Gravitational Instantons, old and new.

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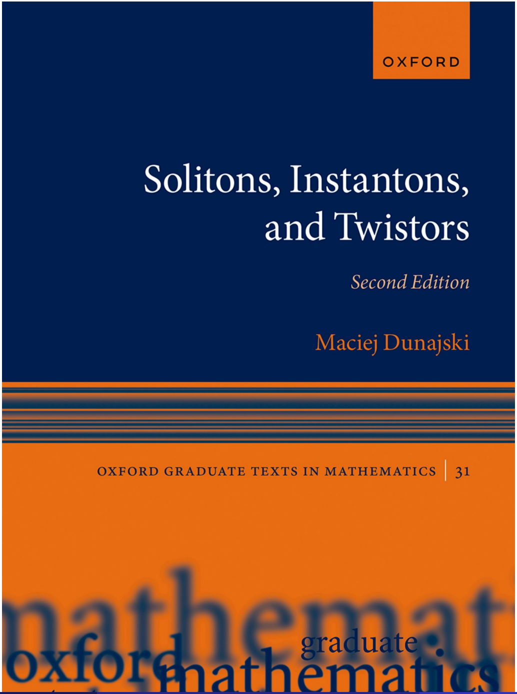
Gravitational Instantons are solutions to the four-dimensional Einstein equations in Riemannian signature which give complete metrics and asymptotically 'look-like' flat space.

- 'Old' (Stephen Hawking 1977)
  - Euclidean Schwarzschild and Kerr (analytically continued Lorentzian black holes).
  - Eguchi–Hanson, Taub–NUT, and multi–centred metric (anti–self–dual solutions with no Lorentzian counterparts. Hyper–Kähler metrics).
  - ALE, ALF and more.
- 'New'
  - Riemannian black hole uniqueness
  - Chen–Teo family, and toric instantons.
  - Twistor theory of Chen–Teo instanton.


OXFORD
Solitons, Instantons, and Twistors
Second Edition
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Euclidean Schwarzschild Metric

- Schwarzschild metric

$$g = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - \left(1 - \frac{2m}{r}\right) dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

  Removable singularity at $r = 2m$ (event horizon). Essential singularity at $r = 0$.

- Euclidean Schwarzschild metric. $t = i\tau, 2m < r < \infty$. Set $\rho = 4m\sqrt{1 - 2m/r}$. Near $\rho = 0$

$$g \sim d\rho^2 + \frac{\rho^2}{16m^2} d\tau^2 + 4m^2(d\theta^2 + \sin^2\theta d\phi^2).$$

  Identify $\tau \sim \tau + 8\pi m$. Flat and regular metric.

- Kerr black hole $\rightarrow$ Euclidean Kerr instanton.

Hawking's gravestone

- Left–invariant one forms on $S^3 = SU(2)$.

$$\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i\sin\theta d\phi), \quad \sigma_3 = d\psi + \cos\theta d\phi$$

$$0 \leq \theta \leq \pi, \quad \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 4\pi.$$

$$d\sigma_1 + \sigma_2 \wedge \sigma_3 = 0, \quad d\sigma_2 + \sigma_3 \wedge \sigma_1 = 0, \quad d\sigma_3 + \sigma_1 \wedge \sigma_2 = 0$$

$$g_{\mathbb{R}^4} = dr^2 + \frac{1}{4}r^2\Big({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2\Big).$$

- Taub–NUT instanton (Hawking 1977)

$$g = \frac{1}{4}\frac{r+m}{r-m}dr^2 + m^2\frac{r-m}{r+m}\sigma_3^2 + \frac{1}{4}(r^2 - m^2)\left(\sigma_1^2 + \sigma_2^2\right).$$

- Anti-self–dual (ASD): $R_{abcd} = \frac{1}{2}{\varepsilon_{ab}}^{pq}R_{cdpq}$. Implies $R_{ab} = 0$. No Lorentzian analogue (Lorentzian: ASD iff flat).
- Large r: S^1 bundle over S^2 (Hopf–fibration with Chern number 1.)

ALF METRICS

- A complete regular four-dimensional Riemannian manifold (M, g) which solves the Einstein equations (possibly with cosmological constant) is called ALF (asymptotically locally flat) if it approaches S^1 bundle over S^2 at infinity.

$$\lim_{r\to\infty} g = (d\tau + 2n\cos\theta d\phi)^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2).$$

- Asymptotically flat (AF)= the S^1 bundle is trivial (so $n = 0$). E.g. Euclidean Schwarzschild and Euclidean Kerr.
- Lorentzian black hole uniqueness (Hawking, Carter, D. Robinson, . . .).
- Conjecture: Riemannian 'black hole uniqueness': Euclidean Schwarzschild and Kerr are the only AF gravitational instantons.

ALE METRICS

- Eguchi–Hanson (EH) instanton (1978)

$$g = \left(1 - \frac{a^4}{r^4}\right)^{-1} dr^2 + \frac{1}{4} r^2 \left(1 - \frac{a^4}{r^4}\right) \sigma_3^2 + \frac{1}{4} r^2 \left(\sigma_1^2 + \sigma_2^2\right)$$

$$r > a, \quad 0 \le \psi \le 2\pi, \quad 0 \le \phi \le 2\pi, \quad 0 \le \theta \le \pi$$

$$\rho^2 = r^2 \left[1 - (a/r)^4\right], \; g \sim \frac{1}{4} \left(d\rho^2 + \rho^2 d\psi^2\right)$$

- A complete regular four-dimensional Riemannian manifold (M, g) which solves the Einstein equations (possibly with cosmological constant) is called ALE (asymptotically locally Euclidean) if it approaches $\mathbb{R}^4/\Gamma$ at infinity, where Γ is a discrete subgroup of $SO(4)$.
- Theorem (Kronheimer 1989): For Γ (cyclic A_N, dihedral D_N, dihedral, tetrahedral, octahedral, and icosahedral) there exists an ALE gravitational instanton. $A_2 : \Gamma = \mathbb{Z}_2$ (Eguchi–Hanson).

Mathematical detour: Hyper–Kähler metrics

- An almost complex structure on a 4–manifold M

$$I : TM \to TM, \quad I^2 = -\mathrm{Id}$$

- $\mathbb{C} \otimes TM = T^{1,0}M \oplus T^{0,1}M$ (eigen-spaces of I with eigenvalues $\pm i$).
- I is a complex structure iff $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. Theorem (Newlander-Nirenberg): I is a complex structure iff there exists a holomorphic atlas and M is a two–dimensional complex manifold.
- (M, g) Riemannian mfd with almost –complex structure I.
  - Hermitian if $g(X, Y) = g(IX, IY)$.
  - Kähler if I is complex, and $d\Omega = 0$, where $\Omega(X, Y) = g(X, IY)$.
  - (M, g) is hyper–Kähler if it is Kähler w.r.t. complex structures I_1, I_2, I_3 such that

$$I_1 I_2 = I_3, \quad I_2 I_3 = I_1, \quad I_3 I_1 = I_2.$$

  - Riemann tensor anti–self–dual (ASD) iff (M, g) hyper–Kähler.
- ASD gravitational instantons = complete hyper–Kähler metrics. Compact hyper–Kähler metrics: four–torus with a flat metric. $K3$ surface: Existence theorem (Yau's proof (1977) of Calabi's conjecture). Not known in closed form.

Multi–Centered metrics

- Gibbons–Hawking 1978: Ansatz: $(V, A)=$ function, 1–form on $\mathbb{R}^3$.

$$g = V({dx_1}^2 + {dx_2}^2 + {dx_3}^2) + V^{-1}(d\tau + A)^2.$$

- Hyper–Kähler

$$\Omega_i = -(d\tau + A) \wedge dx_i + \frac{1}{2} V \epsilon_{ijk} dx_j \wedge dx_k.$$

$$d\Omega_i = 0 \quad \text{iff} \quad dA = \star_3 dV \quad \text{(Abelian Monopole Equation).}$$

- V harmonic on $\mathbb{R}^3$. Multi–centre metrics

$$V = V_0 + \sum_{m=1}^{N} \frac{1}{|\,\mathbf{x} - \mathbf{x}_m\,|}.$$

  - $V_0 = 0, N = 1$. Flat metric.
  - $V_0 = 0, N = 2$. Eguchi–Hanson. $N > 2$ general A_N ALE instantons.
  - $V_0 \neq 0, N = 1$. Taub–NUT. $N > 1$ A_N ALF instantons.

Other developments

- Volume growth of a ball of large radius R. It is of orders R^4, R^3, R^2 and R for respectively ALE, ALF, ALG and ALH instatons (Chen+Chen 2011, Hein 2012).
- Einstein–Maxwell instantons (Euclidean Israel–Willson metrics): lots of AF examples. Whitt (1985), Dunajski–Hartnoll (2007)
- Twistor theory (Penrose 1976, Atiyah–Hitchin–Singer 1978). Hyper–Kahler four–manifolds (ASD Ricci flat metrics) in one-to-one correspondence with three dimensional complex manifolds (twistor spaces) admtting 4-parameter families of rational curves with some additional structure.
    - Gravitational instantons (Hitchin 1979, Kronheimer 1989).
    - Hidden symmetries of heavenly equations: (Dunajski–Mason 2001, 2003).

The Chen–Teo instanton

- Riemannian 'black hole uniqueness' conjecture: Euclidean Schwarzschild and Kerr are the only AF gravitational instantons.
- Chen–Teo (2011, 2015): This conjecture is wrong. `arXiv:1107.0763, arXiv:1504.01235`
- Five parameter family of toric (two commuting Killing vectors) Riemannian Ricci flat metrics interpolating between three–centre Gibbons–Hawking, and Euclidean Plebański–Demianski solutions.
- Contains a two–parameter sub–family of AF gravitational instantons.
- Hermitian (one–sided type D). Aksteiner–L. Andersson (2022).

Chen–Teo family

- A quartic f with four real roots. Set

$$\begin{aligned} f &= f(\xi) = a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 \\ F &= f(x)y^2 - f(y)x^2 \\ H &= (\nu x + y)[(\nu x - y)(a_1 - a_3 xy) - 2(1-\nu)(a_0 - a_4 x^2 y^2)] \\ G &= f(x)[(2\nu - 1)a_4 y^2 + 2\nu a_3 y^3 + a_0 \nu^2] \\ &\quad - f(y)[\nu^2 a_4 x^4 + 2\nu a_1 x + (2\nu - 1)a_0]. \end{aligned}$$

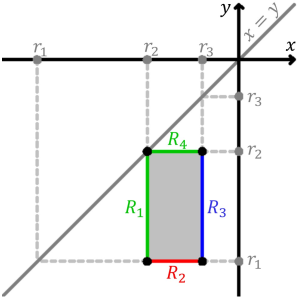
- Ricci–flat metric

$$g = \frac{kH}{(x-y)^3}\Big(\frac{dx^2}{f(x)} - \frac{dy^2}{f(y)} - \frac{f(x)f(y)}{kF}d\phi^2\Big) + \frac{1}{FH(x-y)}(Fd\tau + Gd\phi)^2$$

- Two out of five $(a_0, \ldots, a_4)$ can be fixed by scalings.

REGULARITY

- Real roots of f: $r_1 < r_2 < r_3 < r_4$



- AF Instanton on $M = \mathbb{CP}^2 \setminus S^1$

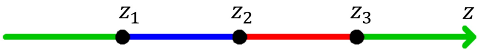
$$
\begin{aligned}
r_1 &= \frac{4s^2(1-s)}{1-2s+2s^2},\ r_2 = -1,\ r_3 = \frac{1-2s}{s(1-2s+2s^2)},\ r_4 = \infty, \\
\nu &= -2s^2, \quad s \in (1/2, \sqrt{2}/2).
\end{aligned}
$$

The Yang equation

- Torus action $K_i = \partial/\partial\phi^i$ where $\phi^i = (\phi, \tau)$,

$$g = \Omega^2(dr^2 + dz^2) + J_{ij}d\phi^i d\phi^j, \quad i, j = 1, 2$$

- $r^2 = \det(J)$ and $*_2 dz = dr$.
- rank$(J(0, z)) = 1$ or 0 at *turning points* where K_i vanish.
- Rod structure



- Ricci–flat condition $\rightarrow$ the Yang equation

$$r^{-1}\partial_r(rJ^{-1}\partial_r J) + \partial_z(J^{-1}\partial_z J) = 0. \quad \text{(Y)}.$$

- (Y) also arises as a reduction of anti–self–dual Yang-Mills (L. Witten 1979, Ward 1983). Look at it next.

Anti–self–dual Yang–Mills (ASDYM)

- Complexified Minkowski space $M_{\mathbb{C}} = \mathbb{C}^4$, coordinates $(W, Z, \widetilde{W}, \widetilde{Z})$

$$ds^2 = 2(dZd\widetilde{Z} - dWd\widetilde{W}), \quad \text{vol} = dW \wedge d\widetilde{W} \wedge dZ \wedge d\widetilde{Z}.$$

- $A \in \Lambda^1(M_{\mathbb{C}}) \otimes \mathfrak{sl}(2), F = dA + A \wedge A$. ASDYM $F = - \star F$.

$$F_{WZ} = 0, \quad F_{\widetilde{W}\widetilde{Z}} = 0, \quad F_{W\widetilde{W}} - F_{Z\widetilde{Z}} = 0.$$

- Gauge choice

$$A = J^{-1}\partial_{\widetilde{W}} J d\widetilde{W} + J^{-1}\partial_{\widetilde{Z}} J d\widetilde{Z}, \quad J = J(W, Z, \widetilde{W}, \widetilde{Z}) \in SL(2, \mathbb{C})$$

$$F = - \star F \quad \text{iff} \quad \partial_Z(J^{-1}\partial_{\widetilde{Z}} J) - \partial_W(J^{-1}\partial_{\widetilde{W}} J) = 0.$$

- Symmetry reduction to (Y)

$$Z = t + z, \quad \widetilde{Z} = t - z, \quad W = re^{i\theta}, \quad \widetilde{W} = re^{-i\theta}, \quad J = J(r, z).$$

- ASDYM $\iff$ connection A is flat on α–planes in $M_{\mathbb{C}}$

$$\alpha = W + \lambda \widetilde{Z}, \quad \beta = Z + \lambda \widetilde{W} \quad \text{(I)}$$

Twistor space $PT \equiv \mathbb{CP}^3 \setminus \mathbb{CP}^1$ with affine coordinates (α, β, λ).

- (I): Points in $M_{\mathbb{C}} = \mathbb{CP}^1$s (twistor lines) in PT. Conformal structure on $M_{\mathbb{C}}$: p_1, p_2 are null separated iff L_1, L_2 intersect.
- Theorem (Ward 1977). $1-1$ correspondence between gauge equivalence classes of A and holomorphic vector bundles $E \to PT$ trivial on twistor lines.
- Cover PT with two open sets: U, where $\lambda \neq \infty$ and $\widetilde{U}$ where $\lambda \neq 0$. Patching matrix: $P_{U\widetilde{U}} = P(\alpha, \beta, \lambda)$. On twistor line. $P_{U\widetilde{U}} = P_U {P_{\widetilde{U}}}^{-1}$.
- ASDYM connection gauge equivalent to

$$A = H^{-1}\partial_Z H \, dZ + H^{-1}\partial_W H \, dW + \widetilde{H}^{-1}\partial_{\widetilde{Z}}\widetilde{H} \, d\widetilde{Z} + \widetilde{H}^{-1}\partial_{\widetilde{W}}\widetilde{H} \, d\widetilde{W}$$

where $H = P_U(\lambda = 0), \widetilde{H} = P_{\widetilde{U}}(\lambda = \infty)$ and $J = \widetilde{H}H^{-1}$.

Twistor bundle for toric Ricci flat metrics

- Back to toric Ricci–flat metrics. K=one of the Killing vectors

$$d\psi = *(K \wedge dK).$$

- Another solution to the Yang equation (Bäcklund transformation)

$$J' = \frac{1}{V}\begin{pmatrix} 1 & -\psi \\ -\psi & \psi^2 - V^2 \end{pmatrix}, \quad V \equiv g(K, K)$$

- Pick a rod on which K is not identically zero. Patching matrix (Fletcher and Woodhouse 1990) $P(z) = J'(0, z)$. Analiticaly continue

$$P(\gamma), \quad \gamma = z + \frac{1}{2}r\Big(\lambda - \frac{1}{\lambda}\Big).$$

Twistor bundle for Chen–Teo

- Dunajski-Tod `arXiv:2405.08170`

$$P(z) = \begin{pmatrix} C_1/C & Q/C \\ Q/C & C_2/C \end{pmatrix},$$

C_1, C_2, C monic cubics, Q quadratic, with coefficients depending on the Chen–Teo parameters.

- Outer rod. Asymptotics near $z = \infty$

$$P \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 2m & 2n \\ 2n & 2m \end{pmatrix} + O(1/z^2),$$

where m, n are mass and nut parameters. For Chen–Teo instanton

$$m = \sqrt{k}\frac{(1+2s^2)^2}{2\sqrt{1-4s^4}}, \quad n = 0$$

(agrees with Kunduri–Lucietti 2021).

Outlook

- Gravitational Instantons are solutions to the four-dimensional Einstein equations in Riemannian signature which give complete metrics and asymptotically 'look-like' flat space.
- ALE, ALF, ALG, ALH (explicit description of the last two classes?)
- Closed form of the compact gravitational instanton on $K3$?
- Does Chen–Teo complete the classification of Hermitian, non–Kähler ALF instantons?
- The status of Euclidean quantum gravity?

Thank You