rm NL}\left(\mathcal{R}_g^2 - \left\langle \mathcal{R}_g^2 \right\rangle\right)$$

$$\mathcal{R} = \frac{1}{\lambda} \ln\left(1 + \lambda \mathcal{R}_g\right)$$

$$\mathcal{P}_{\mathcal{R}} = \mathcal{A}_{\mathcal{R}} \delta\left(\ln k - \ln k_*\right)$$

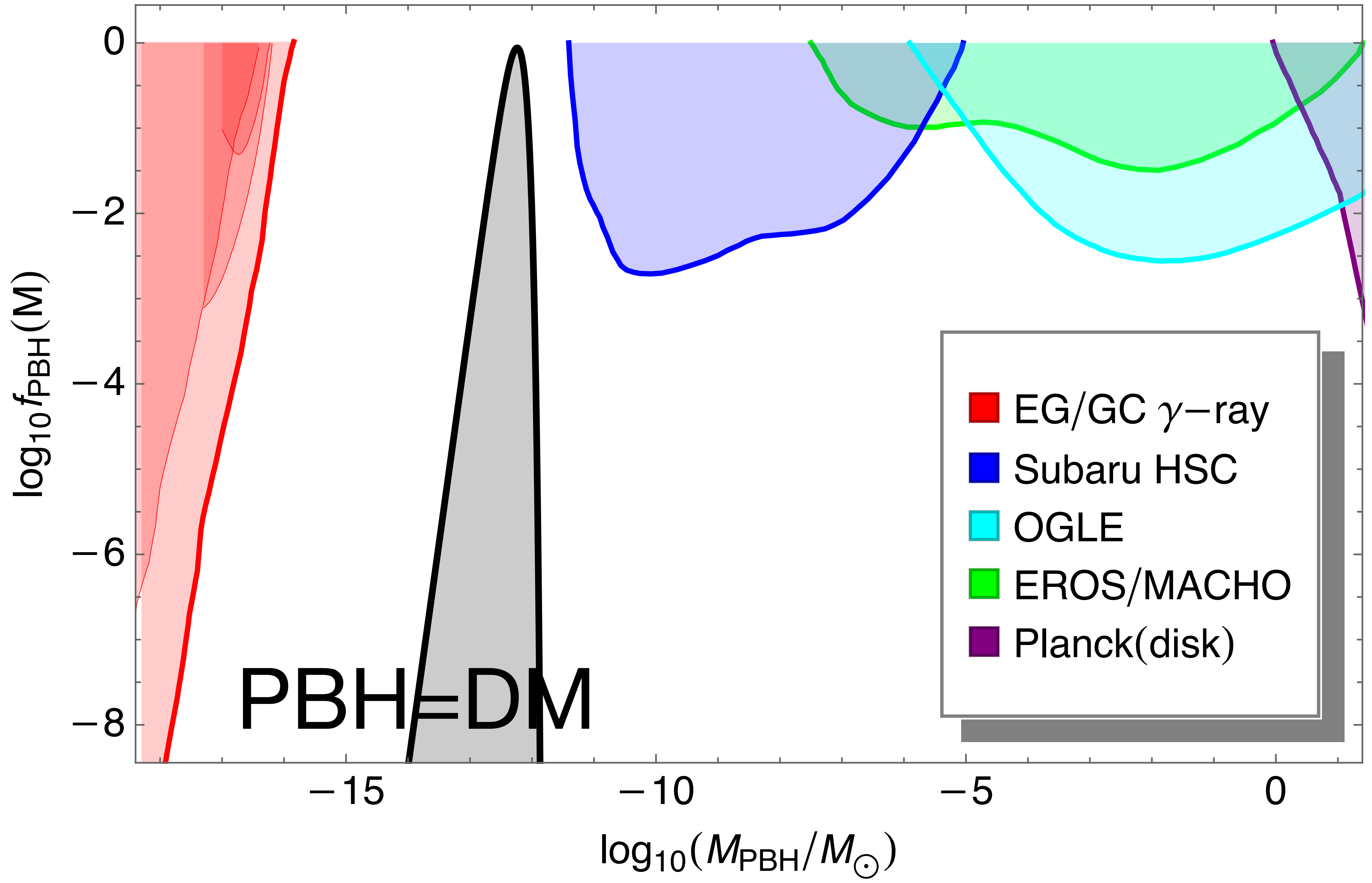


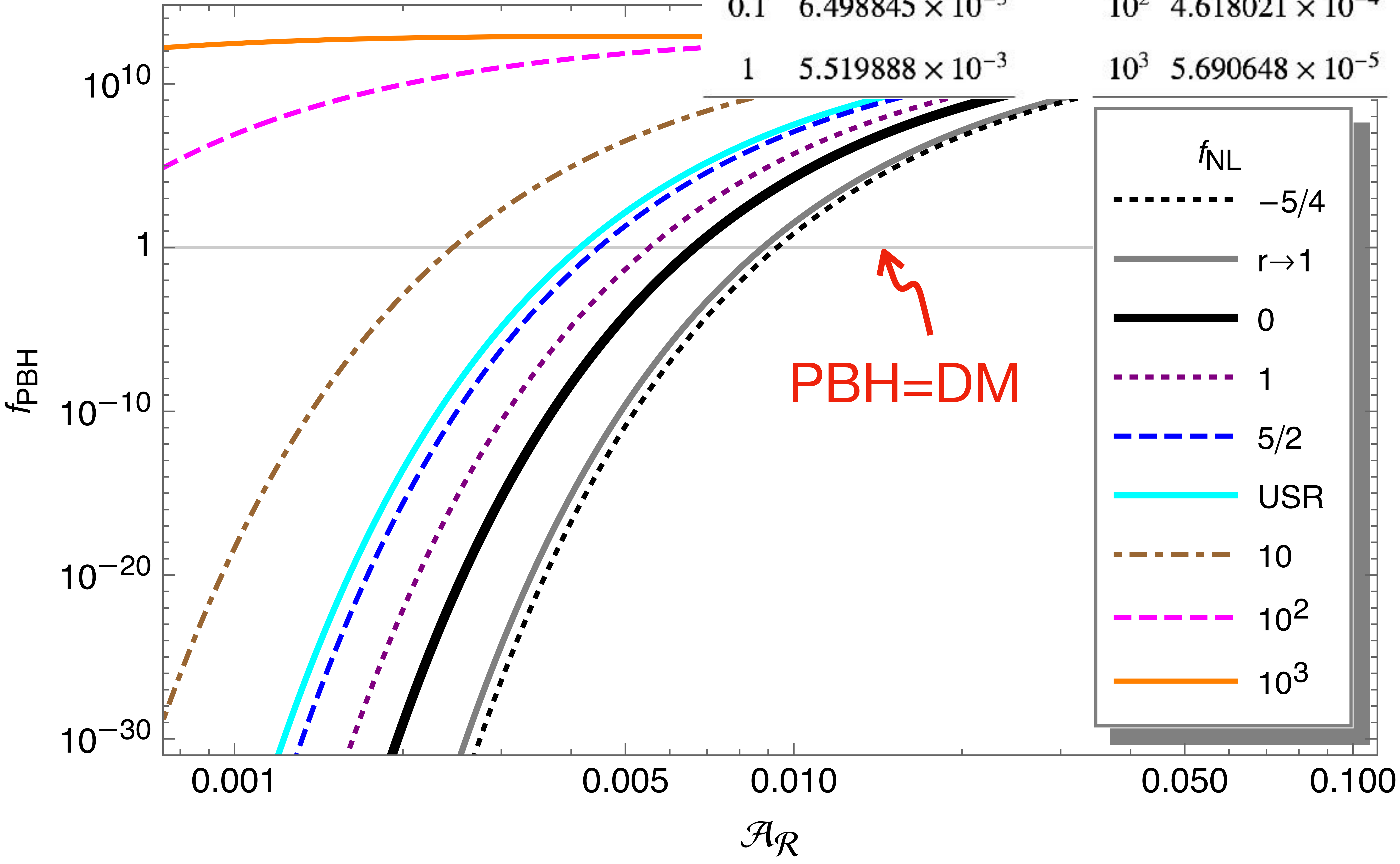
Cai, SP and Sasaki, 1810.11000
SP, 2404.06151

PBH as DM

$f_{\rm NL}$	$\mathcal{A}_\mathcal{R}$
$-5/4$	9.328243×10^{-3}
$r\to1$	8.808810×10^{-3}
0	6.635506×10^{-3}
0.1	6.498845×10^{-3}
1	5.519888×10^{-3}

$f_{\rm NL}$	$\mathcal{A}_\mathcal{R}$
$5/2$	4.479141×10^{-3}
USR	4.151077×10^{-3}
10	2.456822×10^{-3}
10^2	4.618021×10^{-4}
10^3	5.690648×10^{-5}





$$\mathcal{R} = \mathcal{R}_g + \frac{3}{5} f_{\rm NL}\left(\mathcal{R}_g^2 - \langle \mathcal{R}_g^2 \rangle\right)$$

$$\mathcal{R} = \frac{1}{\lambda}\ln\left(1+\lambda\mathcal{R}_g\right)$$

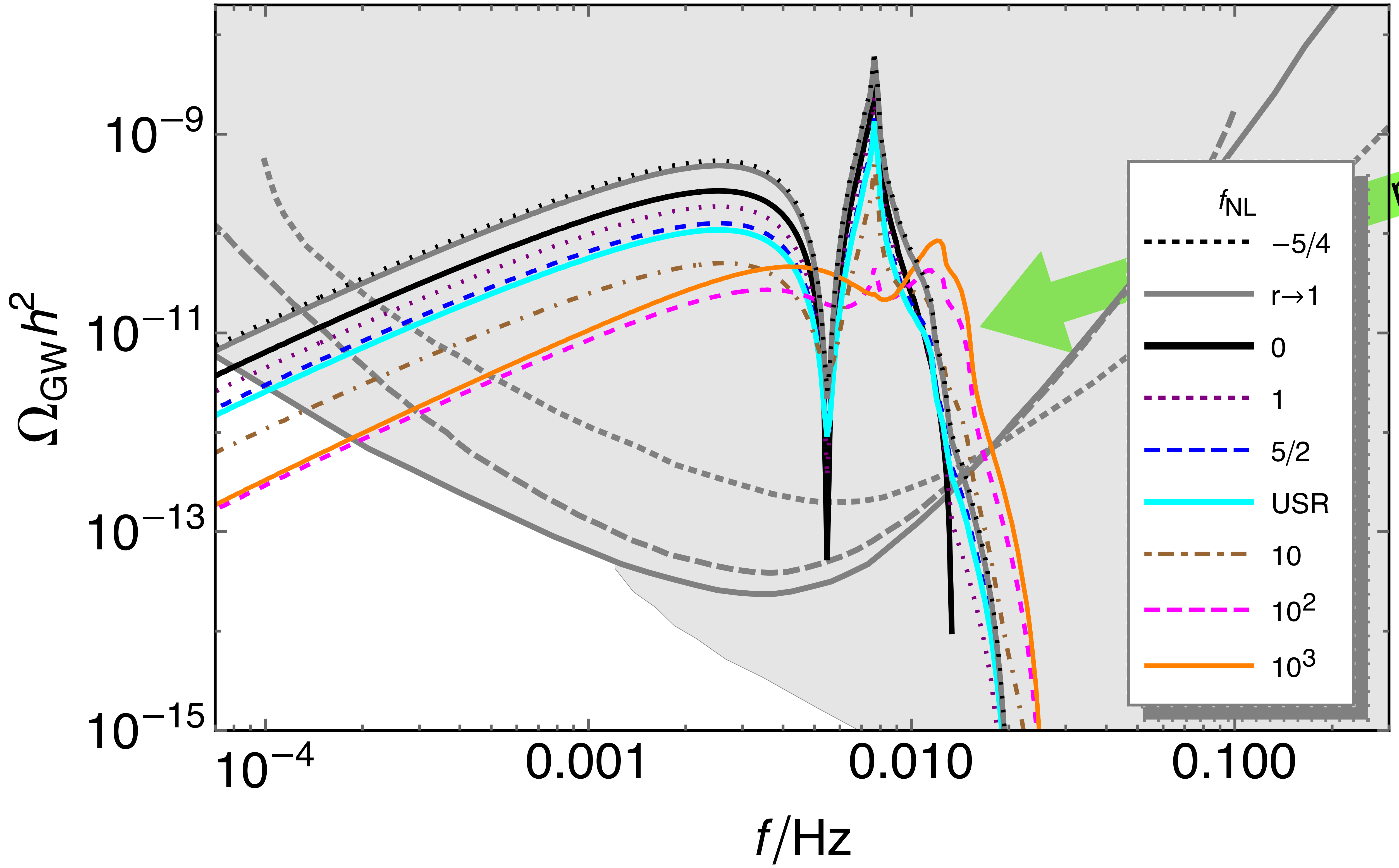
$$\mathcal{P}_\mathcal{R} = \mathcal{A}_\mathcal{R}\,\delta\left(\ln k - \ln k_*\right)$$

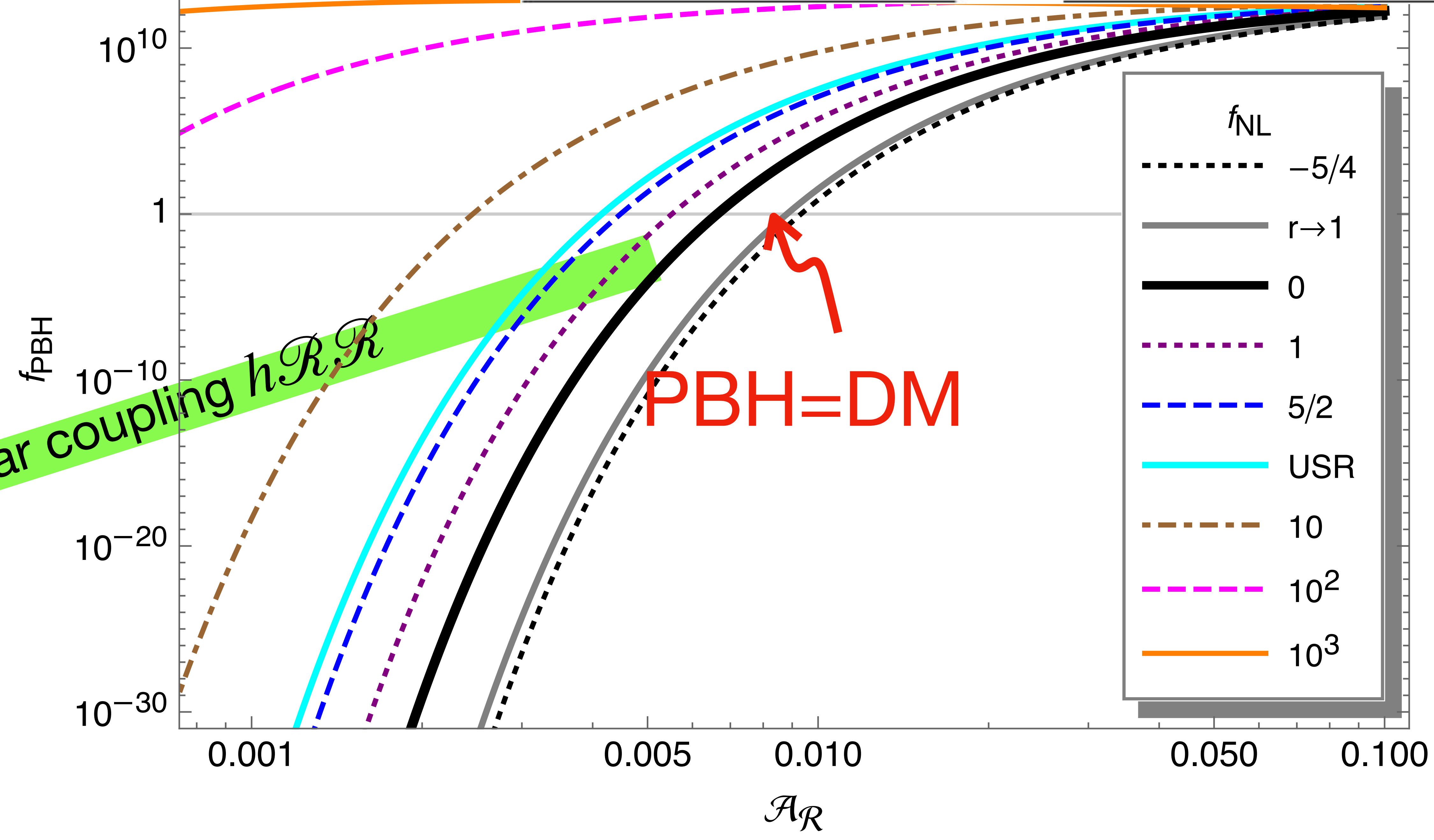
Cai, SP and Sasaki, 1810.11000
SP, 2404.06151

PBH as DM

$f_{\rm NL}$	$\mathcal{A}_\mathcal{R}$	$f_{\rm NL}$	$\mathcal{A}_\mathcal{R}$
$-5/4$	9.328243×10^{-3}	$5/2$	4.479141×10^{-3}
$r\to1$	8.808810×10^{-3}	USR	4.151077×10^{-3}
0	6.635506×10^{-3}	10	2.456822×10^{-3}
0.1	6.498845×10^{-3}	10^2	4.618021×10^{-4}
1	5.519888×10^{-3}	10^3	5.690648×10^{-5}

- Quadratic: Cai, SP, Sasaki 1810.11000; Unal 1811.09151
- Higher orders: $f_{\rm NL}$: Adshead+ 2105.01659; Abe+ 2209.13891; $\tau_{\rm NL}$: Garcia-Saenz+ 2207.14267. $g_{\rm NL}$: Yuan+, 2308.07155; Li+, 2309.07792. $i_{\rm NL}$: Perna+, 2403.06962
- When fixing PBH abundance, NG impact on SGWB is mild





Cai, SP and Sasaki, 1810.11000
SP, 2404.06151

PBH as DM

$$\Omega_{\rm GW,peak}h^2 \approx 1.6\times10^{-5}\max\left[6.4\mathscr{A}_{\mathscr{R}}^2,\ 3.7\mathscr{A}_{\mathscr{R}}^3F_{\rm NL}^2,\ 3.9\mathscr{A}_{\mathscr{R}}^4F_{\rm NL}^4\right]$$

$f_{\rm NL}$	$\mathcal{A}_{\mathcal{R}}$	$f_{\rm NL}$	$\mathcal{A}_{\mathcal{R}}$
$-5/4$	9.328243×10^{-3}	$5/2$	4.479141×10^{-3}
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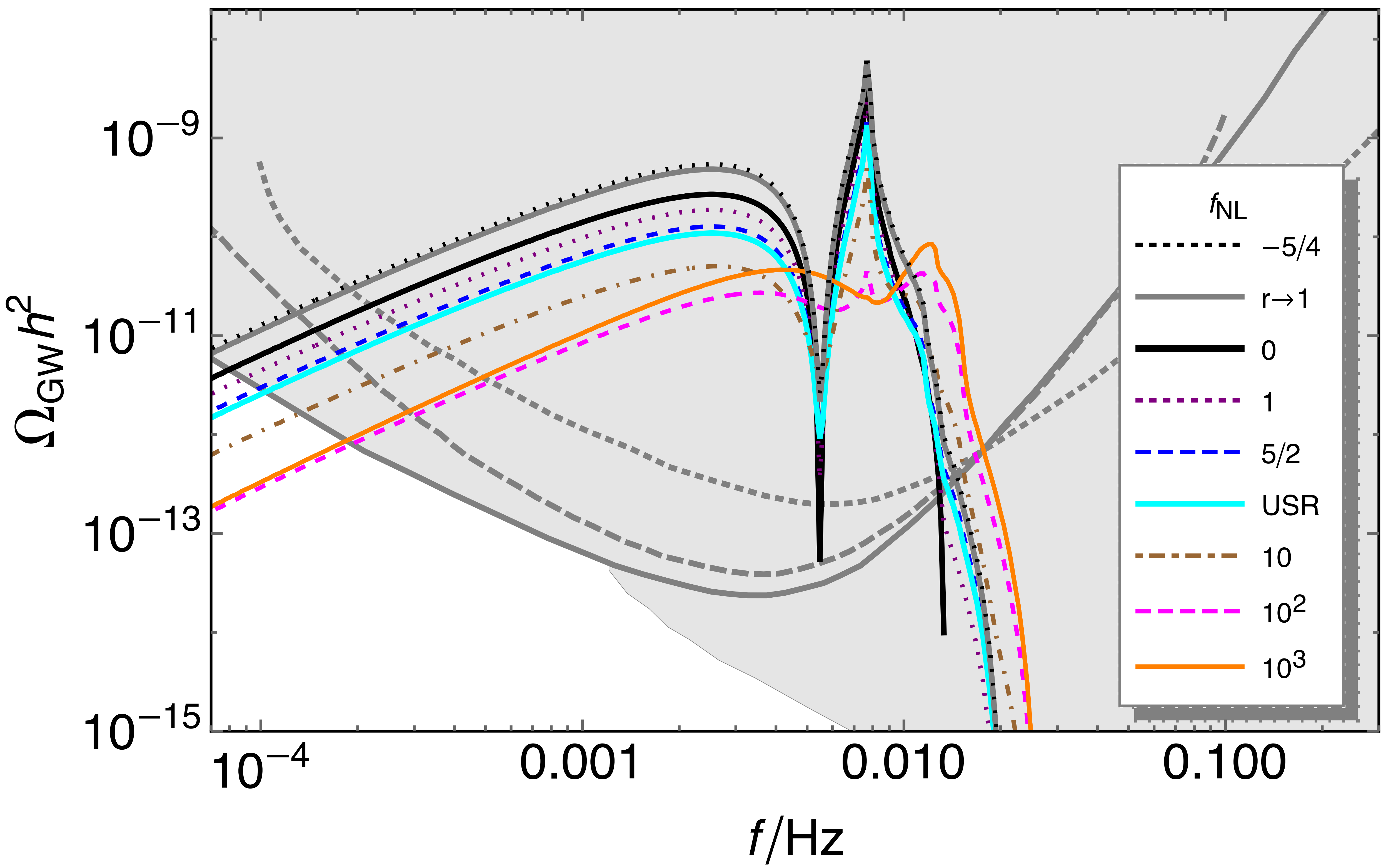
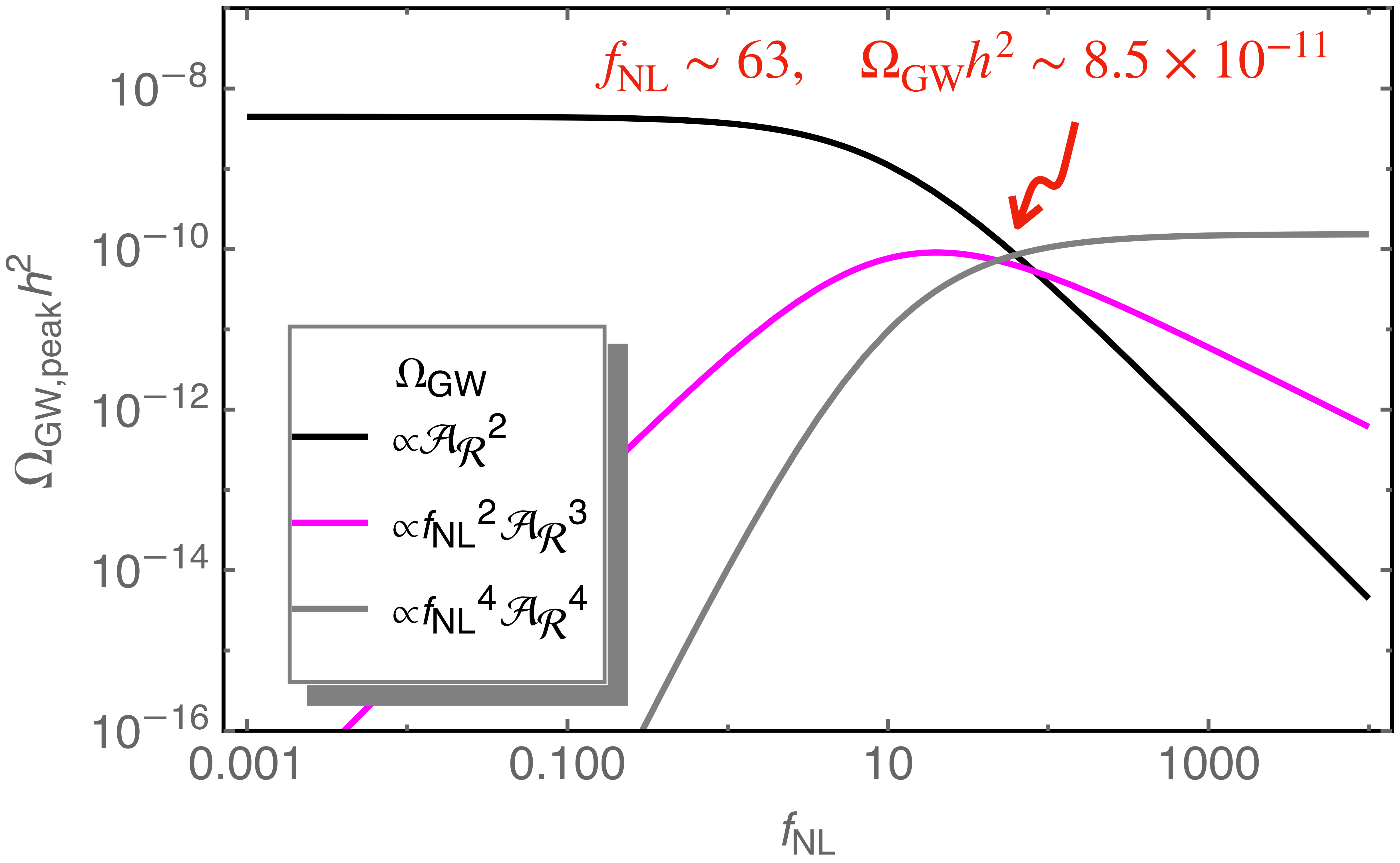
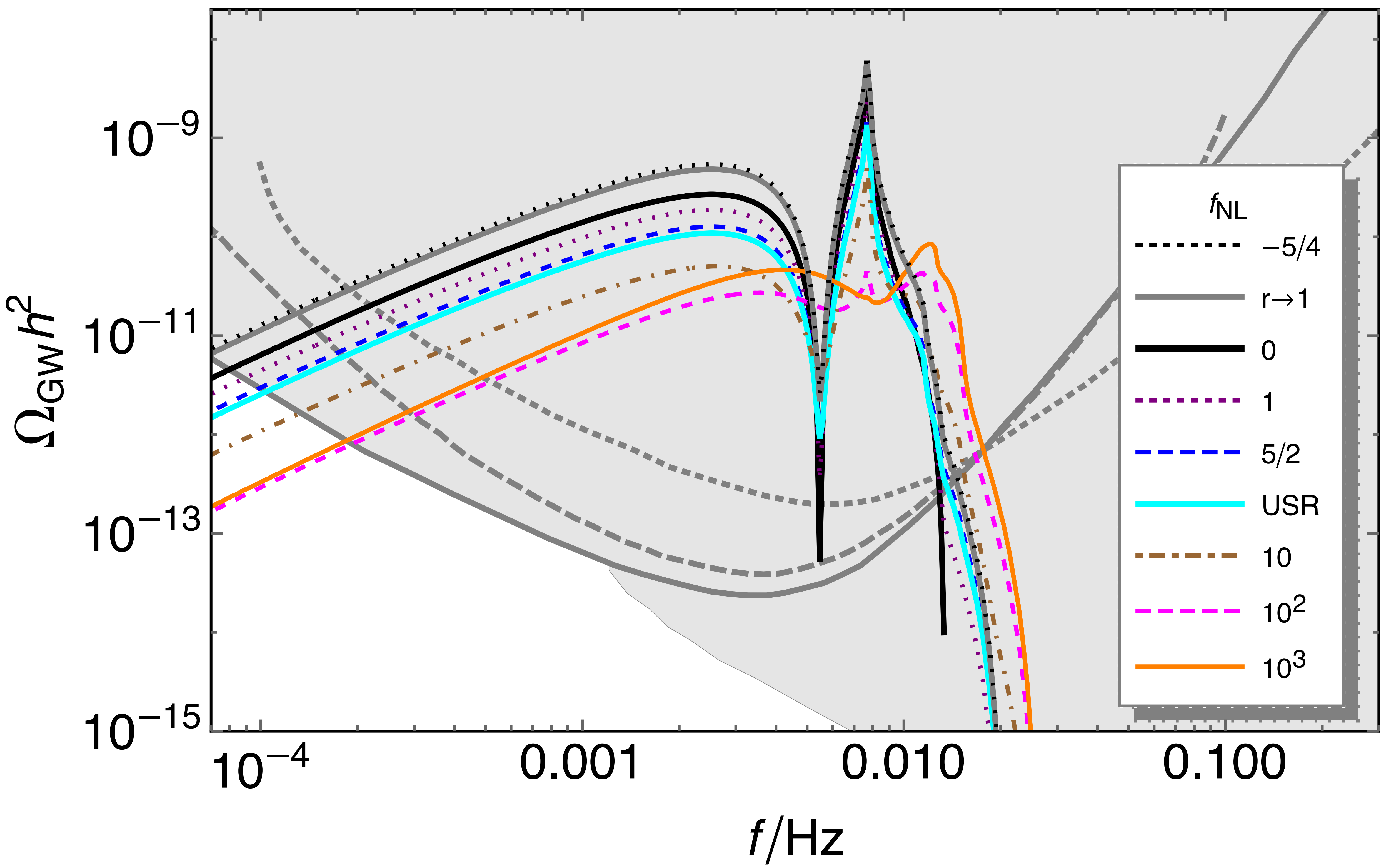


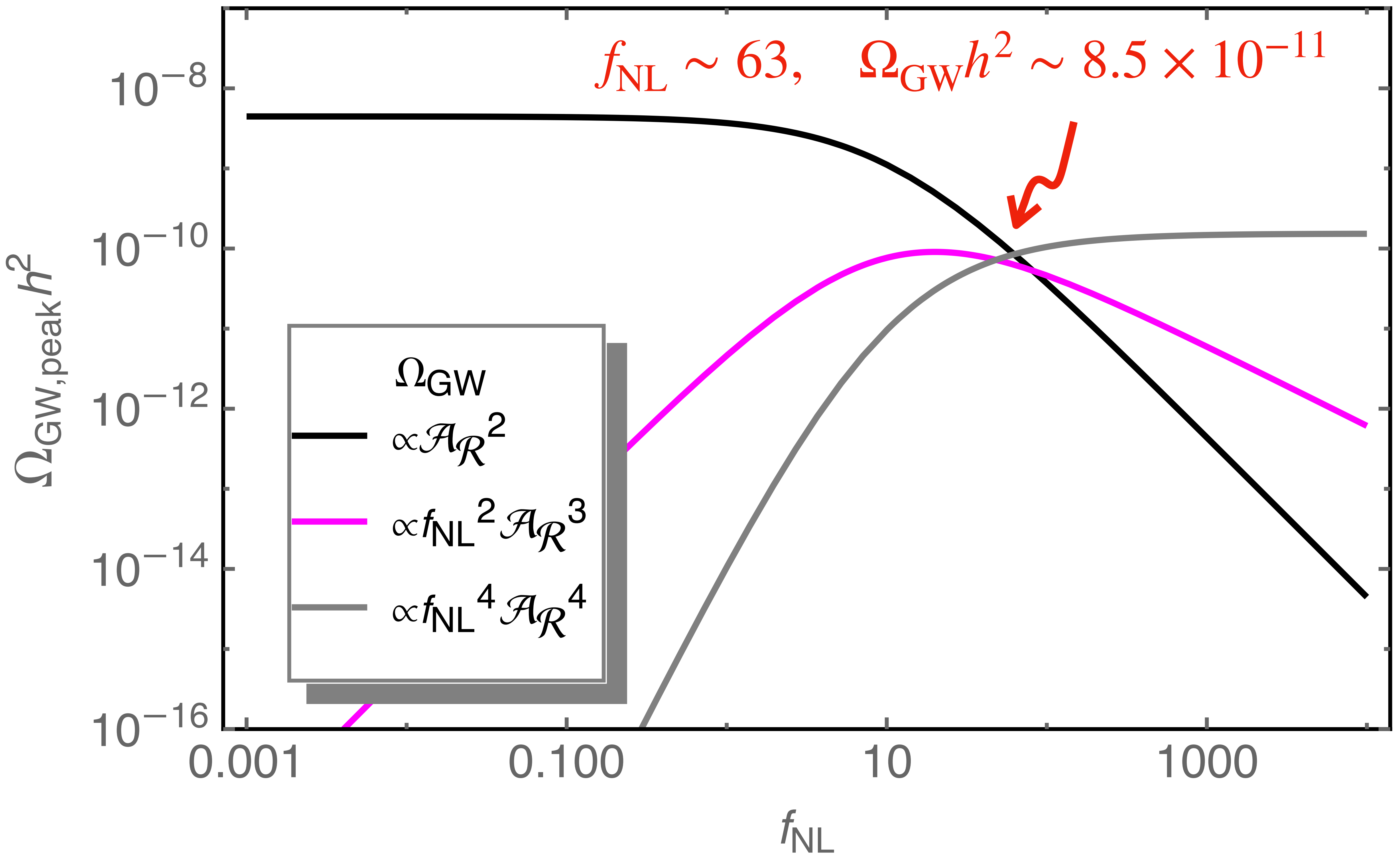


PBH as DM

$$\Omega_{\rm GW,peak}h^2 \gtrsim 8.5\times 10^{-11} > \text{LISA, Taiji, TianQin, BBO, DECIGO}, \cdots$$

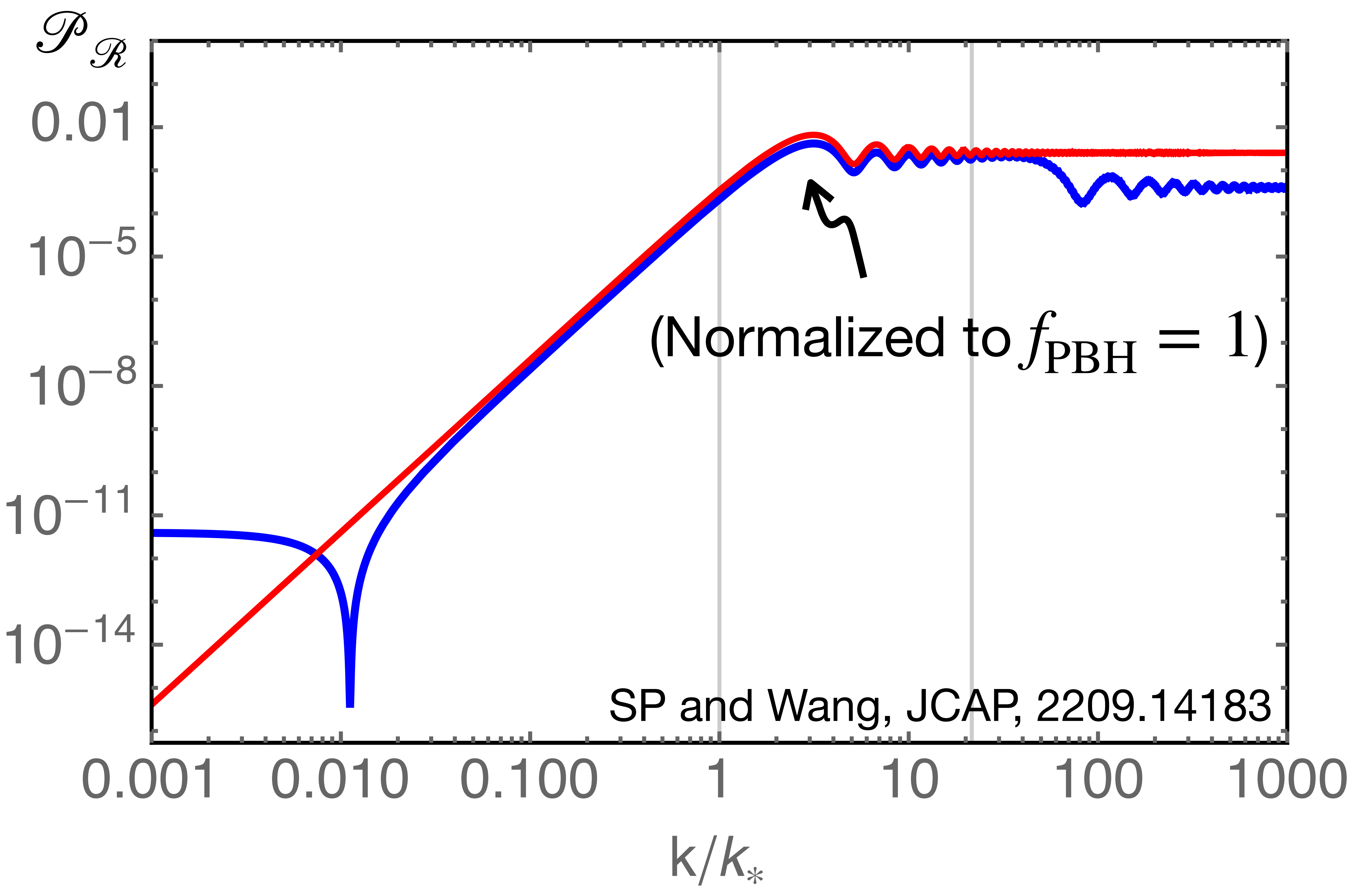
When PBH are all the dark matter, LISA/Taiji/TianQin/BBO/DECIGO can probe the induced GW signal, which is relatively robust against non-Gaussianity.

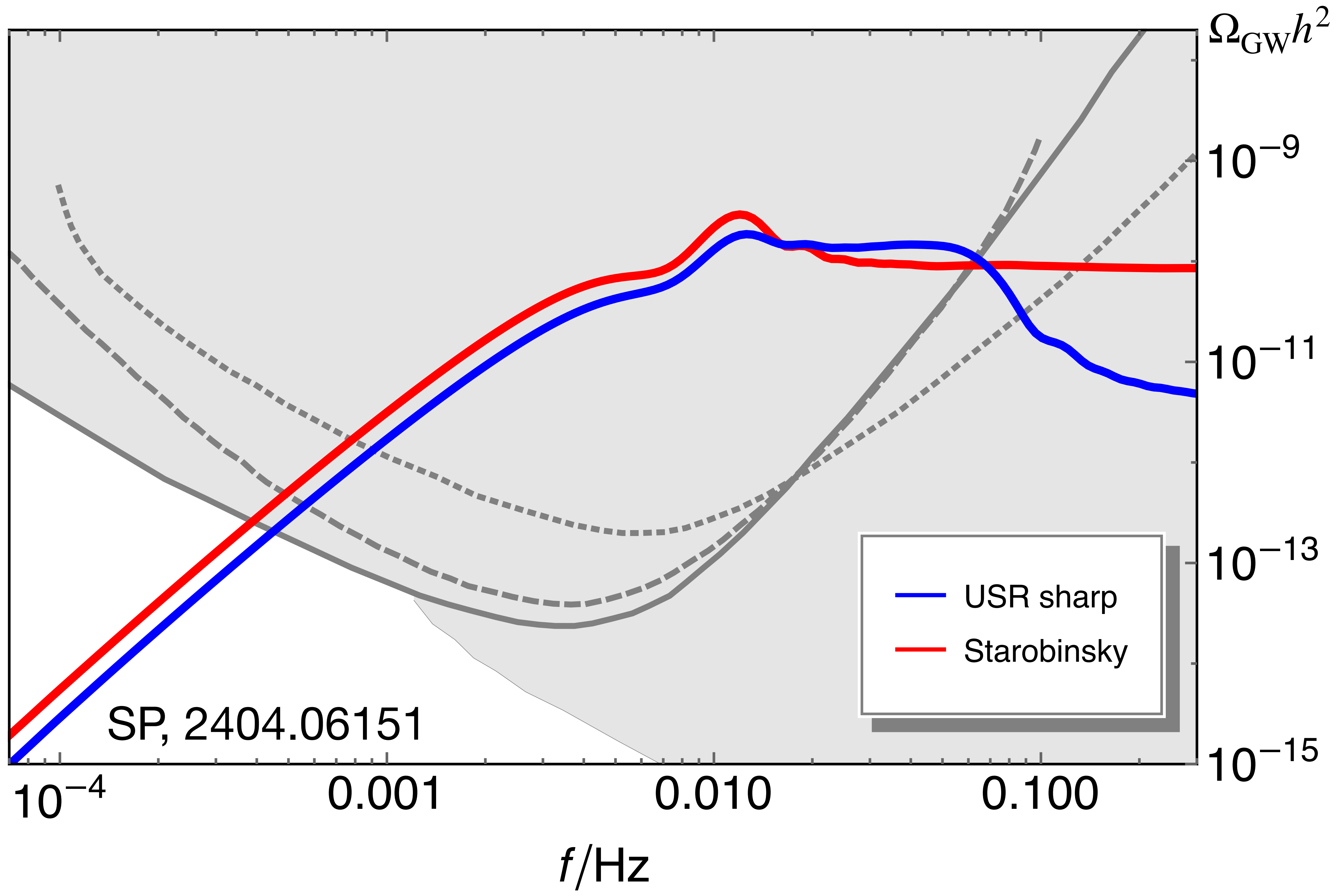




PBH and IGW

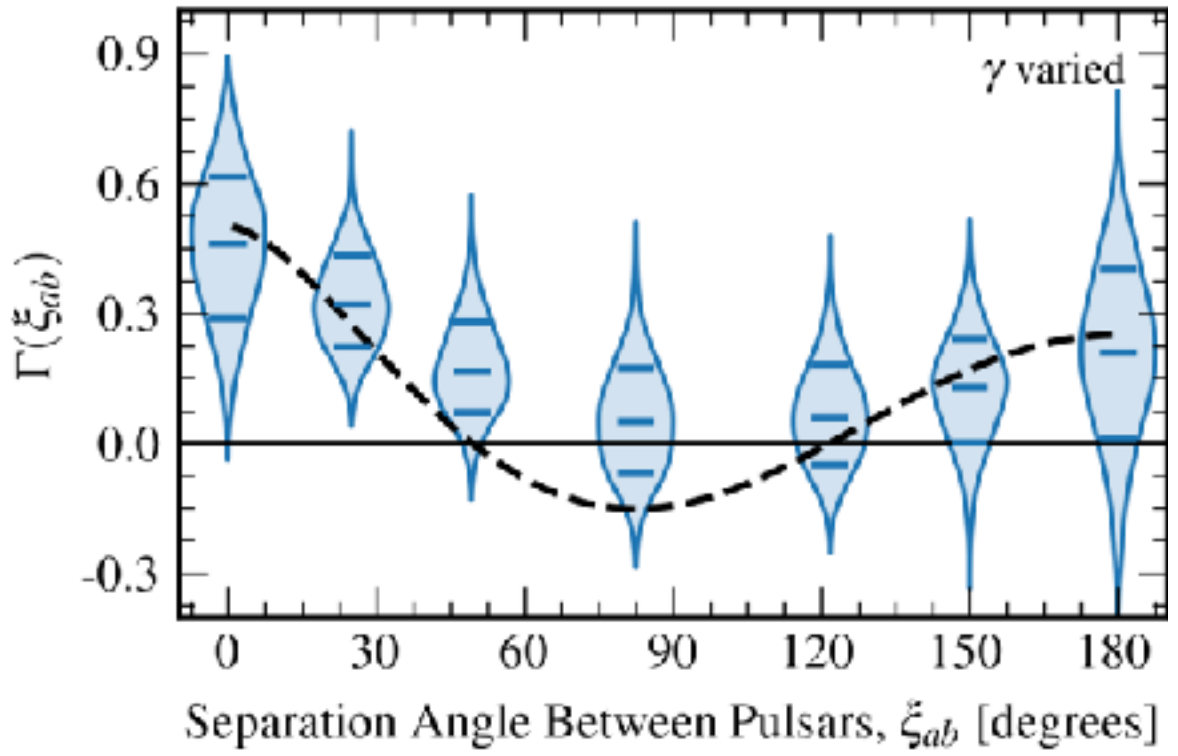
More physical signals: Ultra-slow-roll vs Starobinsky



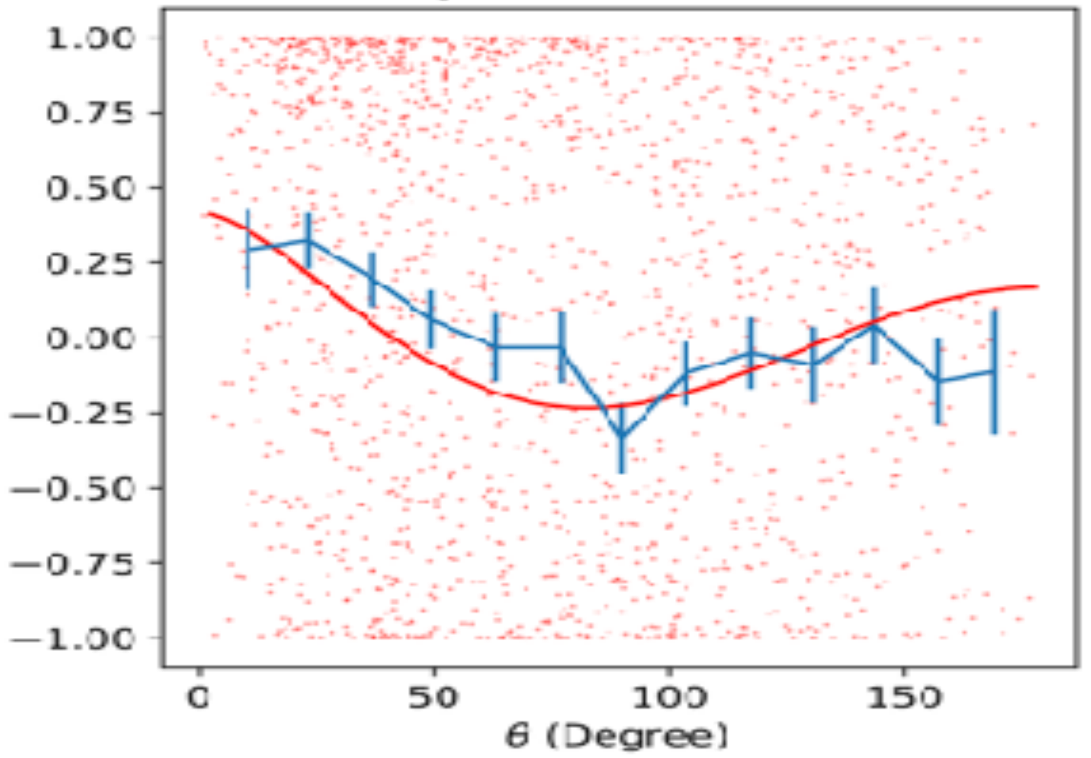


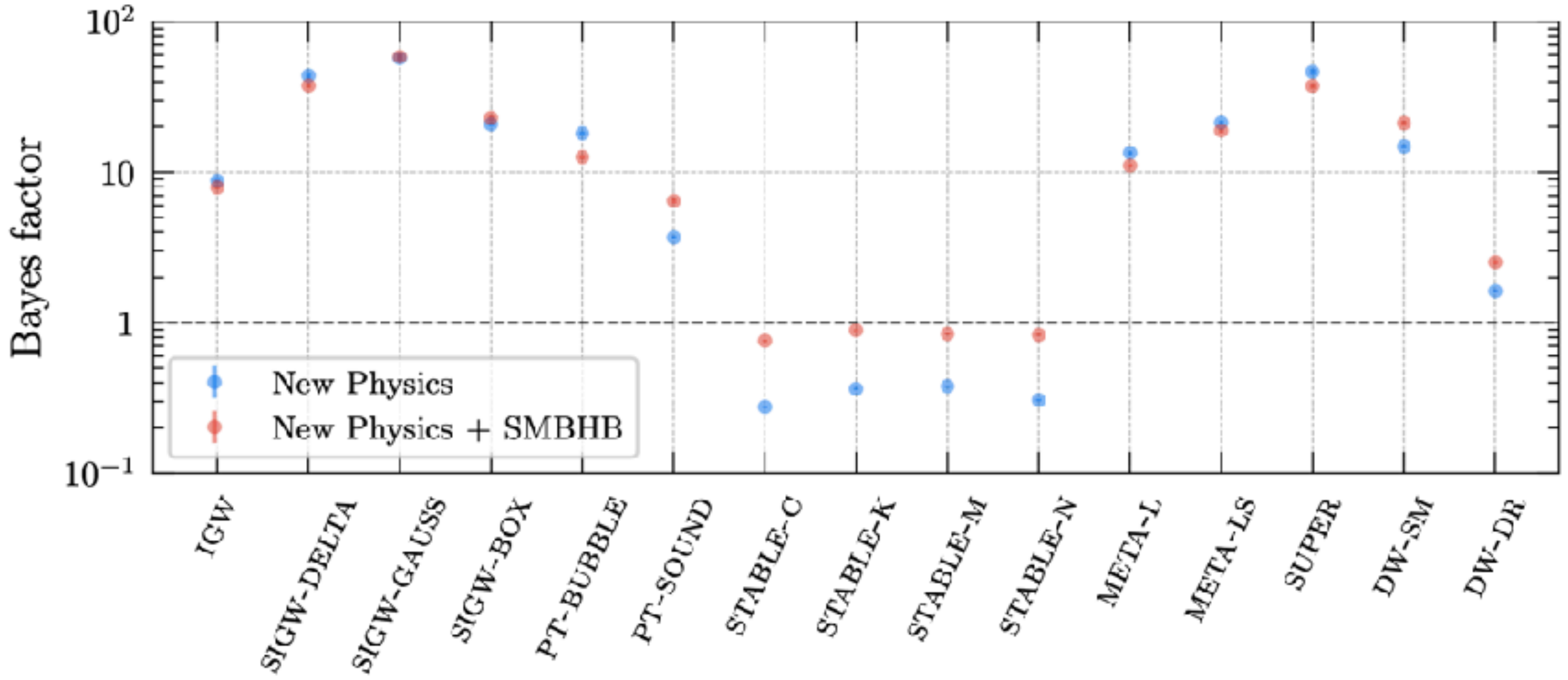
Application: nHz SGWB

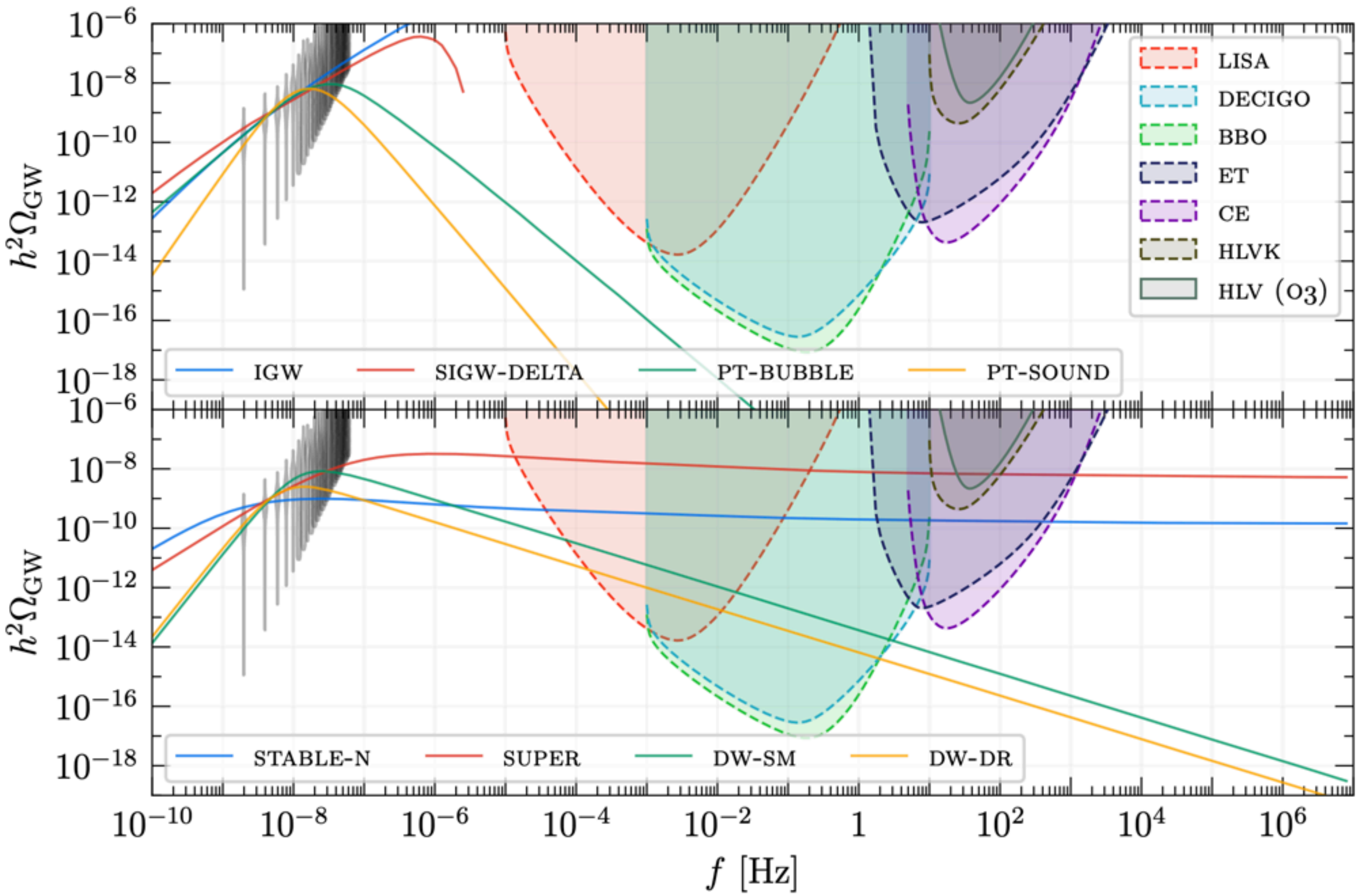
NANOGrav, 2306.16213



CPTA, 2306.16216



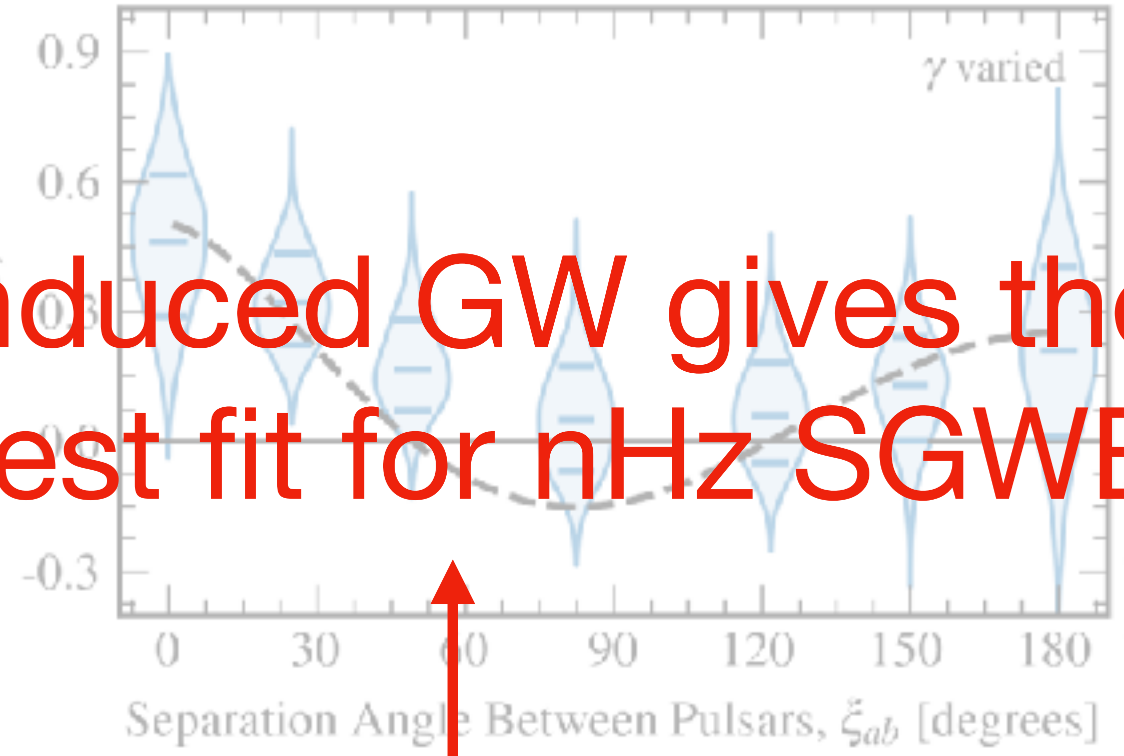




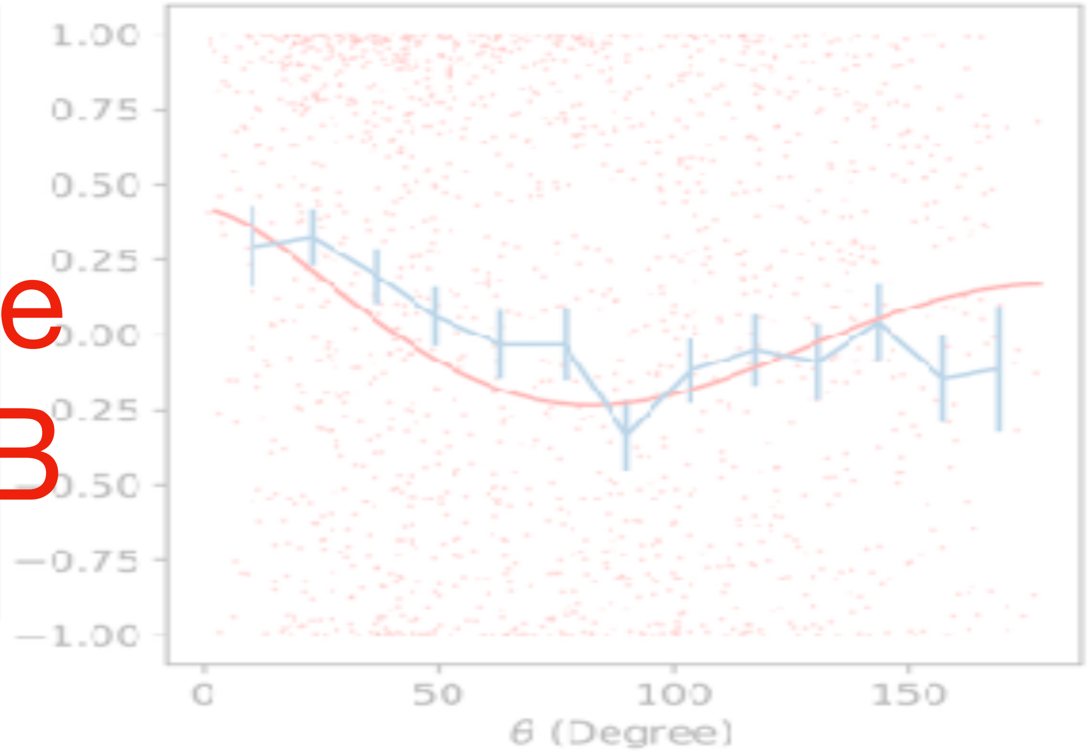
NANOGrav, 2306.16219

Application: nHz SGWB

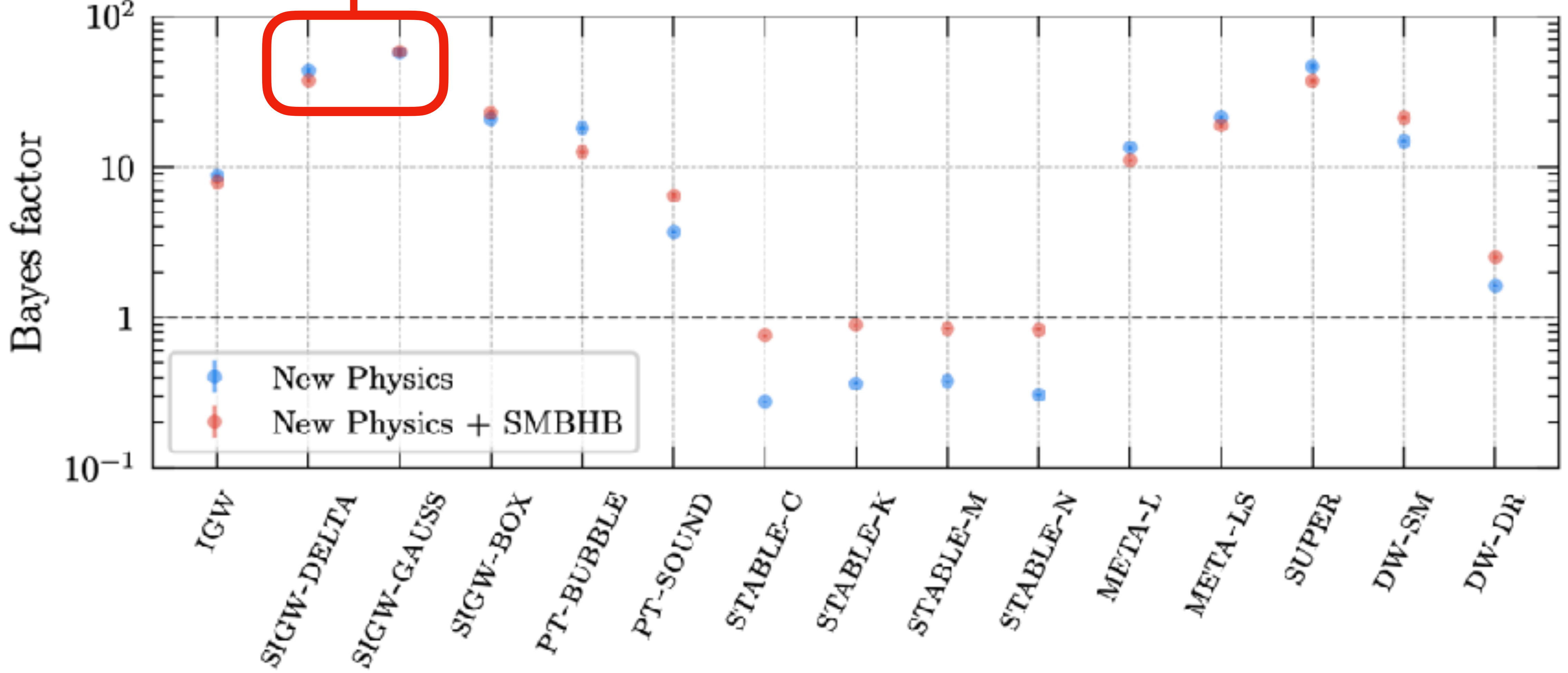
NANOGrav, 2306.16213



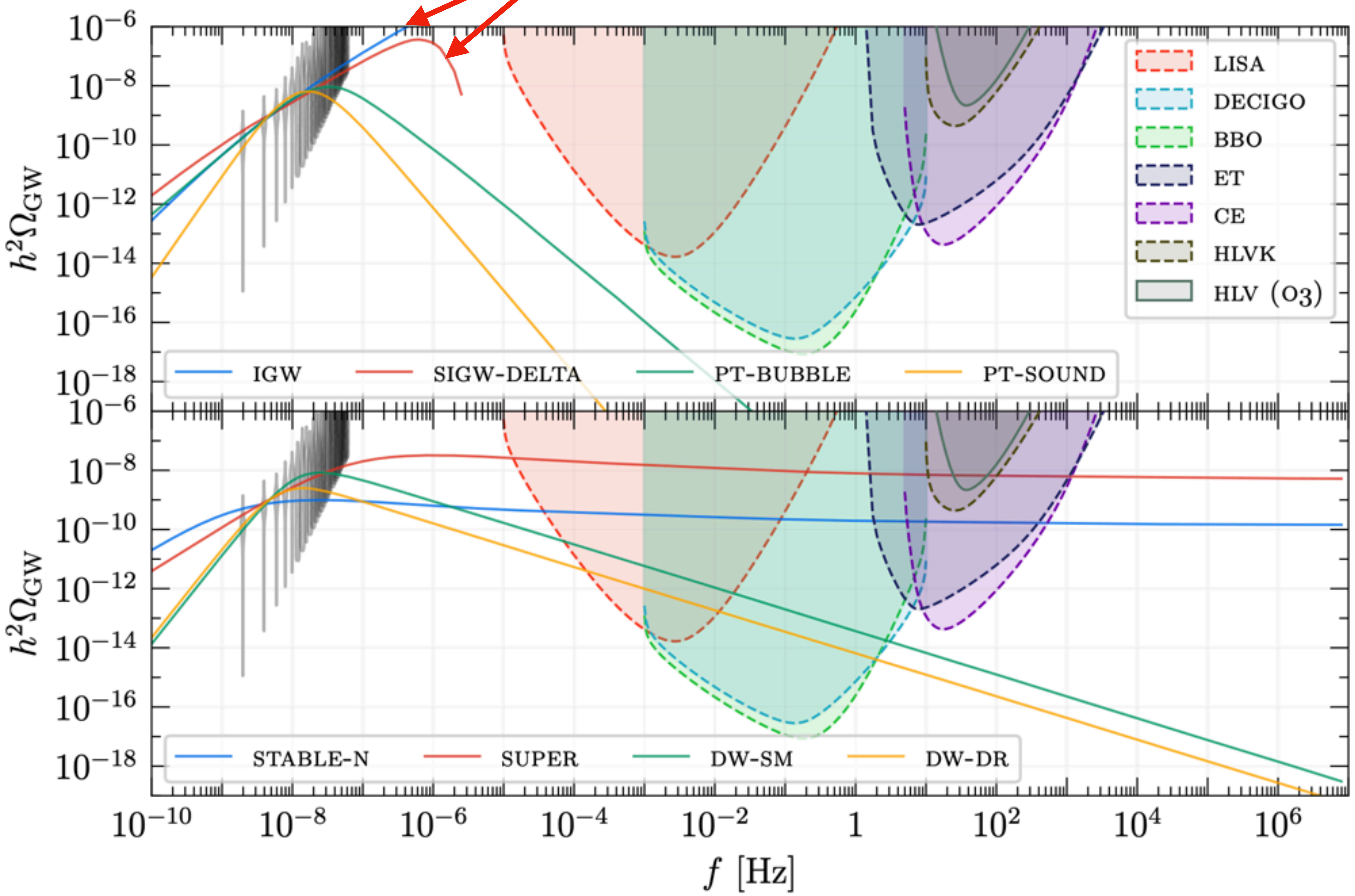
CPTA, 2306.16216



Induced GW gives the best fit for nHz SGWB

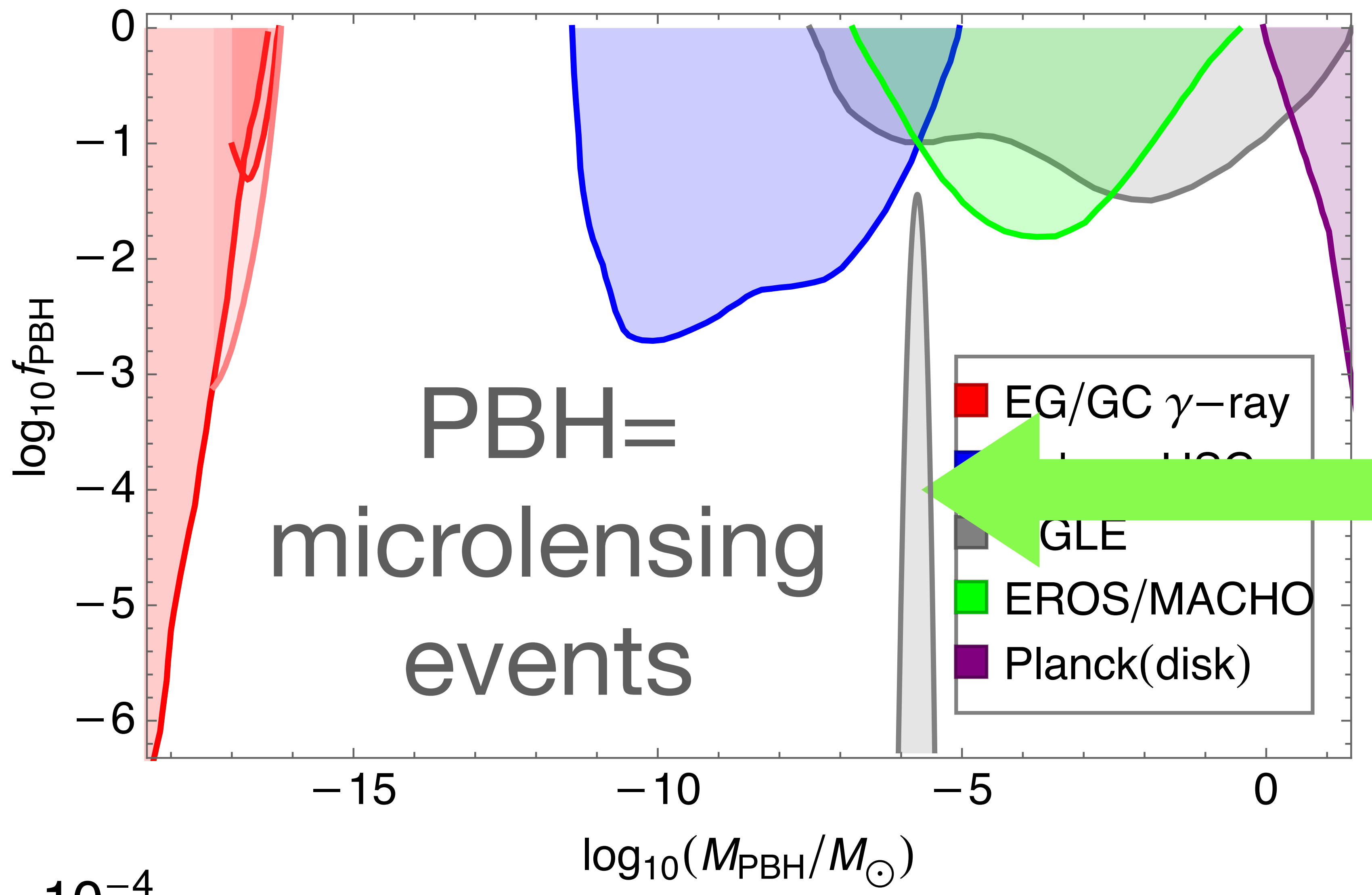


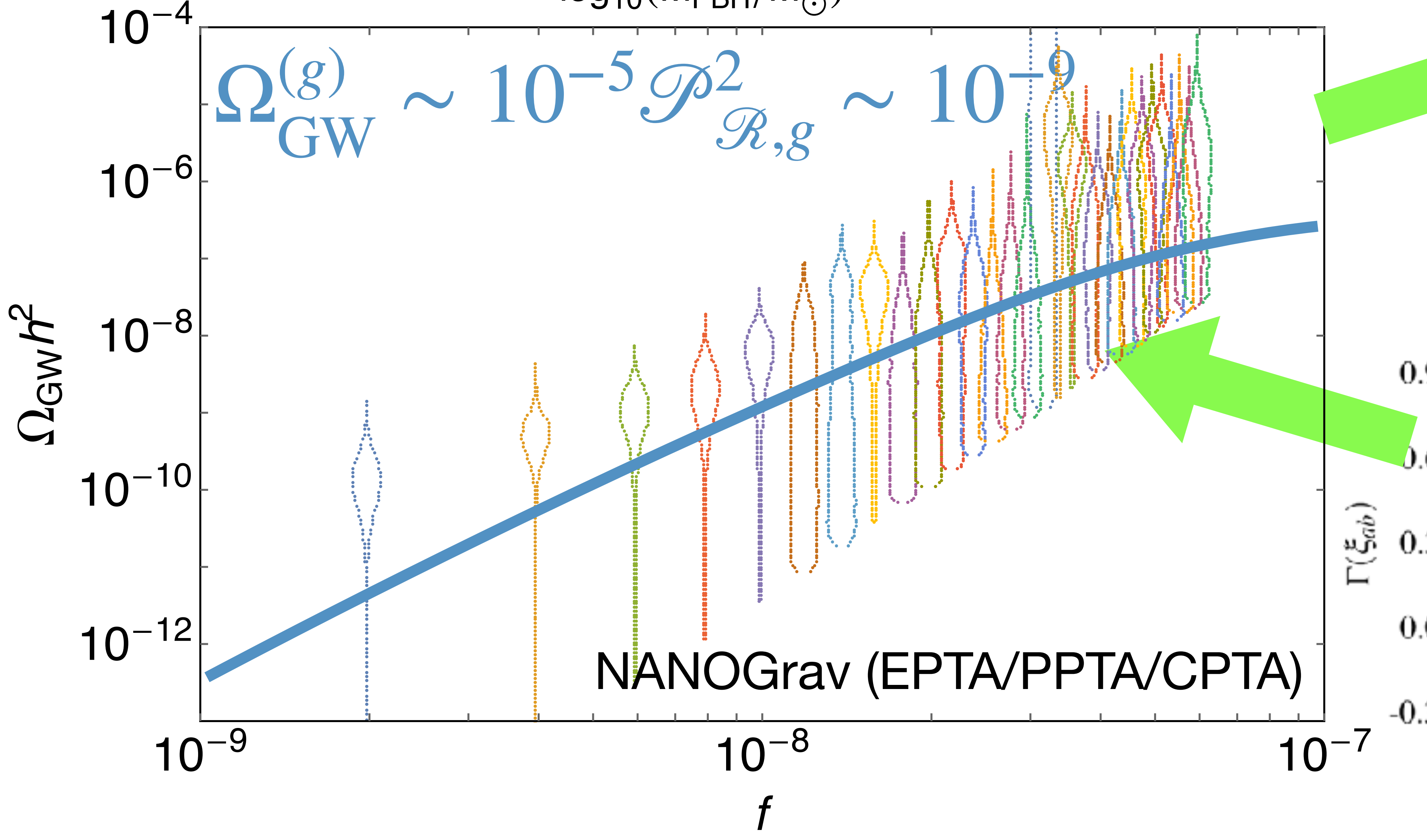
Induced GW

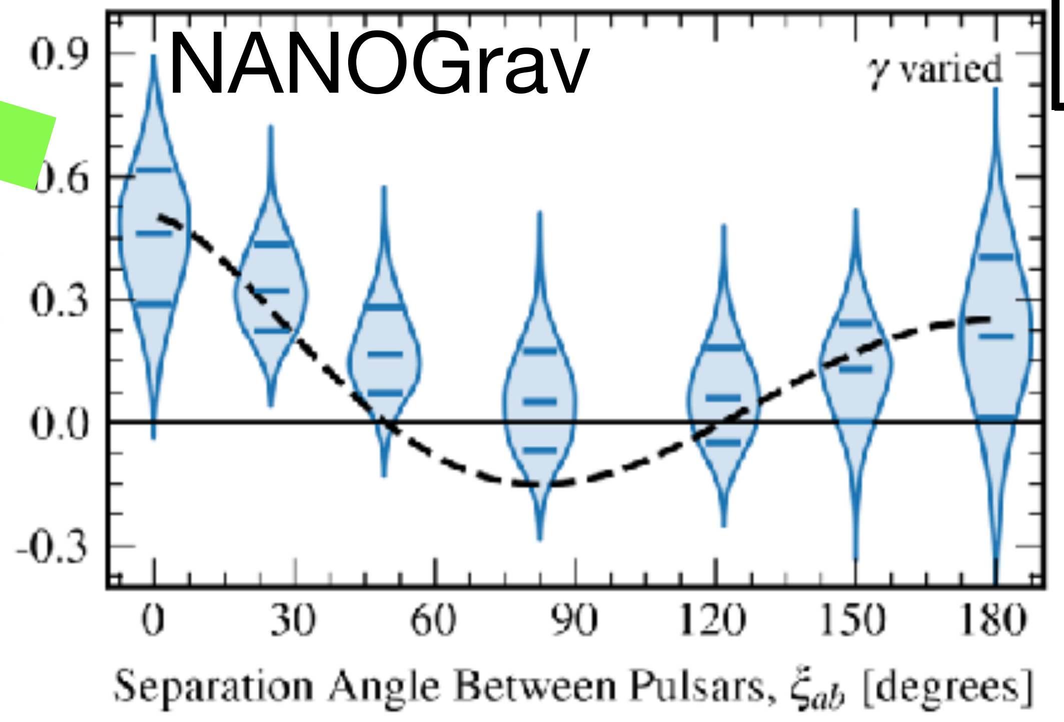


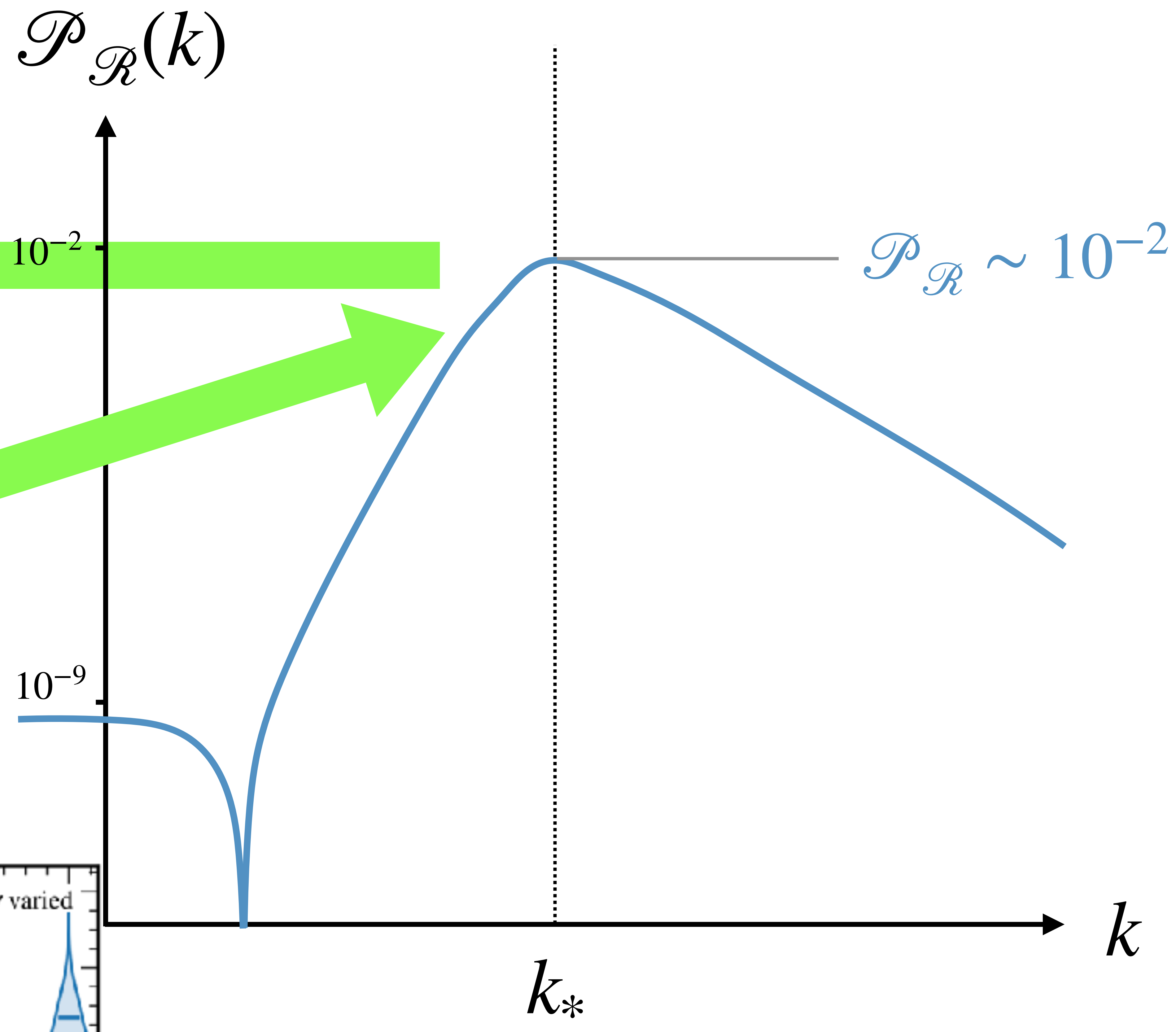
NANOGrav, 2306.16219

Crosscheck by PBH and IGW



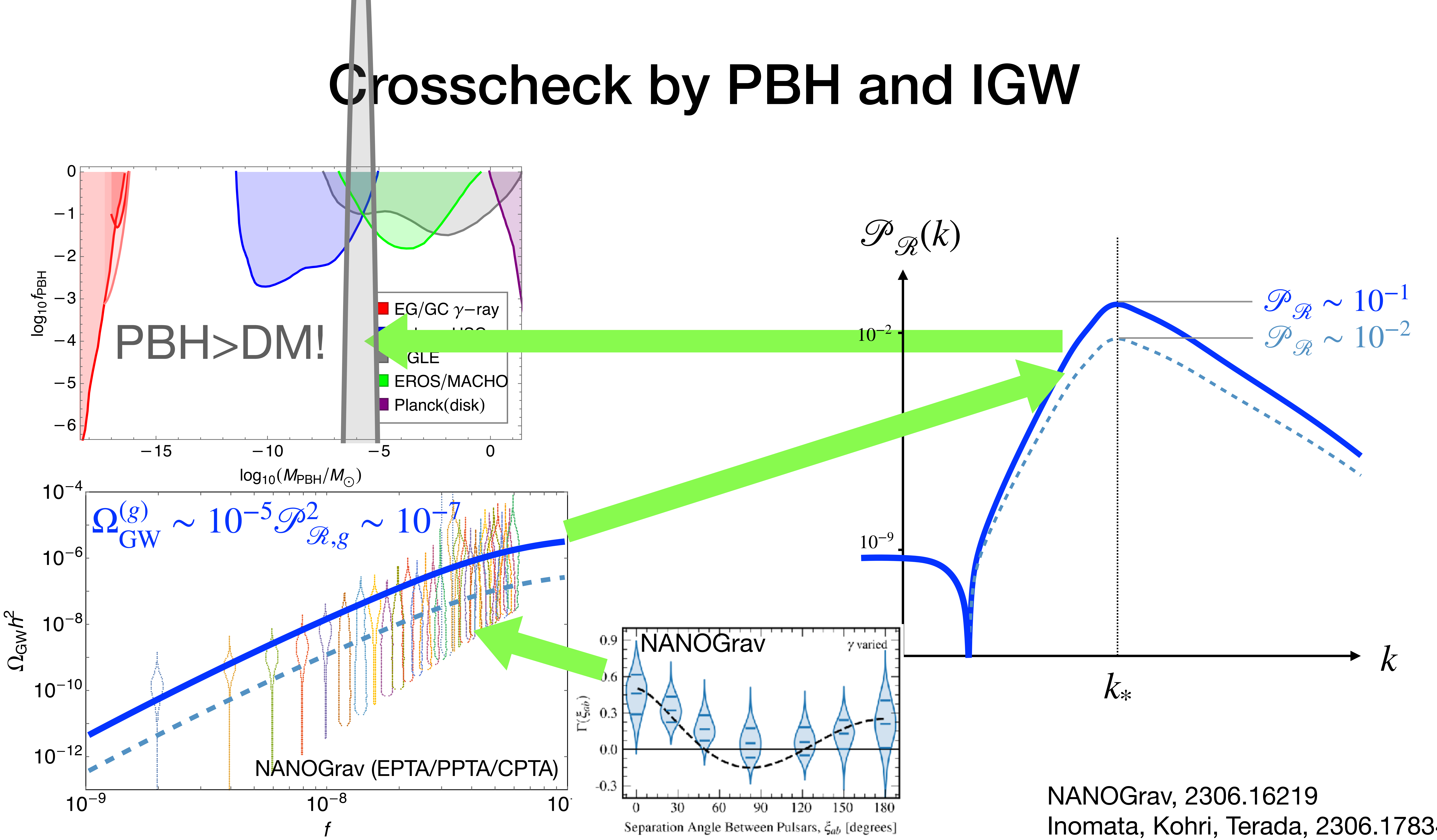






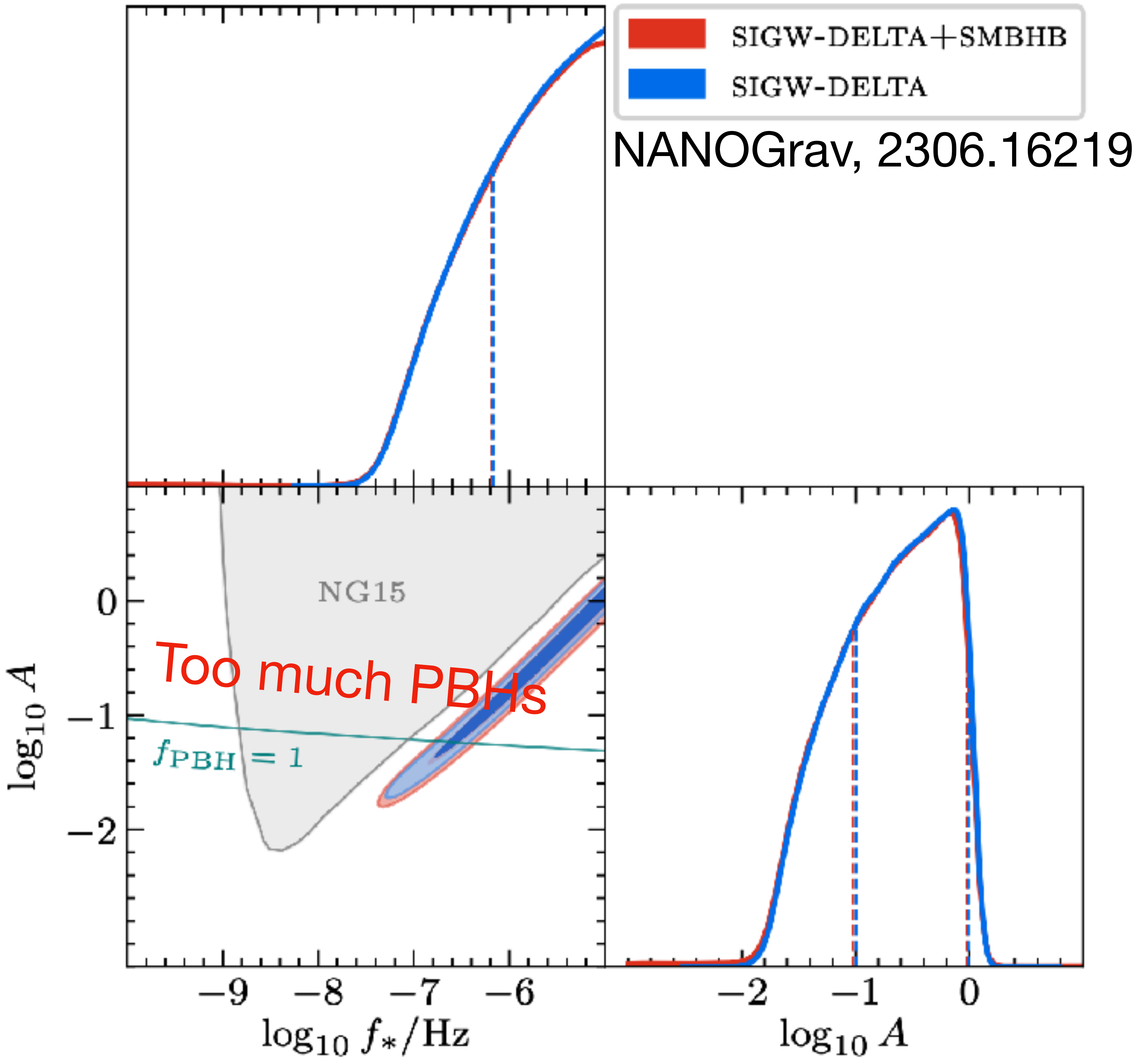
Kohri, Terada, 2009.11853
SP and Domenech, 2010.03976
NANOGrav, 2306.16219
Inomata, Kohri, Terada, 2306.17834

Crosscheck by PBH and IGW



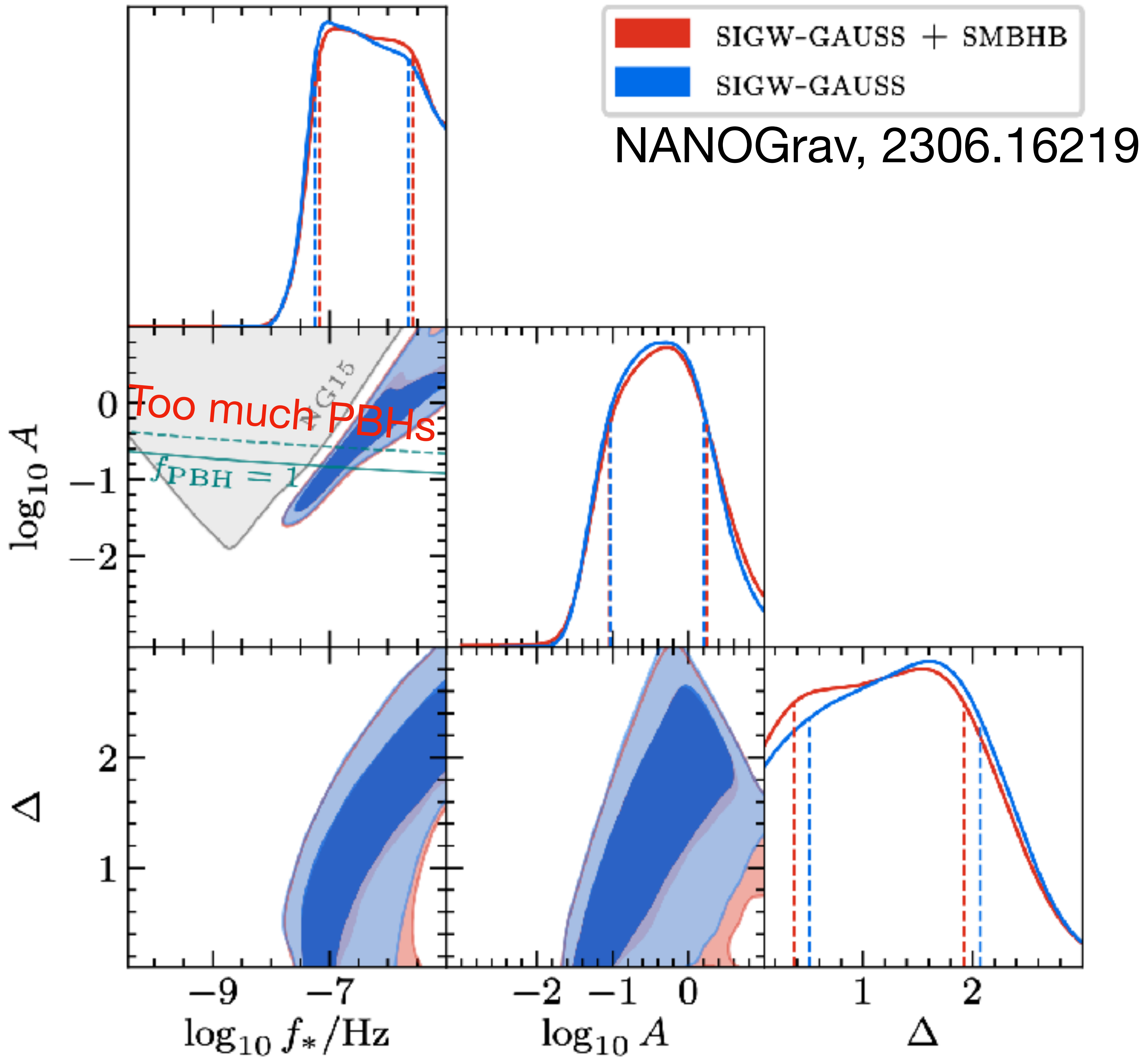
NANOGrav, 2306.16219
Inomata, Kohri, Terada, 2306.17834

IGW as nHz SGWB



$$\mathscr{P}_{\mathscr{R}} = A\delta(\ln k - \ln k_*)$$

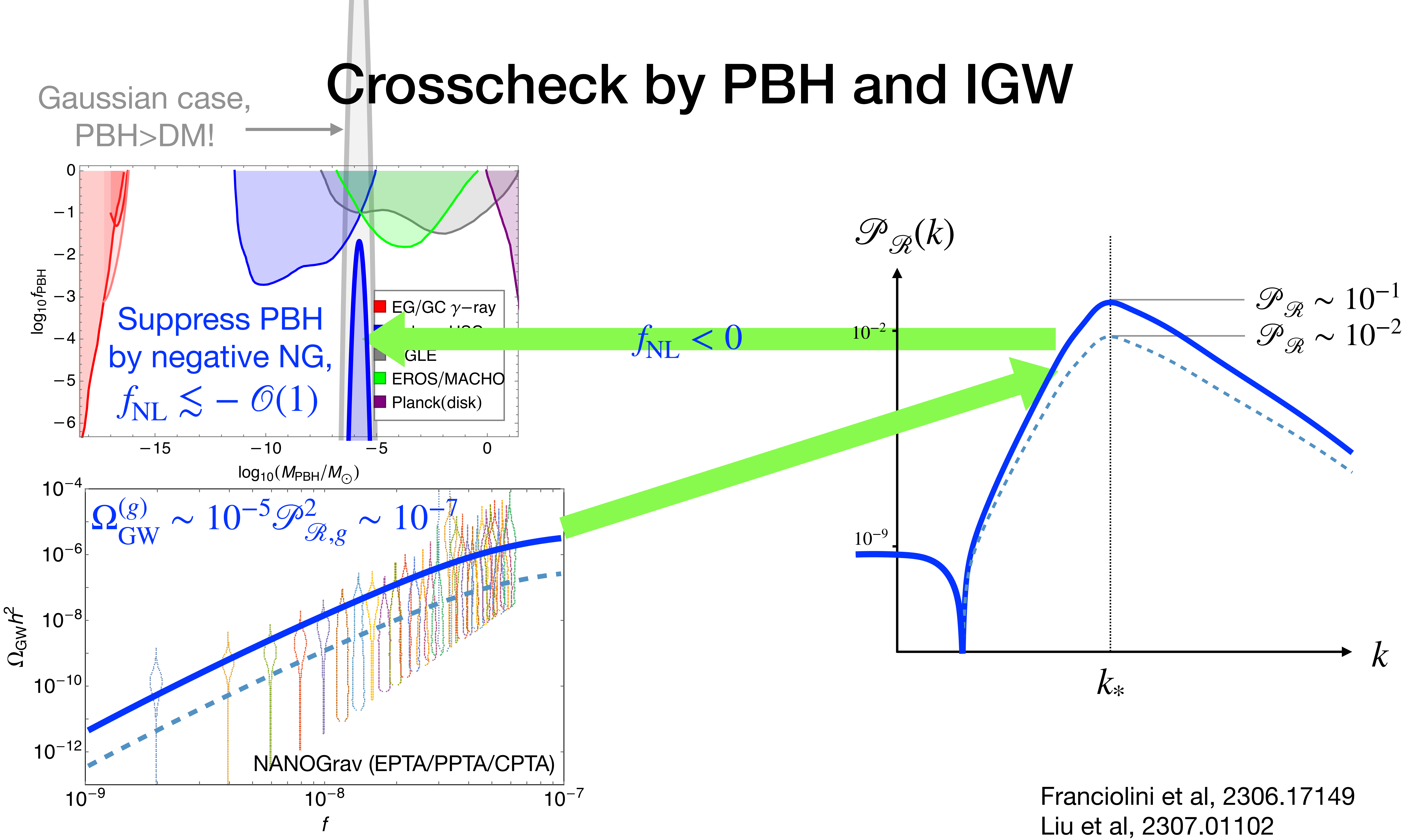
monochromatic



$$\mathscr{P}_{\mathscr{R}} = \frac{A}{\sqrt{2\pi}\Delta} \exp\left(-\frac{(\ln k - \ln k_*)^2}{2\Delta^2}\right)$$

lognormal [SP and Sasaki 2005.12306]

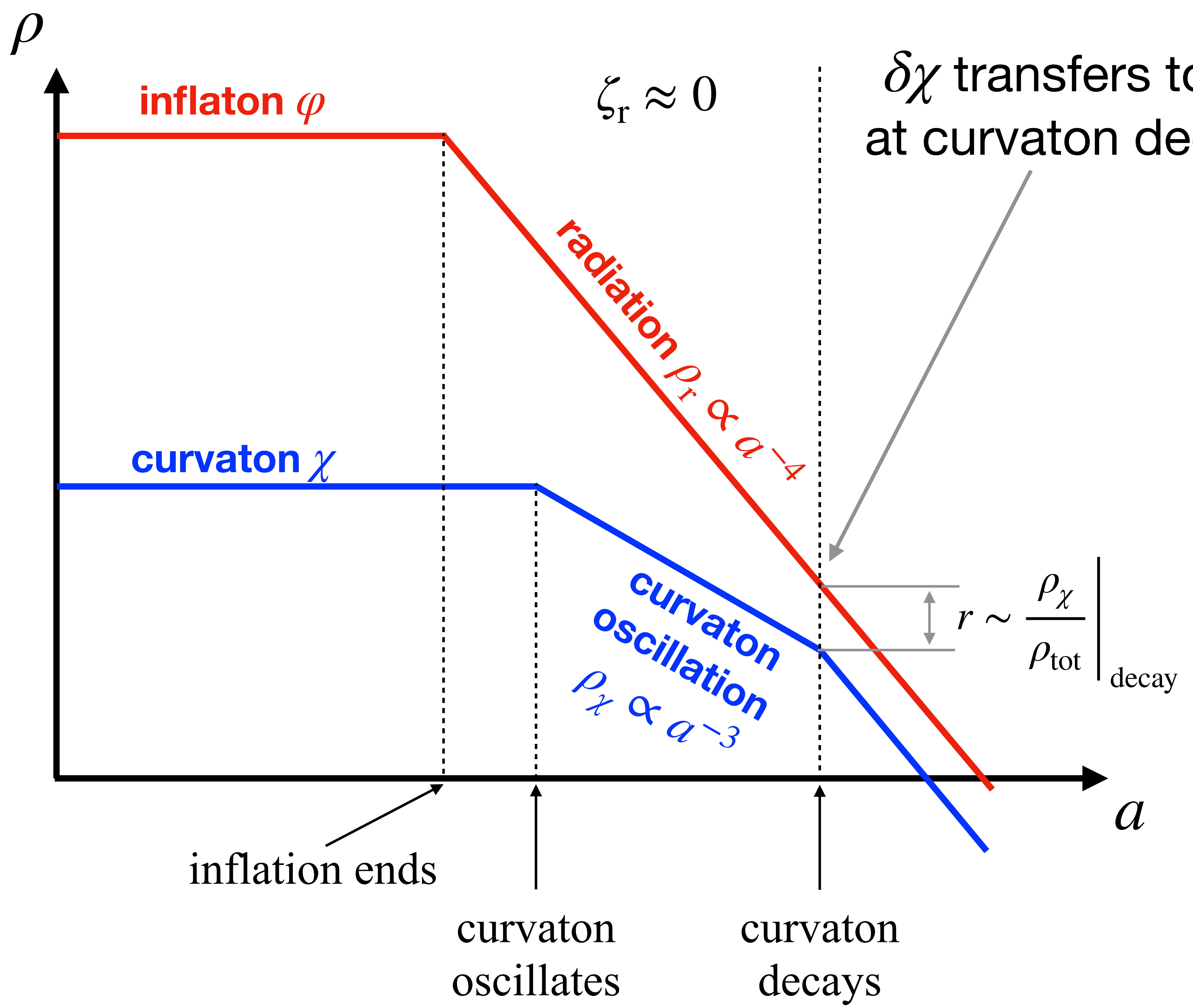
Crosscheck by PBH and IGW



Franciolini et al, 2306.17149
Liu et al, 2307.01102

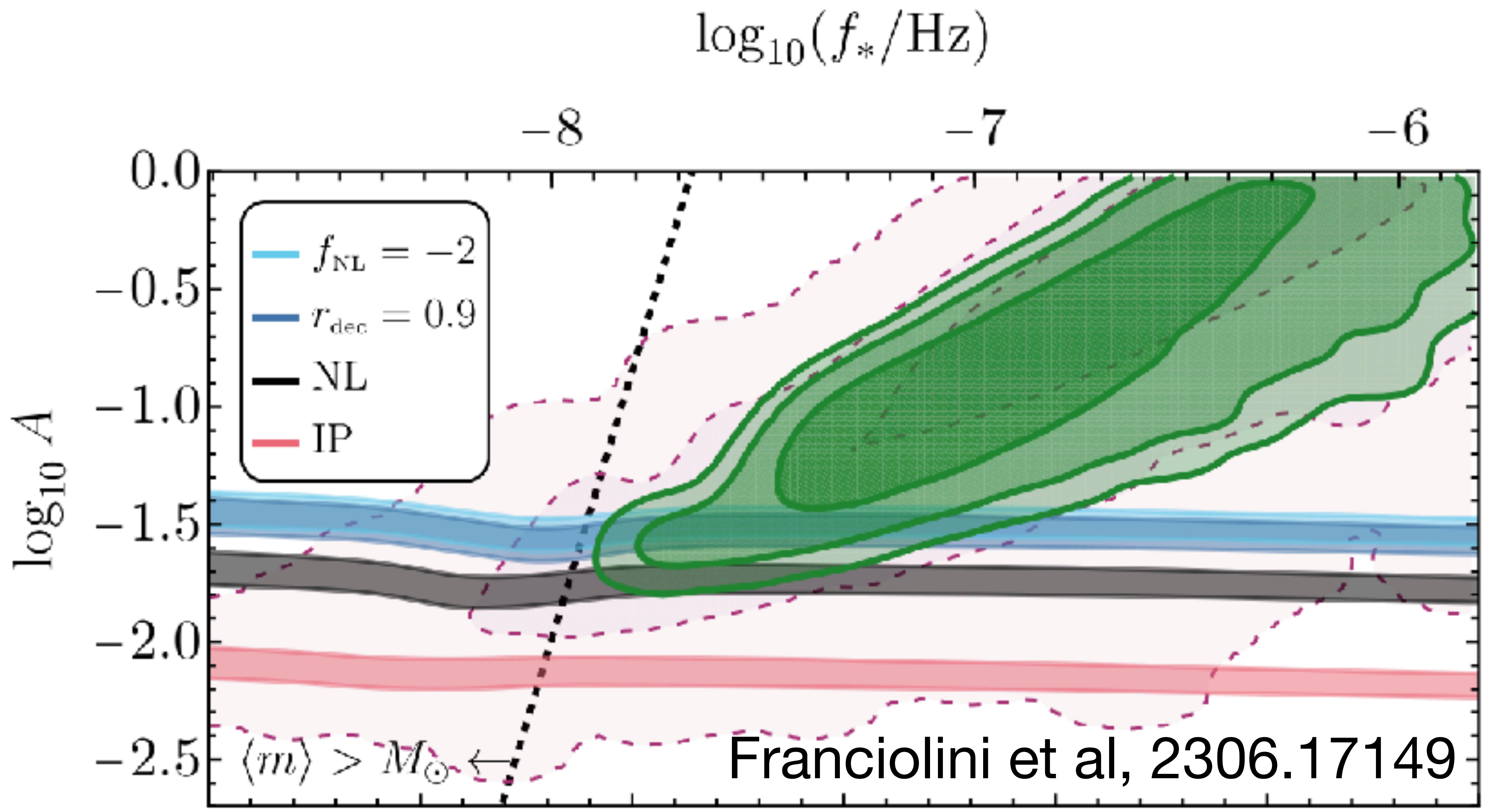
Curvaton Scenario

SP and Sasaki, 2112.12680
Ferrante et al, 2211.01728



$$\zeta = \zeta(\delta\chi/\chi) \longrightarrow \begin{cases} \frac{r}{3}\left[2\frac{\delta\chi}{\chi} + \left(\frac{\delta\chi}{\chi}\right)^2\right] & \text{when } r \ll 1 \\ \frac{2}{3}\ln\left|1+\frac{\delta\chi}{\chi}\right| & \text{when } r \sim 1 \end{cases}$$

- $\zeta(\delta\chi)$ degenerates to a logarithmic relation ($f_{\mathrm{NL}} = -5/4$) when the curvaton dominates.



Franciolini et al, 2306.17149

Discussion

- "Exponential tail" can be extended to heavy-tail PDF $P(\mathscr{R}) \propto \exp(-\lambda |\mathscr{R}|^p)$, with $0 < p < 1$ (Nakama et al. 1609.02245; Namjoo et al. 2112.04520, 2305.19257; Creminelli et al. 2103.09244; Hooper et al. 2308.00756…). This can enhance PBH formation even more, and evade the distortion constraints.

- When $\mathscr{R}$ is a sum of many logarithms, the PDF is more complicated. Cruces, SP, Sasaki, in preparation.

- Non-Gaussianity of other shapes. Matsubara and Sasaki 2208.02941. Clustering, window function, gradient expansion, transition to stochastic approach, Type II PBH, bubble channel.…

Conclusion

- Primordial non-Gaussianity must be taken into account when calculating PBH abundance.
- Fixing PBH abundance, the prediction of induced GW is robust, which is an important scientific goal of LISA/Taiji/TianQin. [Whitebooks, snowmass, Astro2020, etc.]
- For USR, non-Gaussianity can change the spectral shape of the induced GW, which can be used to fix non-Gaussianity.