Nonlinear Klein-Gordon equation with cubic-quintic power nonlinearities

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Introduction

Boson stars

Localized complex scalar field configurations with a finitie energy, bounded by gravity.
Simplest example: (3+1)–dimensional massive Einstein-Klein-Gordon theory with a mass term and without self-interaction.

Q-balls

Arise as a flat spacetime limit of the boson star configuration. They exist only within a restricted interval of values of the angular frequency ω.

Toy model

We consider massive Klein-Gordon equation for a complex scalar field in 3+1 dimensions

$$\phi_{tt} = \Delta\phi - \phi + |\phi|^2\phi - \alpha|\phi|^4\phi, \quad \phi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^3. \qquad (1)$$

We assume $\alpha > 0$.

$$E = \int \left(\frac{1}{2}|\phi_t|^2 + \frac{1}{2}|\Delta\phi^2| + \frac{1}{2}|\phi|^2 - \frac{1}{4}|\phi|^4 + \frac{\alpha}{6}\phi|^6 \right) d^3x.$$

$$Q = \mathfrak{Im} \int \phi\bar{\phi}_t \, d^3x.$$

U(1) global gauge symmetry $\phi \longrightarrow e^{i\vartheta}\phi$

Standing wave solutions

We consider $\phi(x,t) = e^{i\omega t} f(r)$, $\omega \in (0,1)$

$$f'' + \frac{2}{r}f' - (1-\omega^2)f + f^3 - \alpha f^5 = 0, \quad 0 \leq \alpha \leq \frac{3}{16}.$$

$$E = \int \left(\frac{1}{2}f'^2 + \frac{1}{2}(1-\omega^2)f^2 - \frac{1}{4}f^4 + \frac{\alpha}{6}f^6 \right) r^2\, dr,$$

$$Q = \omega \int f^2 r^2\, dr.$$

Let us consider a rescaling $P(r) = \alpha^{1/2} f(\alpha^{1/2} r)$ of the equation

$$f'' + \frac{2}{r} f' - (1 - \omega^2) f + f^3 - \alpha f^5 = 0.$$

Equivalent equation

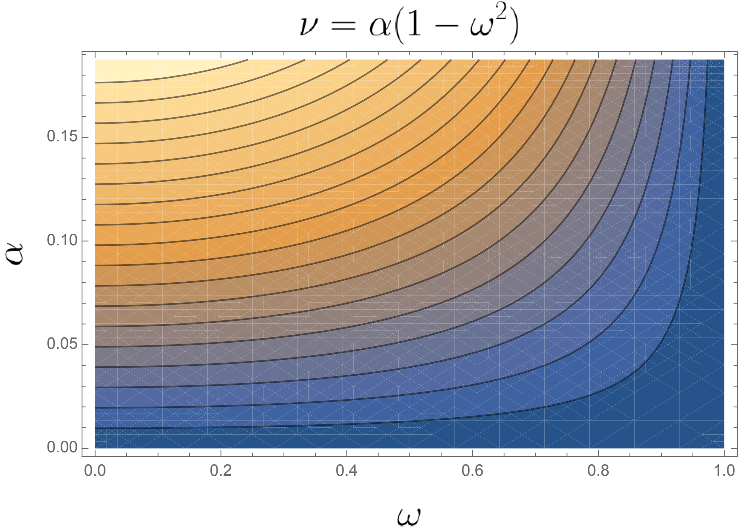
Solutions to the equation (1) are rescaled solutions to the equation

$$\Delta P - \nu\, P + P^3 - P^5 = 0. \tag{2}$$

where

$$\nu = \alpha(1 - \omega^2), \quad f_{\omega,\alpha}(r) = \alpha^{-1/2} P_\nu(\alpha^{-1/2} r).$$

Space of solutions with equivalence classes



Spectral stability analysis

We linearize around a standing wave solutions

$$\phi(t,x) = e^{i\omega t}\left(f(x) + v(t,x)\right),$$

where $v : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a complex perturbation of the solution.

Ignoring all the $\mathcal{O}(v^2)$ terms we arrive at

$$v_{tt} + 2i\omega v_t + (1-\omega^2)v - \Delta v + (f^2 - \alpha f^4)v + (2f^2 - 4\alpha f^4)\Re v = 0,$$

We decompose v into its real and imaginary part

$$\mathbf{v} = (\Re v,\, \Im v).$$

Linearization of the equation

$$\mathbf{v}_{tt} + 2\omega J \mathbf{v}_t + \mathcal{H}\mathbf{v} = 0,$$

$$J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix},$$

Operators L_+ and L_- are given by

$$L_+ = -\Delta + (1-\omega^2) - 3f^2 + 5\alpha f^4,$$
$$L_- = -\Delta + (1-\omega^2) - f^2 + \alpha f^4.$$

Hamiltonian operator

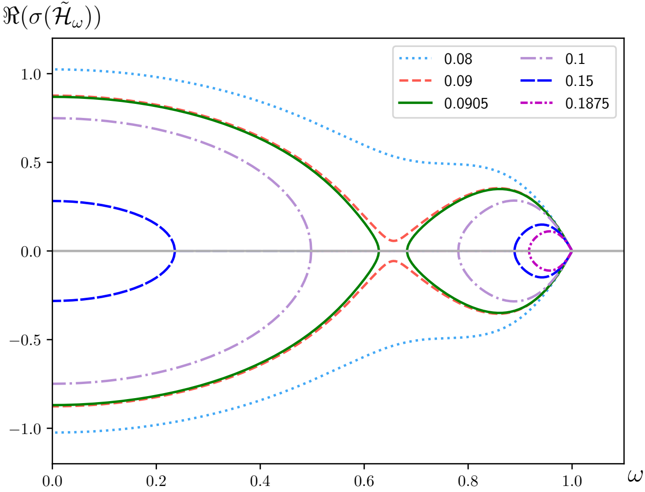
The Hamiltonian operator on a phase space takes the form

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathcal{H} & -2\omega J \end{pmatrix}.$$

Spectral stability

The system is spectrally stable, if the spectrum of $\tilde{H}$ lies in the closed left half-space

$$\sigma(\tilde{H}) \subseteq \{z : \Re z \leq 0\}.$$


$\Re(\sigma(\tilde{\mathcal{H}}_\omega))$
1.0
0.5
0.0
−0.5
−1.0
0.0
0.2
0.4
0.6
0.8
1.0
ω
0.08
0.09
0.0905
0.1
0.15
0.1875

Vakhitov-Kolokolov stability criterion [3]

Let $\omega \in (-1, 1)$ and assume that the equation (1) has a possitive smooth solution $f_\omega(|x|)$ in both x and ω variables, such that

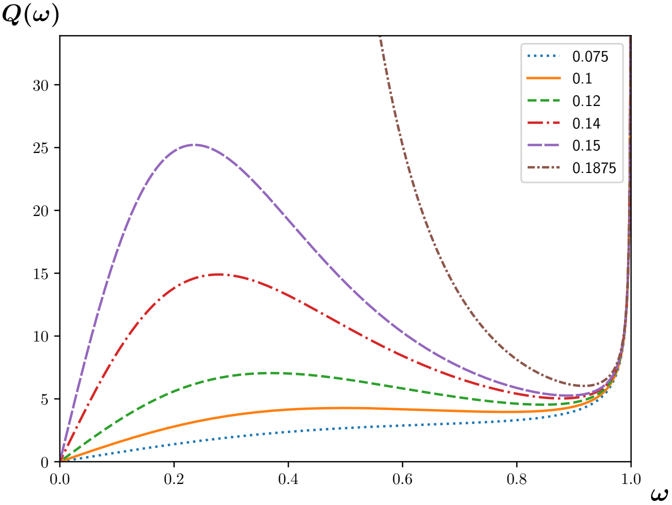
i) $\lim_{r\to\infty} f_\omega(r) = 0$,

ii)
$$n(L_+) = \#\{\lambda \in \sigma(L_+) : \lambda < 0\} = 1,$$

iii) $\ker[L_-] = \mathrm{span}[f_\omega]$.

Then the wave is spectrally stable if and only if

$$\frac{d}{d\omega} Q_\omega \leq 0.$$


$Q(\omega)$
0.075
0.1
0.12
0.14
0.15
0.1875
0
5
10
15
20
25
30
0.0
0.2
0.4
0.6
0.8
1.0
ω

A limit $\nu \to 0$

When $\nu \to 0$, then $\alpha \to 0$ or $\omega \to 1$. Let us note, that taking $f(r) = \beta u(\beta r)$, $\beta = \sqrt{1-\omega^2}$ in equation (1), we get

$$u'' + \frac{2}{r}u' - u + u^3 - \alpha(1-\omega^2)u^5 = 0.$$

Perturbative expansion

Let g be a unique nonnegative radially symmetric solution to

$$-\Delta g + g - g^3 = 0.$$

Then the linearized operator $L : h \mapsto -\Delta h + h - 3g^2 h$ is an isomorphism from H^1_{rad} onto H^{-1}_{rad}. Thus, by the implicit function theorem there exists an expansion

$$u(x) = g(x) - \alpha(1-\omega^2)[L^{-1}g^5](x) + \mathcal{O}(\alpha^2).$$

Charge expansion near $\alpha = 0$ or $\omega = 1$

We have

$$Q(f_\omega) = \frac{\omega}{\sqrt{1-\omega^2}} ||u||_{L^2}^2.$$

Using $L(g + x \cdot \nabla g) = -2g$, we get

$$Q(f_\omega) = \frac{\omega}{\sqrt{1-\omega^2}} ||g||_{L^2}^2 + \frac{1}{2}\alpha\omega\sqrt{1-\omega^2}||g||_{L^6}^6 + \mathcal{O}(\alpha^2).$$

Looking for the inflection point on the curve (ω, Q_ω) we solve the equations

$$\frac{d}{d\omega} Q_\omega = 0, \quad \frac{d^2}{d\omega^2} Q_\omega = 0,$$

for ω and α, which gives us

$$\alpha = \frac{16||g||_{L^2}^2}{||g||_{L^6}^6}, \quad \omega = \sqrt{3}/2. \quad (\alpha_*, \omega_*) = (0.0902, 0.657).$$

A limit $\nu \to \frac{3}{16}$

When $\nu = \alpha(1-\omega^2) \to \frac{3}{16}$, the solutions become more and more step-like and the support of the solutions grows. One can take advantage of asymptotics $Q(\omega)$ provided in [1]

$$\int_{\mathbb{R}^3} |P_\nu|^2 \sim (\frac{3}{16} - \nu)^{-3}, \text{as} \quad \nu \to \frac{3}{16}.$$

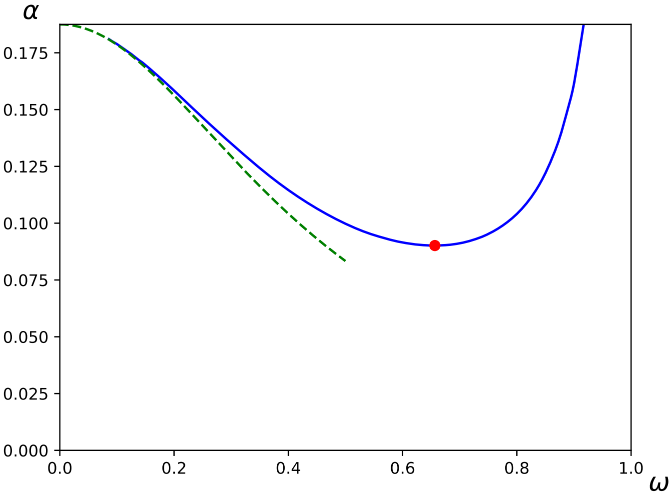
Using the equivalence relationship we can write

$$Q(\omega) \sim \omega \int_{\mathbb{R}^3} \alpha^{-1} \, |P_\nu(\alpha^{-1/2} r)|^2 \, r^2 \, dr$$

Performing the coordinate change and solving the equation $Q'(\omega) = 0$ we obtain the following relation

$$\alpha = \frac{3}{16(5\omega^2 + 1)}.$$

Stability island of the stationary solutions



Thank you for your attention

- Killip R et al. (2017) Solitons and Scattering for the Cubic–Quintic Nonlinear Schrödinger Equation on $\mathbb{R}^3$ *Arch Rational Mech Anal* 225.1 pp. 469–548.
- Killip R, Murphy J and Visan M (2021) Scattering for the Cubic-Quintic NLS: Crossing the Virial Threshold *SIAM J. Math. Anal.* 53.5 pp. 5803–5812
- Demirkaya A et al. (2014) Spectral Stability Analysis for Standing Waves of a Perturbed Klein-Gordon Equation