Dynamics of conformal cubic scalar field in asymptotically-AdS black hole spacetimes

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Model

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega^2$$

$$V(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2} \xrightarrow{\ell = 1} V(r) = 1 + r^2 - \frac{r_H}{r}\left(1 + r_H^2\right)$$

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$$\Box_g \phi + 2\phi - \lambda \phi^3 = 0$$

conformal equation: permits Robin BC

$$\lim_{r\to\infty} \left(r^2 \partial_r (r\phi) + b(r\phi)\right) = 0$$

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focusing case $\lambda = -1$

defocusing case $\lambda = 1$

- spherical symmetry
- 2 parameter model (y_H, b)

Setup

null coordinate

$$\mathrm{d}v = \mathrm{d}t + \frac{\mathrm{d}r}{V}$$

compactification

$$y = \frac{1}{r} \qquad \Phi(y) = r\,\phi(r)$$

$$V(y) = 1 + \frac{1}{y^2} - \frac{y}{y_H}\left(1 + \frac{1}{y_H^2}\right)$$

where

$$y_H = \frac{1}{r_H}$$

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$$2\partial_y\partial_v\Phi - \partial_y\left(y^2\,V(y)\,\partial_y\Phi\right) - \left(y\,V'(y) + \frac{2}{y^2}\right)\Phi + \lambda\Phi^3 = 0$$

Robin BC

$$\left(-\partial_v\Phi + \partial_y\Phi - b\Phi\right)\big|_{y=0} = 0$$

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$$\left(-\partial_v\Phi + \partial_y\Phi - b\Phi\right)\Big|_{y=0} = 0$$

energy “loss” formula

$$\frac{\mathrm{d}E}{\mathrm{d}v} = -\left(\partial_v\Phi\right)^2\Big|_{y=y_H} - \frac{b}{2}\,\partial_v\left(\Phi^2\right)\Big|_{y=0}$$

Linear equation

$$2\partial_y\partial_v\Phi - \partial_y\left(y^2\,V(y)\,\partial_y\Phi\right) - \left(y\,V'(y) + \frac{2}{y^2}\right)\Phi = 0$$

$$\Phi(v,y) = e^{i\omega v}\psi(y)$$

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$$\xi = 3 + 2y_H^2 + i\sqrt{3+4y_H^2}$$

$$\psi(x) = C\,\mathrm{HeunG}\left(\frac{\bar{\xi}}{\xi}, \frac{2(1+y_H^2)}{\xi}, 1, 1, 1 + \frac{2i\,y_H\,\omega}{3+y_H^2}, 1 + \frac{\bar{\xi}\,y_H\,\omega}{(3+y_H^2)\sqrt{3+4y_H^2}}, \frac{2(y_H-y)(1+y_H^2)}{y_H\,\xi}\right)$$

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static solution $\omega = 0$

$$b\left(y_H\right) = -\frac{2\left(1+y_H^2\right)\mathrm{HeunG}'\left(\frac{\bar{\xi}}{\xi}, \frac{2(1+y_H^2)}{\xi}, 1, 1, 1, 1, \frac{2(1+y_H^2)}{\xi}\right)}{y_H\,\xi\,\mathrm{HeunG}\left(\frac{\bar{\xi}}{\xi}, \frac{2(1+y_H^2)}{\xi}, 1, 1, 1, 1, \frac{2(1+y_H^2)}{\xi}\right)}$$

Linear equation

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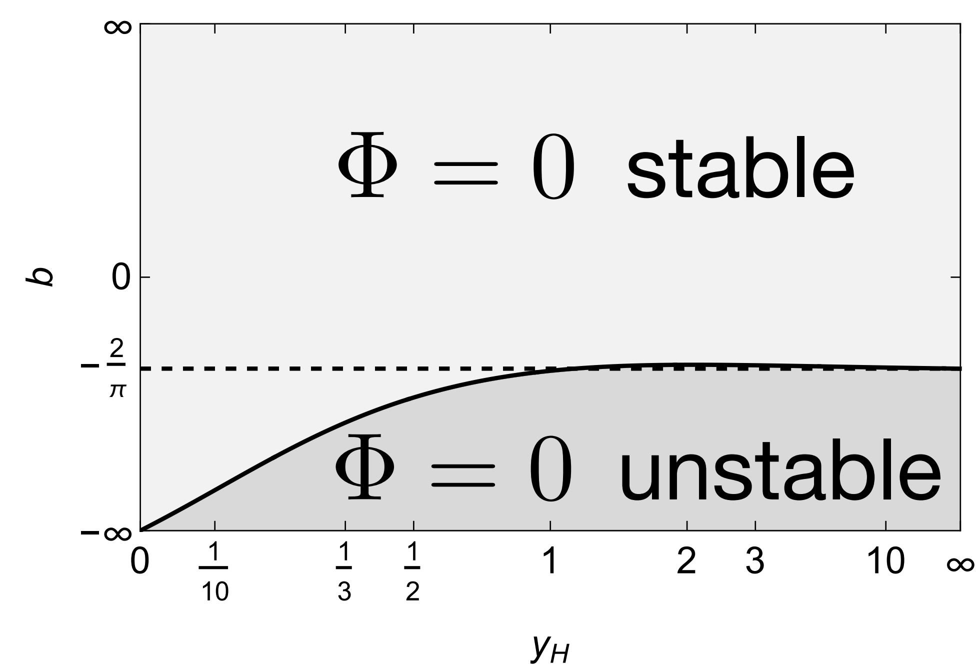
static solution $\omega = 0$

$$\lim_{y_H\to\infty} b\left(y_H\right) = -\frac{2}{\pi}$$

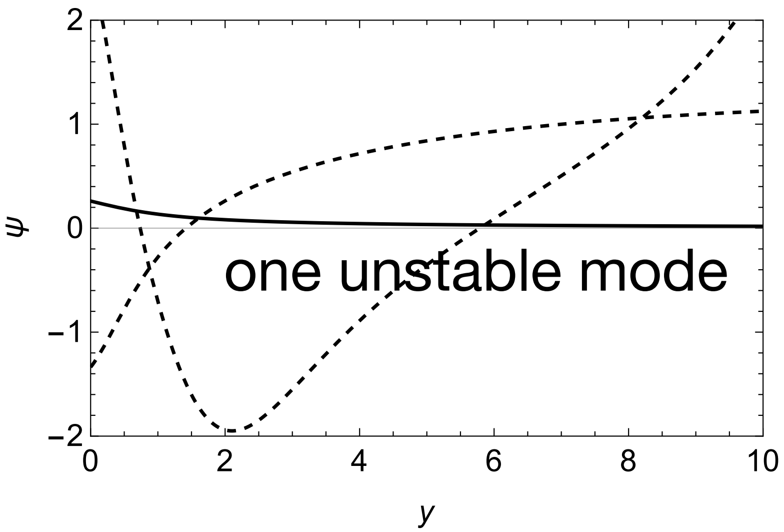


$$b\left(y_H\right) = -\frac{2\left(1+y_H^2\right)\mathrm{HeunG}'\left(\frac{\bar\xi}{\xi}, \frac{2(1+y_H^2)}{\xi}, 1, 1, 1, 1, \frac{2(1+y_H^2)}{\xi}\right)}{y_H\,\xi\,\mathrm{HeunG}\left(\frac{\bar\xi}{\xi}, \frac{2(1+y_H^2)}{\xi}, 1, 1, 1, 1, \frac{2(1+y_H^2)}{\xi}\right)}$$

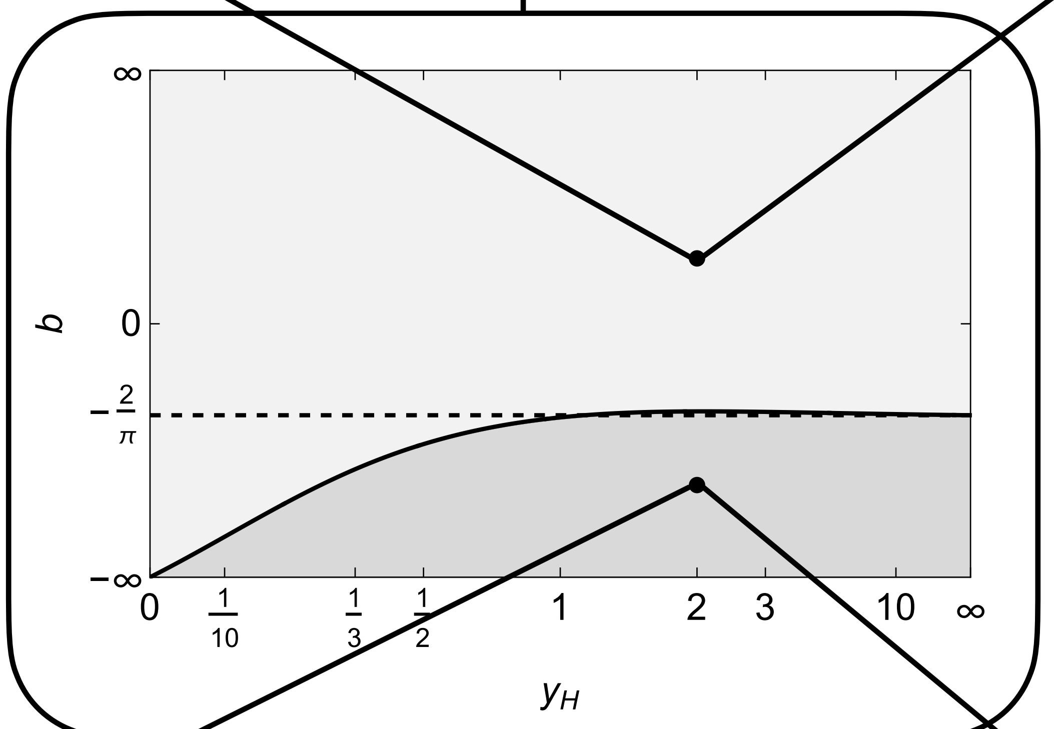
Static solutions

focusing $\lambda = -1$

defocusing $\lambda = 1$

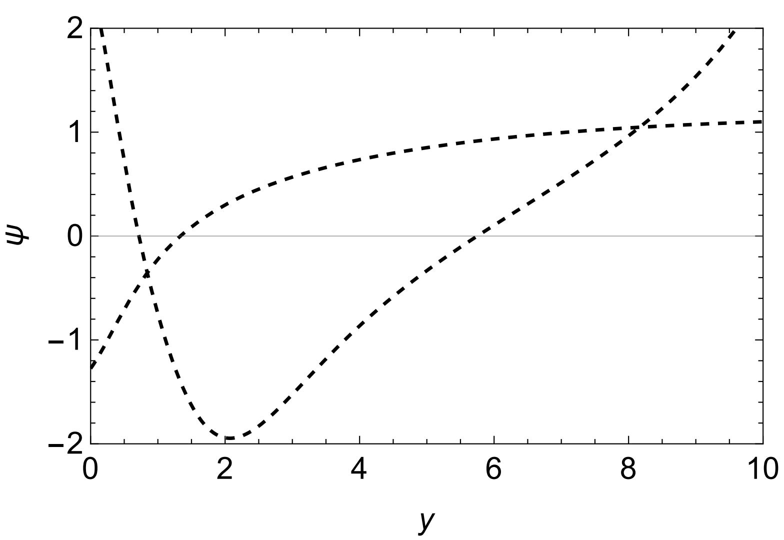


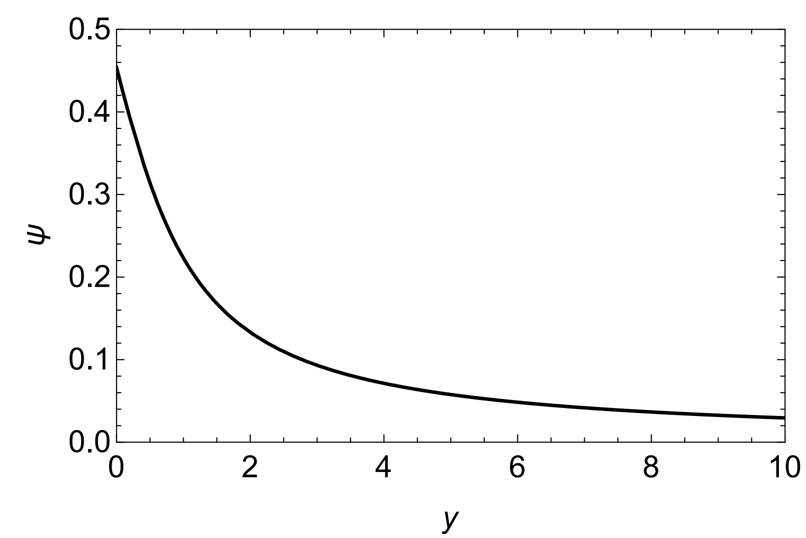
no static solutions



the whole ladder of unstable static solutions

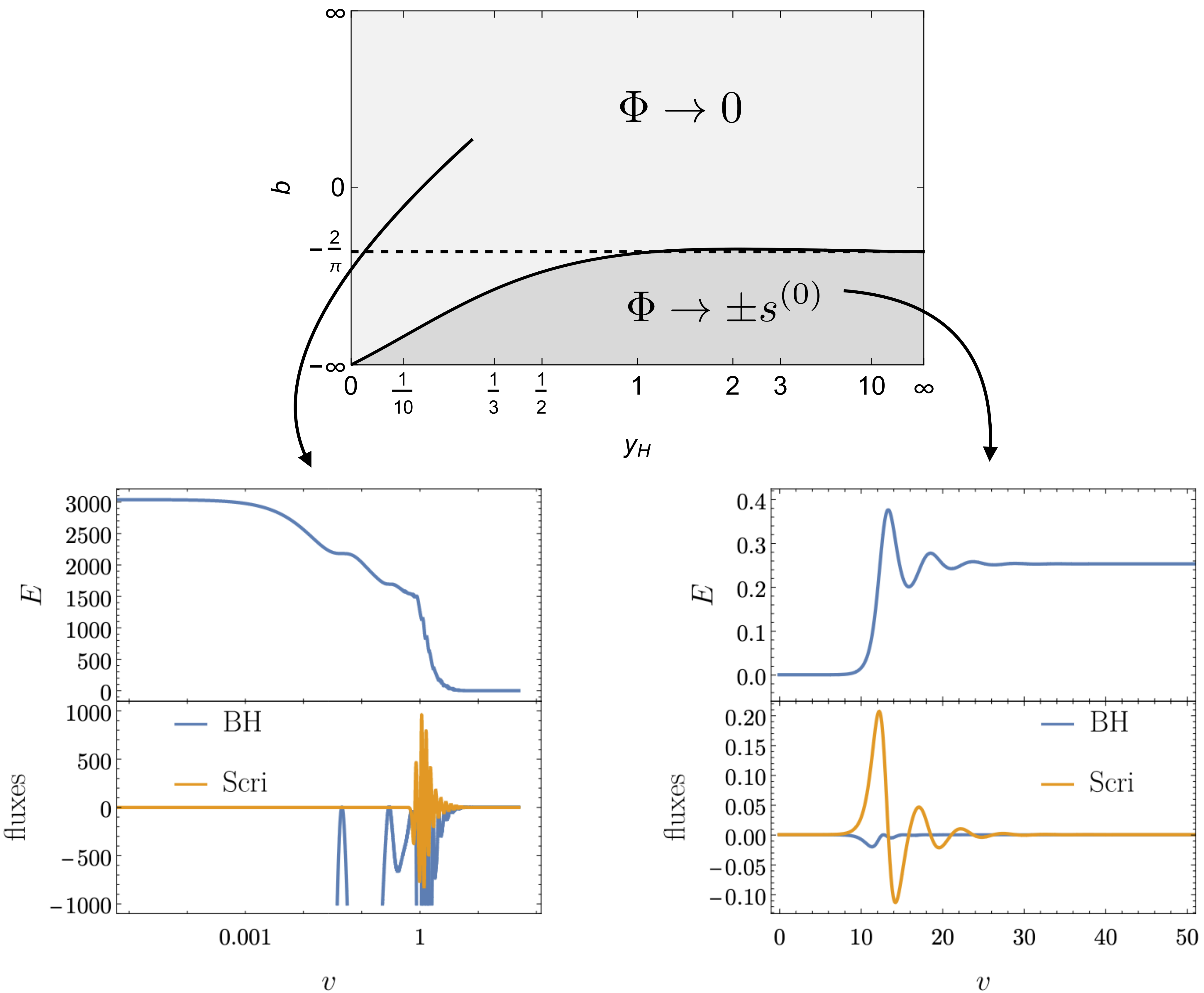
single static solution





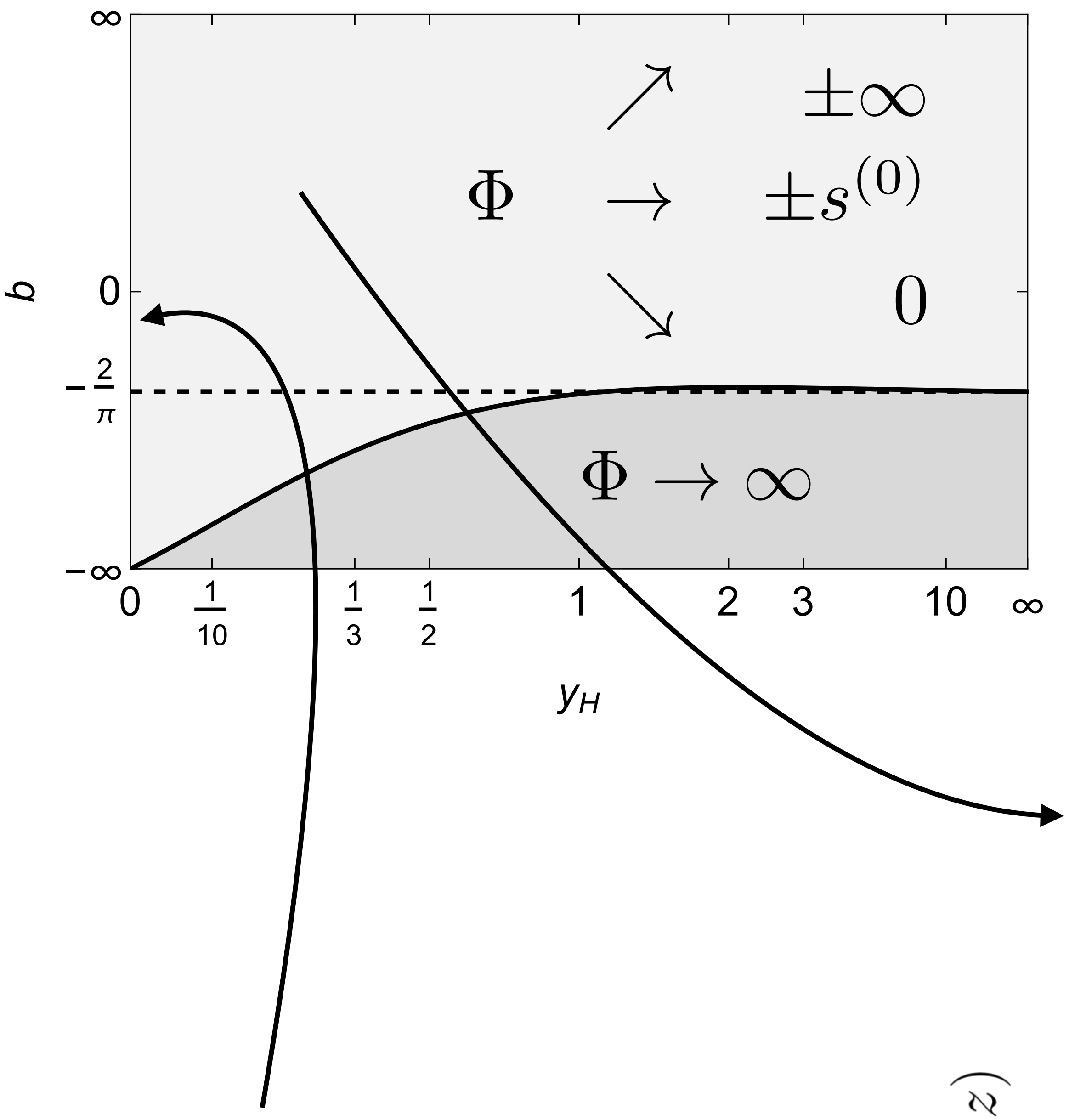
Dynamics for defocusing case

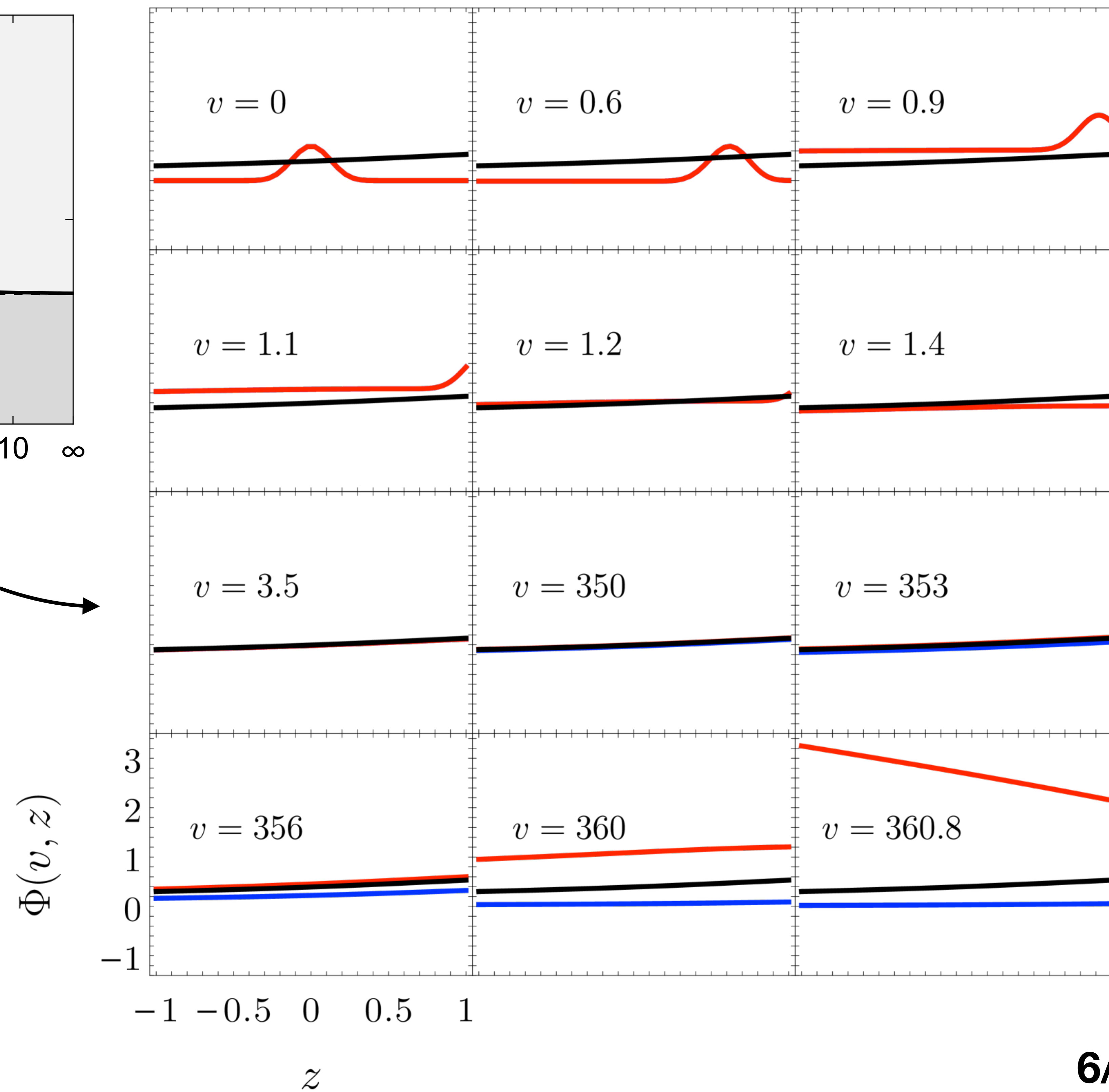
$\lambda = 1$



Dynamics for focusing case

$\lambda = -1$





Large BH limit

Reissner-Nordström-anti-de Sitter

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega^2$$

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}$$

Horizons at r_H and σr_H $\qquad \ell = 1$

$$V(r) = 1 - \frac{(1+\sigma)r_H \left(1 + \left(1+\sigma^2\right) r_H^2\right)}{r} + \frac{\sigma r_H^2 \left(1 + \left(1+\sigma+\sigma^2\right) r_H^2\right)}{r^2} + r^2$$

3 parameter model (y_H, b, σ)

Reissner-Nordström-anti-de Sitter

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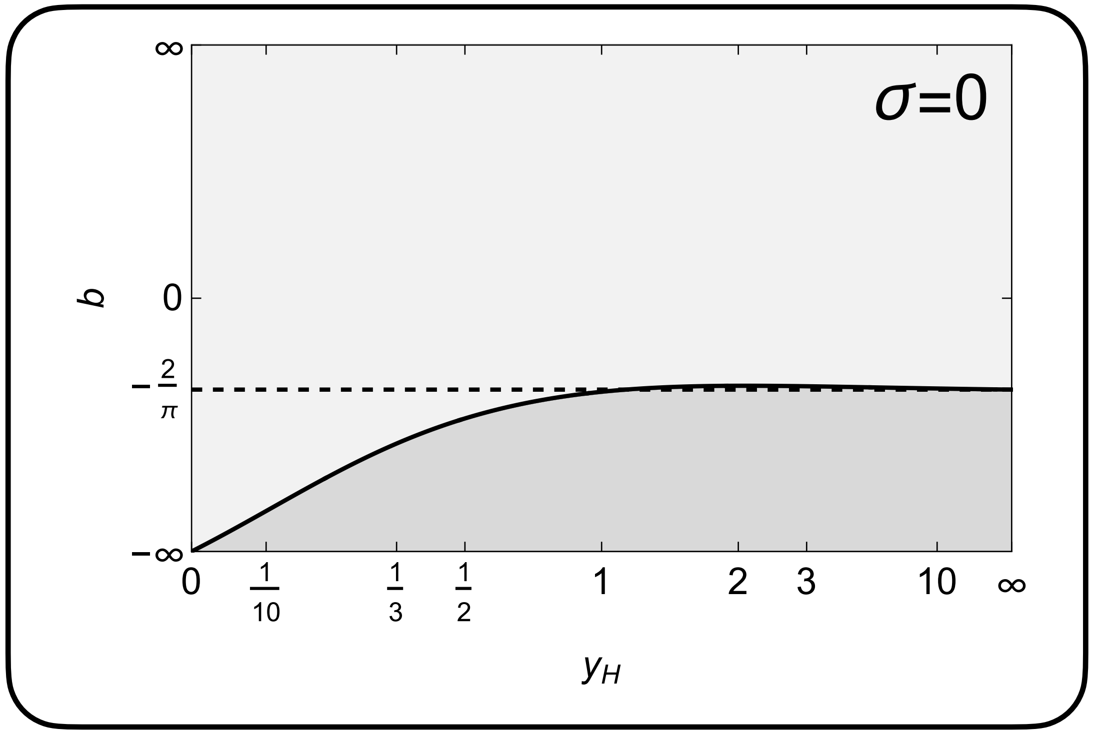
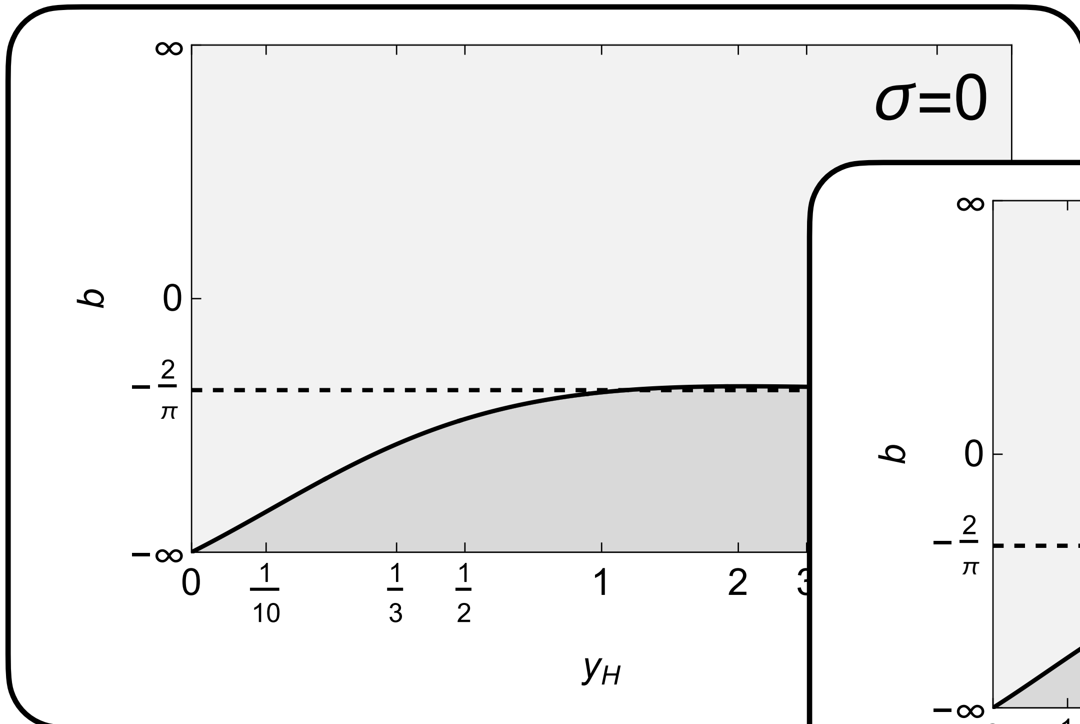
3 parameter model (y_H, b, σ)

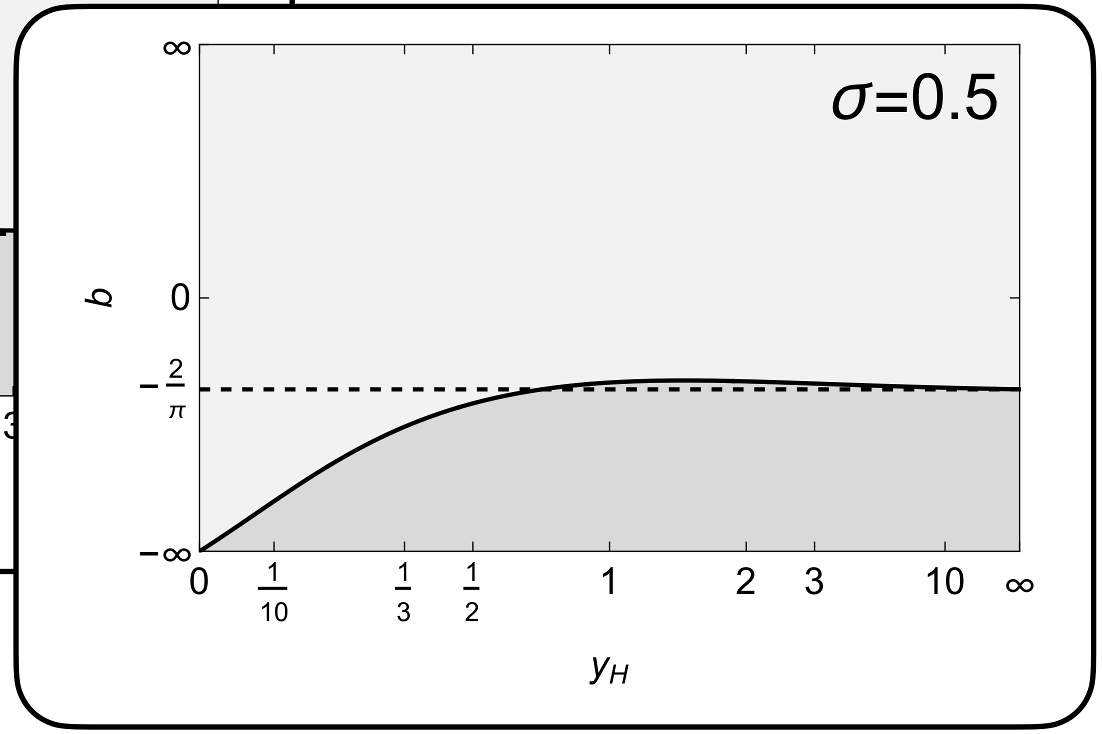
Extremal case $\sigma = 1$

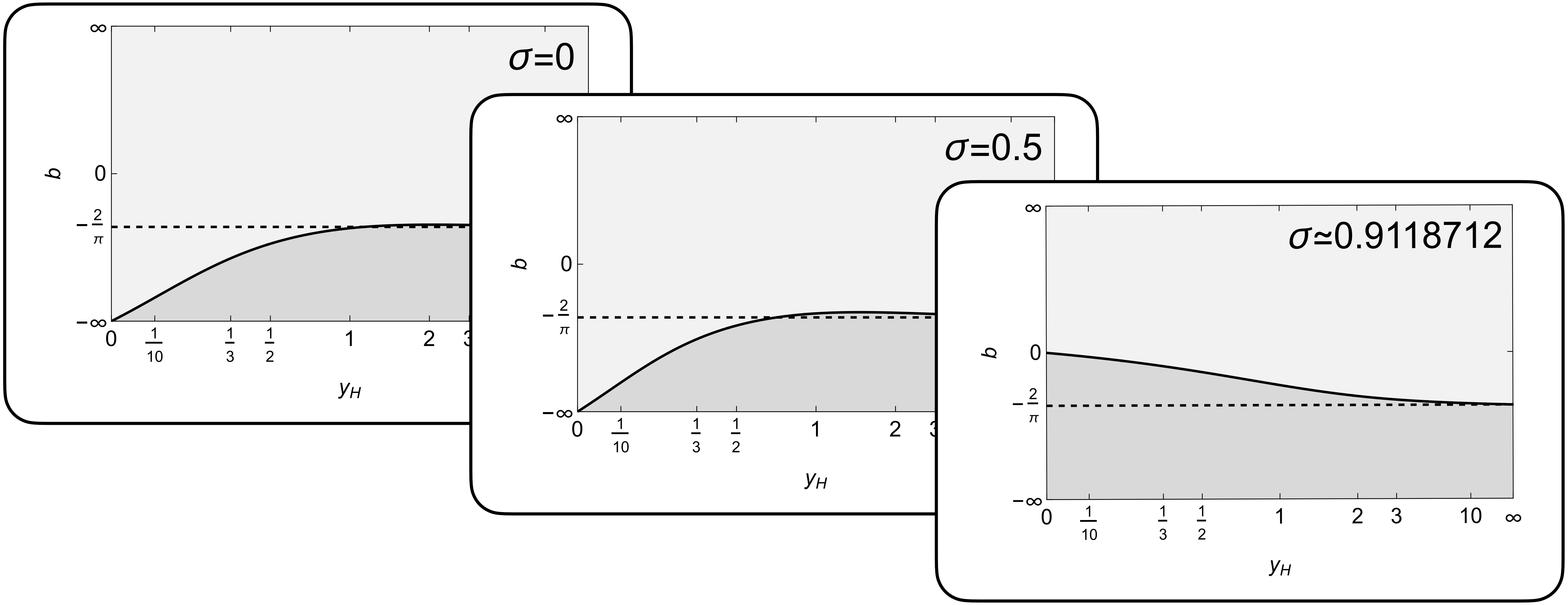
Lemma

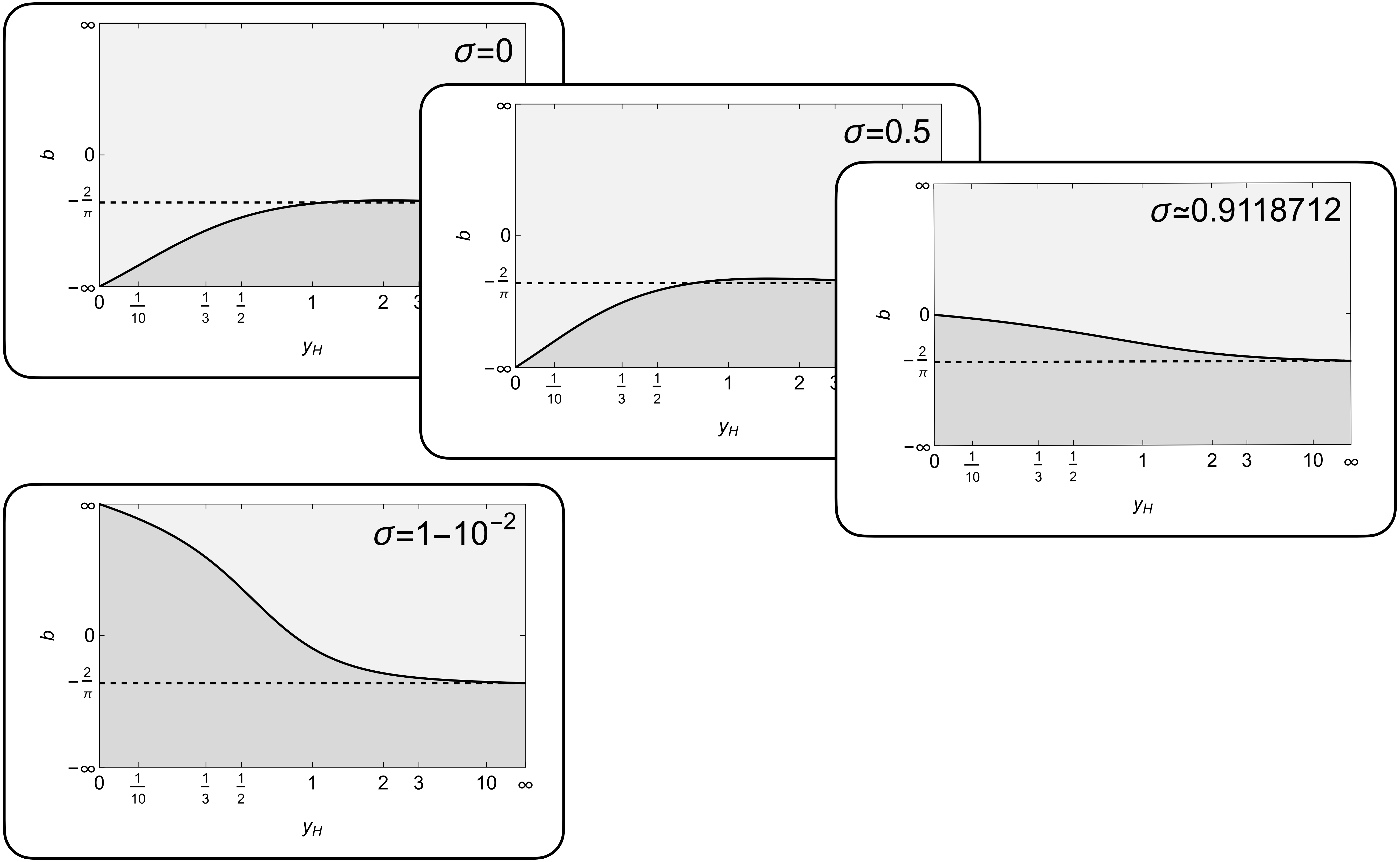
There are no nontrivial regular static solutions for $\lambda = 0$, $\lambda = 1$, and $\lambda = -1$.

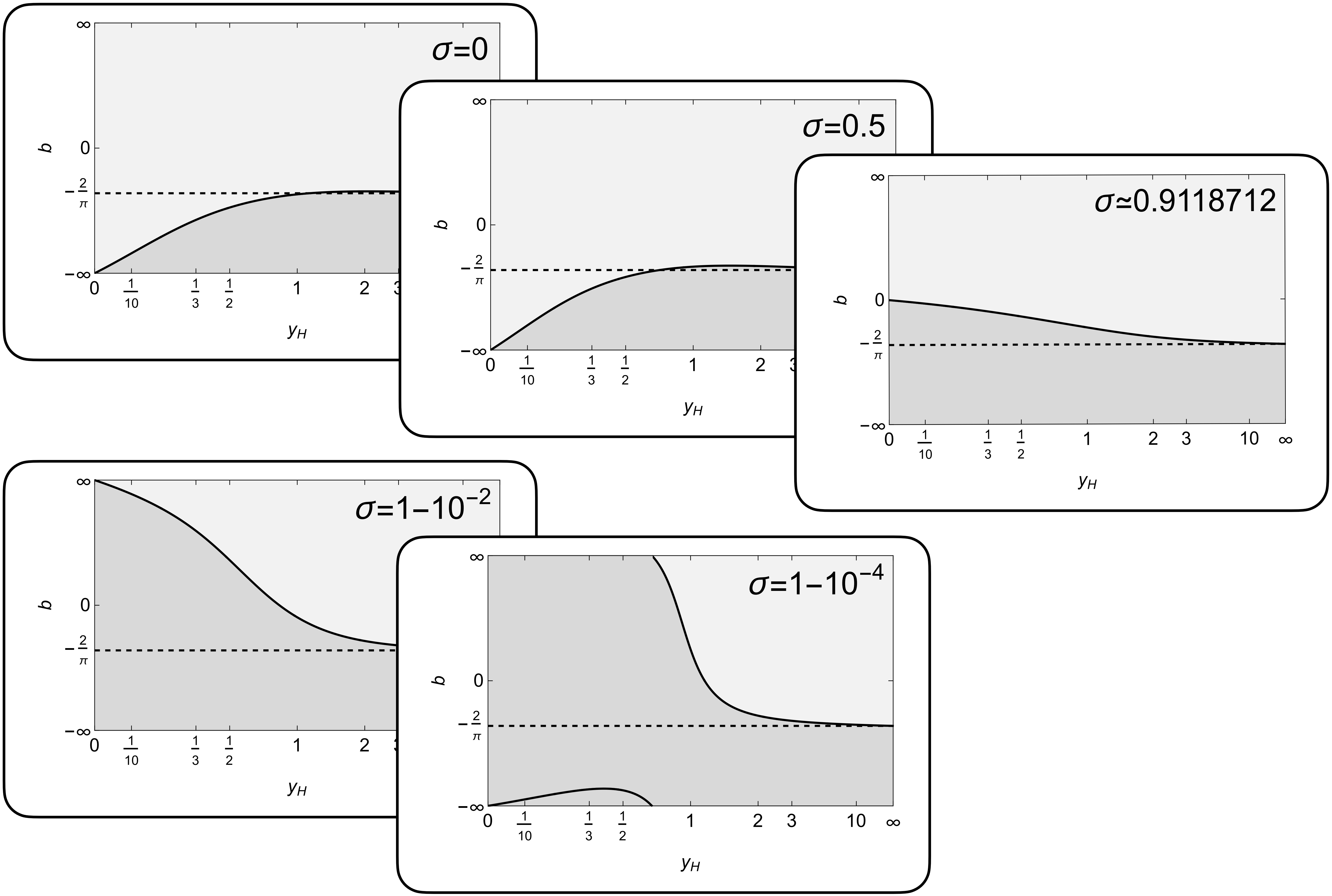

σ=0
b
∞
0
−2/π
−∞
0
1/10
1/3
1/2
1
2
3
10
∞
y_H


σ=0
∞
b
0
−2/π
−∞
0
1/10
1/3
1/2
1
2
3
y_H

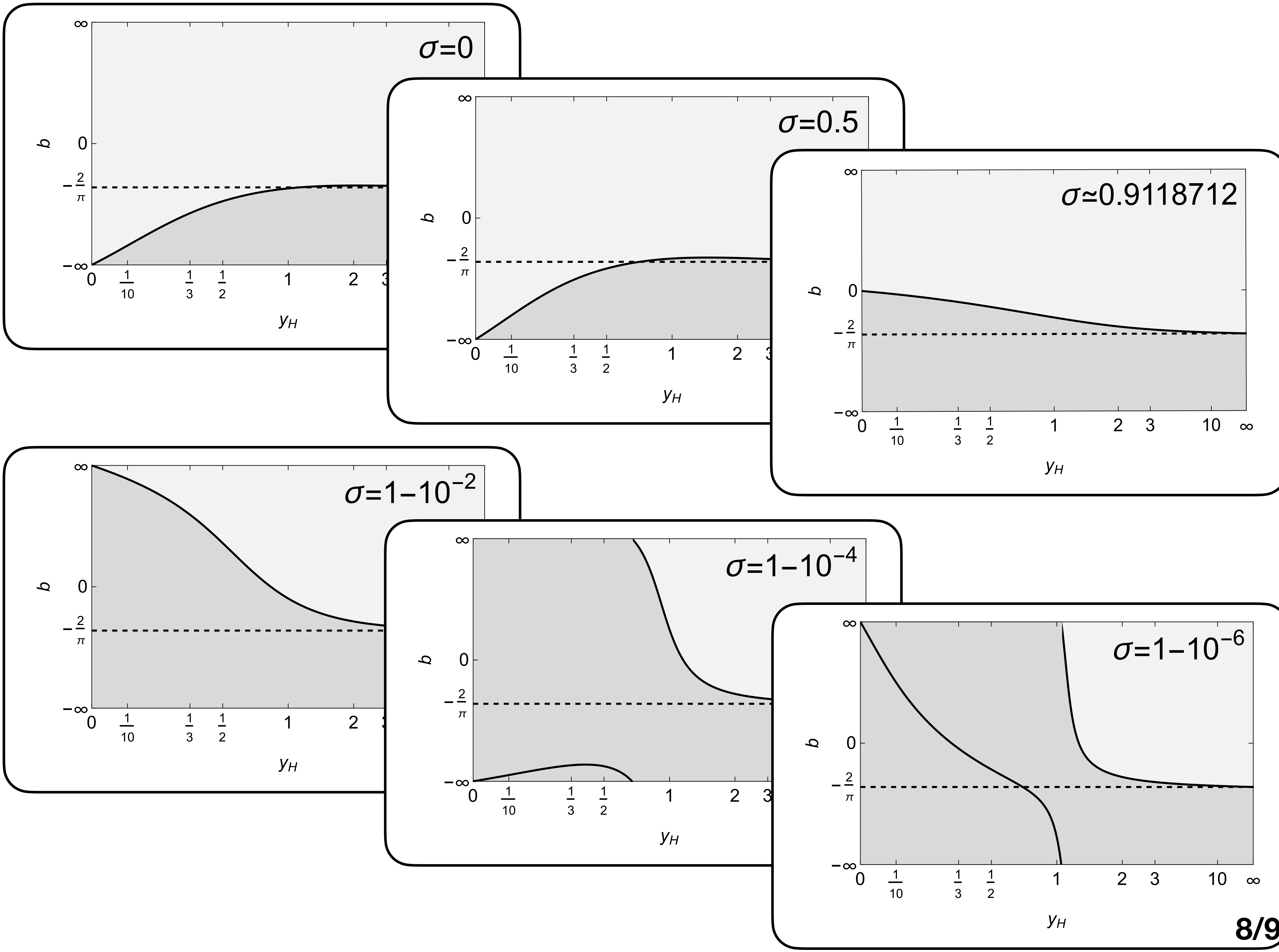

σ=0.5
∞
b
0
−2/π
−∞
0
1/10
1/3
1/2
1
2
3
10
∞
y_H


σ=0
σ=0.5
σ≃0.9118712
b
y_H
∞
0
$-\frac{2}{\pi}$
−∞
0
$\frac{1}{10}$
$\frac{1}{3}$
$\frac{1}{2}$
1
2
3
10
∞


σ=0
σ=0.5
σ≃0.9118712
σ=1−10−2
b
y_H
∞
0
−2/π
−∞
0
1/10
1/3
1/2
1
2
3
10
∞



σ=0
σ=0.5
σ≃0.9118712
σ=1−10^−2
σ=1−10^−4
b
y_H
∞
0
−2/π
−∞
0
1/10
1/3
1/2
1
2
3
10
∞


σ=0
σ=0.5
σ≃0.9118712
σ=1−10^−2
σ=1−10^−4
σ=1−10^−6
b
y_H
∞
0
−2/π
−∞
0
1/10
1/3
1/2
1
2
3
10
∞

Summary

- Static solutions play a key role in the dynamics of the system.
- In SAdS with sufficiently large black holes, small field configurations are stable.
- Large charges seem to introduce significant qualitative differences.

For details see:

FF and M. Maliborski, *Dynamics of nonlinear scalar field with Robin boundary condition on the Schwarzschild–anti–de Sitter background*, Phys. Rev. D **109**, 044015 (2024).

for the charged case: FF and M. Maliborski, in preparation

Thank you for your attention