

# Entanglement Negativity in $T\bar{T}$ -Deformed $CFT_2$ and Holography

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# Outline

- 1 Introduction and Motivation
- 2 Overview of Entanglement measures
- 3 Part I: Field theoretic calculations of the Entanglement Negativity
- 4 Part II: Holographic calculations of the Entanglement Negativity
- 5 Conclusions and Future Directions

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# Introduction and Motivation

## Entanglement in Quantum Field Theories

### Conformal Field Theories (CFTs)

- ▶ Highly constrained from symmetries
- ▶ Critical phenomena in phase transitions

### $T\bar{T}$ -Deformed CFTs

- ▶ Irrelevant deformation with exactly solvable spectrum
- ▶ Aid from CFT machinery
- ▶ Generalization of AdS/CFT Correspondence

Objective: To understand the mixed state entanglement structure of quantum states in  $T\bar{T}$ -Deformed CFTs

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# Overview of entanglement measures

- Consider a bipartite system  $A \cup B$  in a pure state specified by the density matrix  $\rho$ .
- **Entanglement entropy (EE)**: von Neumann entropy of the reduced density matrix  $\rho_A = \text{Tr}_B(\rho)$ ,

$$S(A) = -\text{Tr} \rho_A \ln \rho_A.$$

- **EE** is a good entanglement measure only for pure quantum states.
- Now, consider a bipartite mixed state  $\rho_A$ , where  $A = A_1 \cup A_2$ .
- **EE** fails to characterize the entanglement between  $A_1$  and  $A_2$  as it mixes the classical and quantum correlations [[Calabrese and Cardy 09'](#)].

- The **positive partial transpose criteria** for the separability of mixed states: if a bipartite mixed state is separable, then the partial transpose of its density matrix ( $\rho^{PT}$ ) is non-negative.
- **Entanglement negativity (EN)**: The logarithmic entanglement negativity characterizes the entanglement in the mixed state  $\rho_A$  [Vidal and Werner 02'],

$$\mathcal{E} \equiv \ln \text{Tr} |\rho_A^{T_2}|. \quad (1)$$

- ◊ Here the partial transpose of  $\rho_A$  with respect to  $A_2$  is given by,

$$\langle e_i^{(1)} e_j^{(2)} | \rho_A^{T_2} | e_m^{(1)} e_n^{(2)} \rangle = \langle e_i^{(1)} e_n^{(2)} | \rho_A | e_m^{(1)} e_j^{(2)} \rangle.$$

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*Part I: Field theoretic calculations of the  
Entanglement Negativity*

# Entanglement Negativity in $\text{CFT}_{1+1}$ : Replica technique

- The EN is difficult to compute for systems with infinite degrees of freedom such as conformal field theories.
- Using the path integral formulation of density matrices, a *replica technique* was developed to calculate the entanglement measures such as EE and EN [Calabrese and Cardy 09', Calabrese et al. 12'].

## Methodology

- Consider the ground state of a CFT with a central charge  $c$ , defined on a two-dimensional complex plane  $\mathcal{N}$  with coordinates  $(x, \tau)$ .
- We wish to compute the EN for the mixed state  $\rho_A$  such that  $A_1 = [u_1, v_1]$ ,  $A_2 = [u_2, v_2]$  are two disjoint spatial intervals and  $B = A^c$ , as depicted in the figure below,

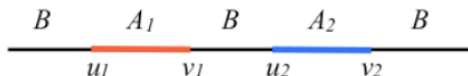


Figure: Two disjoint spatial intervals  $A_1, A_2$  (figure taken from [Calabrese et al. 12']).

- Take  $n$  identical copies of the system and  $\text{Tr}(\rho_A^{T_2})^n$  is given by,

$$\text{Tr}(\rho_A^{T_2})^n = \frac{Z[\mathcal{N}_n]}{(Z[\mathcal{N}])^n}, \quad (2)$$

where  $Z[\mathcal{N}_n]$  and  $Z[\mathcal{N}]$  are the partition functions on the replica manifold and the original manifold respectively.



Figure: Path integral representation of  $\text{Tr}(\rho_A^{T_2})^n$  for  $n = 3$ . [Calabrese et al. 12']

- Finally, the partition function in eq. (2) may be recasted into the correlation function of primary operators on a complex plane such that [Calabrese et al. 12’],

$$\text{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(u_1) \bar{\mathcal{T}}_n(v_1) \bar{\mathcal{T}}_n(u_2) \mathcal{T}_n(v_2) \rangle_{\mathbb{C}}. \quad (3)$$

- ◊ Here  $\mathcal{T}_n$  and  $\bar{\mathcal{T}}_n$  are the twist and anti-twist operators placed on the end points of the intervals  $A_1$  and  $A_2$  with their conformal dimensions given by [Calabrese and Cardy 04’, Calabrese and Cardy 09’],

$$h_{\mathcal{T}_n} = h_{\bar{\mathcal{T}}_n} = \frac{c}{12} \left( n - \frac{1}{n} \right). \quad (4)$$

- The EN may be determined as,

$$\mathcal{E} = \lim_{n \rightarrow 1} \ln \text{Tr}(\rho_A^{T_2})^n, \quad \forall n \in 2\mathbb{Z}. \quad (5)$$

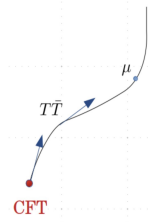
# $T\bar{T}$ -Deformed $CFT_2$ s

- Describes an *irrelevant* deformation of a 2D CFT by a double trace operator comprised of stress energy tensors.
- The action of a  $T\bar{T}$ -Deformed  $CFT_2$  is described by a flow equation with a deformation parameter  $\mu$  [Zamolodchikov 04']:

$$\frac{\partial S_{\text{QFT}}^{(\mu)}}{\partial \mu} = \int d^2w (T_{ww}T_{\bar{w}\bar{w}} - T_{w\bar{w}}^2)_\mu, \quad S_{\text{QFT}}^{(\mu)} \Big|_{\mu=0} = S_{\text{CFT}}.$$

- Perturbatively, for a small deformation parameter  $\mu$

$$S_{\text{QFT}}^{(\mu)} = S_{\text{CFT}} + \mu \int d^2w (T_{ww}T_{\bar{w}\bar{w}} - T_{w\bar{w}}^2)_{\mu=0}.$$



**Figure:** RG flow of a CFT from a fixed point because of the  $T\bar{T}$  operator.

# Entanglement Negativity in $T\bar{T}$ -Deformed $CFT_2$ s

- Consider a  $T\bar{T}$ -deformed  $CFT_2$  in an excited state at a finite temperature  $1/\beta$  living on a cylindrical manifold  $\mathcal{M}$ .
- We setup the coordinates on  $\mathcal{M}$  as  $w = x + i\tau$  and  $\bar{w} = x - i\tau$ , where  $x \in (-\infty, \infty)$  and  $\tau \in [0, \beta]$  with  $\tau \sim \tau + \beta$ .
- Applying the replica technique, the partition function of the deformed theory (on the Riemann surface  $\mathcal{M}_n$ ) in the path integral representation:

$$Z[\mathcal{M}_n] = \int_{\mathcal{M}_n} \mathcal{D}\phi e^{-S_{\text{QFT}}^{(\mu)}[\phi]}. \quad (6)$$

- Using eq. (2) and the action of  $T\bar{T}$ -deformed CFT, we obtained the EN between two spatial intervals  $A$  and  $B$  as,

$$\mathcal{E}^{(\mu)}(A : B) = \mathcal{E}_{\text{CFT}}(A : B) + \lim_{n \rightarrow 1} \log \left[ \frac{(1 - \mu \int_{\mathcal{M}_n} \langle T\bar{T} \rangle_{\mathcal{M}_n})}{(1 - \mu \int_{\mathcal{M}} \langle T\bar{T} \rangle_{\mathcal{M}})^n} \right]. \quad (7)$$

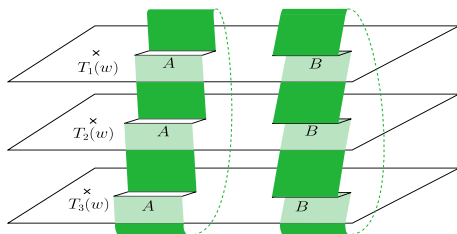
- Upto the first order of the deformation parameter  $\mu$  [LC, Basu, Paul 23'],

$$\mathcal{E}^{(\mu)}(A : B) = \mathcal{E}_{\text{CFT}}(A : B) - \mu \lim_{n \rightarrow 1} \left[ \int_{\mathcal{M}_n} \langle T\bar{T} \rangle_{\mathcal{M}_n} - n \int_{\mathcal{M}} \langle T\bar{T} \rangle_{\mathcal{M}} \right]. \quad (8)$$

# Application to various configurations

- To calculate the EN for two disjoint spatial intervals  $A = [x_1, x_2]$  and  $B = [x_3, x_4]$  in a thermal  $T\bar{T}$ -deformed  $CFT_2$ , essentially we need to solve for  $\langle T\bar{T} \rangle_{\mathcal{M}_n}$ .

$$\int_{\mathcal{M}_n} \langle T\bar{T} \rangle_{\mathcal{M}_n} = \int_{\mathcal{M}} \frac{1}{n} \frac{\langle T^{(n)}(w)\bar{T}^{(n)}(\bar{w})\mathcal{T}_n(w_1, \bar{w}_1)\bar{\mathcal{T}}_n(w_2, \bar{w}_2)\bar{\mathcal{T}}_n(w_3, \bar{w}_3)\mathcal{T}_n(w_4, \bar{w}_4) \rangle_{\mathcal{M}}}{\langle \mathcal{T}_n(w_1, \bar{w}_1)\bar{\mathcal{T}}_n(w_2, \bar{w}_2)\bar{\mathcal{T}}_n(w_3, \bar{w}_3)\mathcal{T}_n(w_4, \bar{w}_4) \rangle_{\mathcal{M}}}.$$



**Figure:** A schematic of the Riemann surface ( $n = 3$ ) for two disjoint intervals in a  $T\bar{T}$ -deformed  $CFT_2$ .

- Next we utilize the Ward identities for the stress energy tensor and the large central charge analysis of four point functions to obtain the first order correction in the EN:

$$\delta\mathcal{E}(A : B) = -\frac{\mu c^2 \pi^4}{12\beta^3} \left[ x_{31} \coth \frac{\pi x_{31}}{\beta} + x_{42} \coth \frac{\pi x_{42}}{\beta} - x_{41} \coth \frac{\pi x_{41}}{\beta} - x_{32} \coth \frac{\pi x_{32}}{\beta} \right] + \delta\mathcal{E}_{\text{cross}} .$$

- Utilizing our general formula for the EN, we also determine the EN for the case of two adjacent intervals and a single interval.
- For two adjacent intervals  $A = [x_1, x_2]$  and  $B = [x_2, x_3]$ ,

$$\delta\mathcal{E}(A : B) = -\frac{\mu c^2 \pi^4}{12\beta^3} \left[ x_{21} \coth \frac{\pi x_{21}}{\beta} + x_{32} \coth \frac{\pi x_{32}}{\beta} - x_{31} \coth \frac{\pi x_{31}}{\beta} \right] + \delta\mathcal{E}_{\text{cross}} .$$

- For a single interval of length  $\ell$ ,

$$\delta\mathcal{E}(A : A^c) = -\frac{\mu \pi^4 c^2 \ell}{6\beta^3} \left[ -1 + \coth \frac{\pi \ell}{\beta} - e^{-\frac{2\pi \ell}{\beta}} \frac{f' \left( e^{-\frac{2\pi \ell}{\beta}} \right)}{f \left( e^{-\frac{2\pi \ell}{\beta}} \right)} \right] + \delta\mathcal{E}_{\text{cross}} ,$$

here  $f$  is a function of the cross ratios.

- Notice that the usual UV divergence of the EN is still encoded in the CFT result, the first order correction in the EN due to  $T\bar{T}$  deformation is finite.



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*Part II: Holographic calculations of the  
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# Holographic Duality

- The holographic duality or AdS/CFT correspondence conjectures that a theory of gravity in a  $d + 1$  - dimensional Anti de-Sitter spacetime is completely equivalent to a  $d$  - dimensional quantum (conformal) field theory living on the boundary of the AdS spacetime.
- A natural implication of the holographic duality is that the observables on one side have a one to one correspondence with the quantities on the other side.

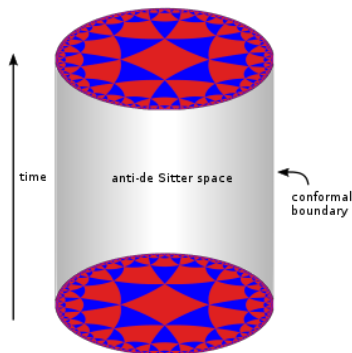
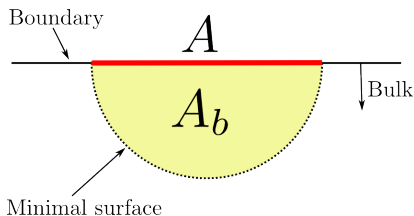


Figure: Holographic spacetime.

# Holographic Entanglement Entropy: A brief overview

- The Ryu-Takayanagi proposal states that the entanglement entropy of a subregion  $A$  in the boundary CFT is given by the area of a static bulk minimal codimension-2 surface  $\tilde{A}$  homologous to  $A$  as [Ryu and Takayanagi 06'],

$$S_A = \frac{1}{4G_N} \text{Area}(\tilde{A}).$$



**Figure:** Holographic prescription for the entanglement entropy (figure taken from [Faulkner et al. 13']).

# Cut-off AdS<sub>3</sub>/T $\bar{T}$ - deformed CFT<sub>2</sub> Proposal and the Holographic Entanglement Entropy

- According to the holographic proposal [Verlinde et al. 18'], a T $\bar{T}$ -Deformed CFT<sub>2</sub> (with  $\mu > 0$ ) is dual to an AdS<sub>3</sub> geometry with a finite radial cutoff

$$r_C = \sqrt{\frac{6R^4}{\pi c\mu}}.$$

- The thermal CFT<sub>2</sub> with T $\bar{T}$ -deformation is dual to a BTZ black hole (of horizon radius  $r_H$ ) in the finite radius bulk geometry, with the metric

$$ds^2 = \frac{r^2 - r_H^2}{R^2} dt^2 + \frac{R^2}{r^2 - r_H^2} dr^2 + r^2 d\tilde{x}^2. \quad (9)$$

- The dual T $\bar{T}$ -deformed CFT<sub>2</sub> is located at the cut-off radius  $r_C$  and hence the metric of the background manifold is conformal to the flat metric as follows

$$ds^2 = dt^2 + \frac{d\tilde{x}^2}{1 - \frac{r_H^2}{r_C^2}} \equiv dt^2 + dx^2, \quad (10)$$

where  $x = \tilde{x} \left(1 - \frac{r_H^2}{r_C^2}\right)^{-1/2}$  is the spatial coordinate in the CFT<sub>2</sub>.

- The authors in [Chen et al. 18', Jeong et al. 19'] showed that the Ryu-Takayanagi formula still holds in the dual finite radius geometry for the holographic EE of bipartite pure states in a  $T\bar{T}$ -deformed  $CFT_2$  at high temperatures.
- The length of the minimal spacelike surface (geodesic) homologous to a subsystem  $A = [x_i, x_j]$  in the deformed  $CFT_2$  at a temperature  $1/\beta$  was computed to be [Chen et al. 18', Jeong et al. 19'],

$$\mathcal{L}_{ij} = R \log \left( \mathcal{A}(x_i, x_j) + \sqrt{\mathcal{A}(x_i, x_j)^2 - 1} \right), \quad (11)$$

where

$$\mathcal{A}(x_i, x_j) \equiv 1 + \frac{2r_C^2}{r_H^2} \sinh^2 \left( \frac{\pi|x_i - x_j|}{\beta} \sqrt{1 - \frac{r_H^2}{r_C^2}} \right). \quad (12)$$

- The holographic EE for various bipartite states may be calculated using the above expression for geodesic lengths.

# Holographic Entanglement Negativity in $T\bar{T}$ -Deformed $CFT_2$ s

- The holographic construction for the EN of two disjoint intervals  $A$  and  $B$  in a  $CFT_2$  concerns an algebraic sum of the lengths of bulk minimal spacelike geodesics homologous to various combination of subsystems [Malvimat et al. 18', D. Basu et al. 20'],

$$\mathcal{E}(A : B) = \frac{3}{16G_N} (\mathcal{L}_{AUC} + \mathcal{L}_{BUC} - \mathcal{L}_C - \mathcal{L}_{AUBUC}), \quad (13)$$

where  $C$  is another interval sandwiched between  $A$  and  $B$ .

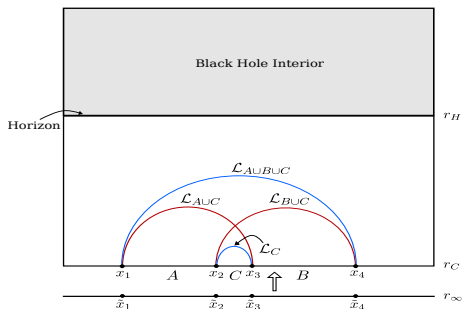


Figure: Ryu-Takayanagi surfaces for two disjoint intervals in a finite radius bulk geometry.

- Utilizing the above construction for the holographic entanglement negativity, we obtain the EN for two disjoint intervals  $A$  and  $B$  as [LC, Basu, Paul 23'],

$$\mathcal{E}^{(\mu)}(A : B) = \frac{3R}{16G_N} \log \left[ \frac{(\mathcal{A}(x_1, x_3) + \sqrt{\mathcal{A}(x_1, x_3)^2 - 1}) (\mathcal{A}(x_2, x_4) + \sqrt{\mathcal{A}(x_2, x_4)^2 - 1})}{(\mathcal{A}(x_2, x_3) + \sqrt{\mathcal{A}(x_2, x_3)^2 - 1}) (\mathcal{A}(x_1, x_4) + \sqrt{\mathcal{A}(x_1, x_4)^2 - 1})} \right]. \quad (14)$$

- Upon solving, it precisely matches with the field theoretical result.
- In a similar manner, we computed the holographic entanglement negativity for the configuration of a single interval and two adjacent intervals and found an agreement from the field theory side [LC, Basu, Paul 23'].



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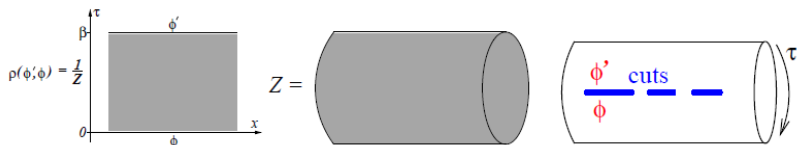
# Conclusions and Future Directions

- We constructed a general formula for the EN of arbitrary bipartite mixed states in  $T\bar{T}$ -deformed  $CFT_2$ , up to first order of the deformation parameter  $\mu$ .
- Our EN results perfectly matches with certain combinations of geodesic lengths in finite radius  $AdS_3$  BTZ black hole background verifying the Cut-off  $AdS_3/T\bar{T}$ -deformed  $CFT_2$  proposal of [Verlinde et al. 18’].
- **Future directions:**
  - ▶ It will be interesting to apply our work in studying near critical phenomena relevant to phase transitions in corresponding condensed matter systems.
  - ▶ Other integrable deformations of CFTs may be explored, possibly leading to the construction of insightful dual gravitational theories and hence, finding more examples of the holographic principle.

*Thank You!*

# Backup

- For a general density operator ( $\rho$ ) in a basis of quantum states  $\{|a\rangle\}$ ,  
 $\rho = \sum_a p_a |a\rangle \langle a|$ , where  $p_a$  is the probability corresponding to the state  $|a\rangle$ .  
If  $\rho^2 = \rho$ ,  $\text{Tr}(\rho^2) = 1 \implies$  system is in a pure state.  
If  $\rho^2 \neq \rho$ ,  $\text{Tr}(\rho^2) < 1 \implies$  system is in a mixed state.
- Path integral representation of density matrices:



**Figure 1:** First, representation of a density matrix  $\rho$ . Second, the partition function  $Z = \text{Tr}(\rho)$ . Finally, the reduced density matrix  $\rho_A$  where  $A = [u_1, v_1] \cup [u_1, v_1] \cup \dots$  (figure taken from [Calabrese and Cardy 09']).