Comparison of the known generalizations of quadrupole formulas

Denis Dobkowski-Ryłko

64. Cracow School of Theoretical Physics

Zakopane, June 20th, 2024

Caltech archives — letter to John Tate (editor of the Physical Review), written on Feb 18th, 1937:

You neglected to keep me informed on the paper submitted last summer by your most distinguished contributor. But I shall nevertheless let you in on the subsequent history. It was sent (without [...] correction [...] pointed out by your referee) to another journal, and when it came back in galley proofs was completely revised because I had been able to convince him in the meantime that it proved the opposite of what he thought.

Einstein battle with the Physical Review

1916 & 1918 - Einstein published papers, where he calculated the gravitational-wave field and radiated energy of a time-dependent source. He used linearized Einstein's equations, slow-motion approximation and obtained the famous quadrupole formula.

1933 - Einstein began working together with Rosen in Princeton, they were looking for exact solutions with plane waves. Probably because they were not convinced by the linear approximation.

June 1st, 1936 - Einstein & Rosen submit a paper to the Physical Review

Do gravitational waves exist?

They thought they had found an exact solution of the field equations describing plane GW, but because the solution had singularity it could not be physically valid ⇒ GW don't exist!

July 23rd, 1936 - John Tate (*Physical Review* Editor) returns the manuscript to Einstein with a mild request:

…would be glad to have [Einstein's] reaction to the various comments and criticisms the referee has made.

July 23rd, 1936 - Einstein wrote back, withdrawing the paper:
We (Mr. Rosen and I) had sent you our manuscript for publication and had not authorized you to show it to specialists before it is printed. I see no reason to address the—in any case erroneous—comments of your anonymous expert. On the basis of this incident I prefer to publish the paper elsewhere.

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Journal of the Franklin Institute in Philadelphia.

Early 1937 - paper with radically altered conclusions appears in the Journal of the Franklin Institute in Philadelphia.

ON GRAVITATIONAL WAVES.

BY

A. EINSTEIN and N. ROSEN.

ABSTRACT.

The rigorous solution for cylindrical gravitational waves is given. For the convenience of the reader the theory of gravitational waves and their production, already known in principle, is given in the first part of this paper. After encountering relationships which cast doubt on the existence of *rigorous* solutions for undulatory gravitational fields, we investigate rigorously the case of cylindrical gravitational waves. It turns out that rigorous solutions exist and that the problem reduces to the usual cylindrical waves in euclidean space.

I. APPROXIMATE SOLUTION OF THE PROBLEM OF PLANE WAVES AND THE PRODUCTION OF GRAVITATIONAL WAVES.

It is well known that the approximate method of integration of the gravitational equations of the general relativity theory leads to the existence of gravitational waves. The method used is as follows: We start with the equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -T_{\mu\nu}. \qquad (1)$$

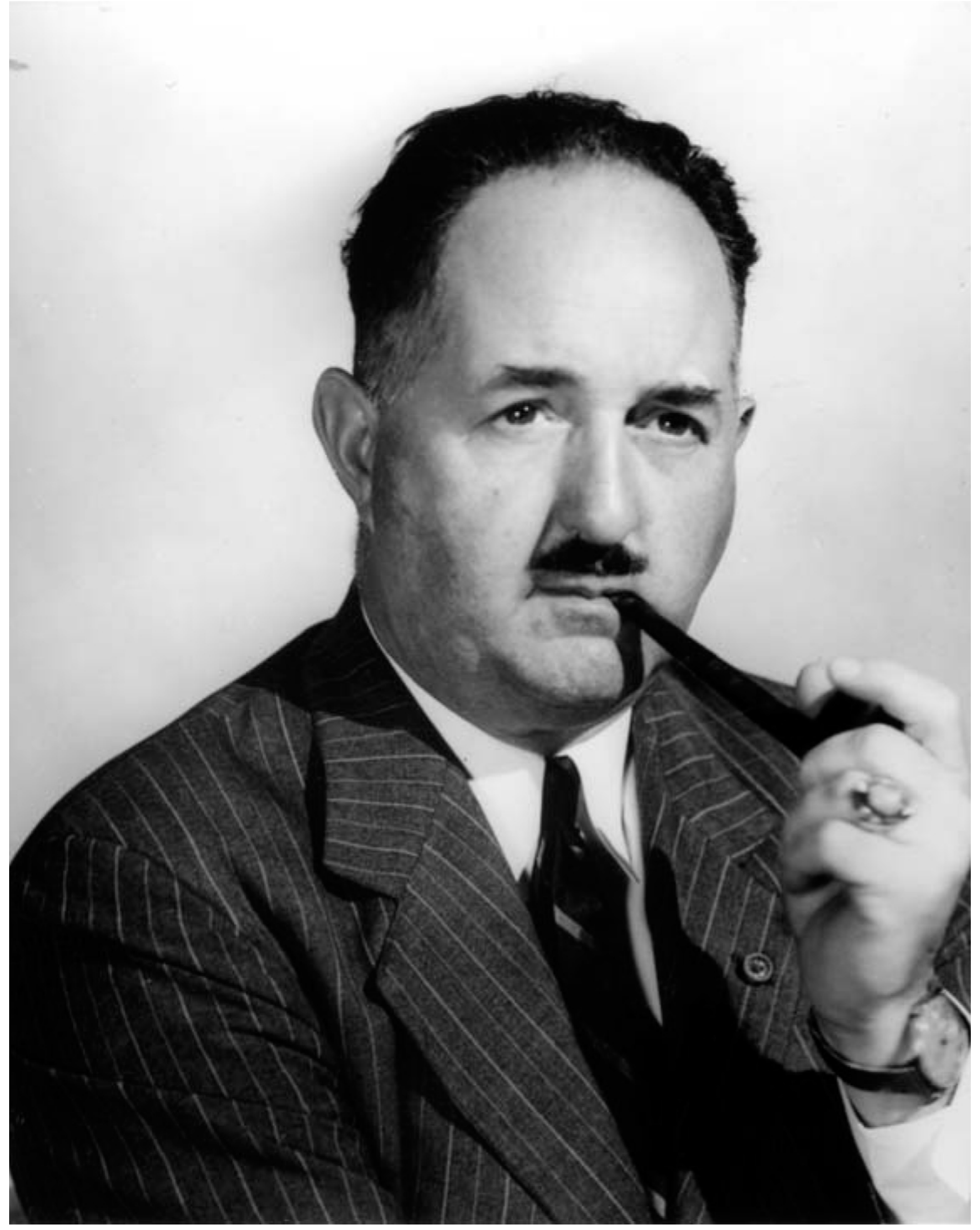
We consider that the $g_{\mu\nu}$ are replaced by the expressions

$$g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}, \qquad (2)$$

where

Physics Today. 2005;58(9):43-48. doi:10.1063/1.2117822

2005 - Daniel Kenefick (*Einstein versus the Physical Review*, 2005) revealed that the referee was a well-known Princeton & Caltech relativist Howard P. Robertson.



1936

NAME	DATE IN	REFEREE	DATE IN	TO AUTHOR	TO N.Y.	ISSUE	REJECTED
[illegible]	5/24	[illegible] 6/4	6/8				6/12
Einstein & Rosen	6/1	Robertson 7/6	7/17	7/23			
[illegible]	[illegible]				4/14	MAY 15, 1936	
Gamow & Teller	3/28		4/16	4/8	4/17/36	JUNE 15, 1936	

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Time changing matter source emitting gravitational radiation in de Sitter spacetime

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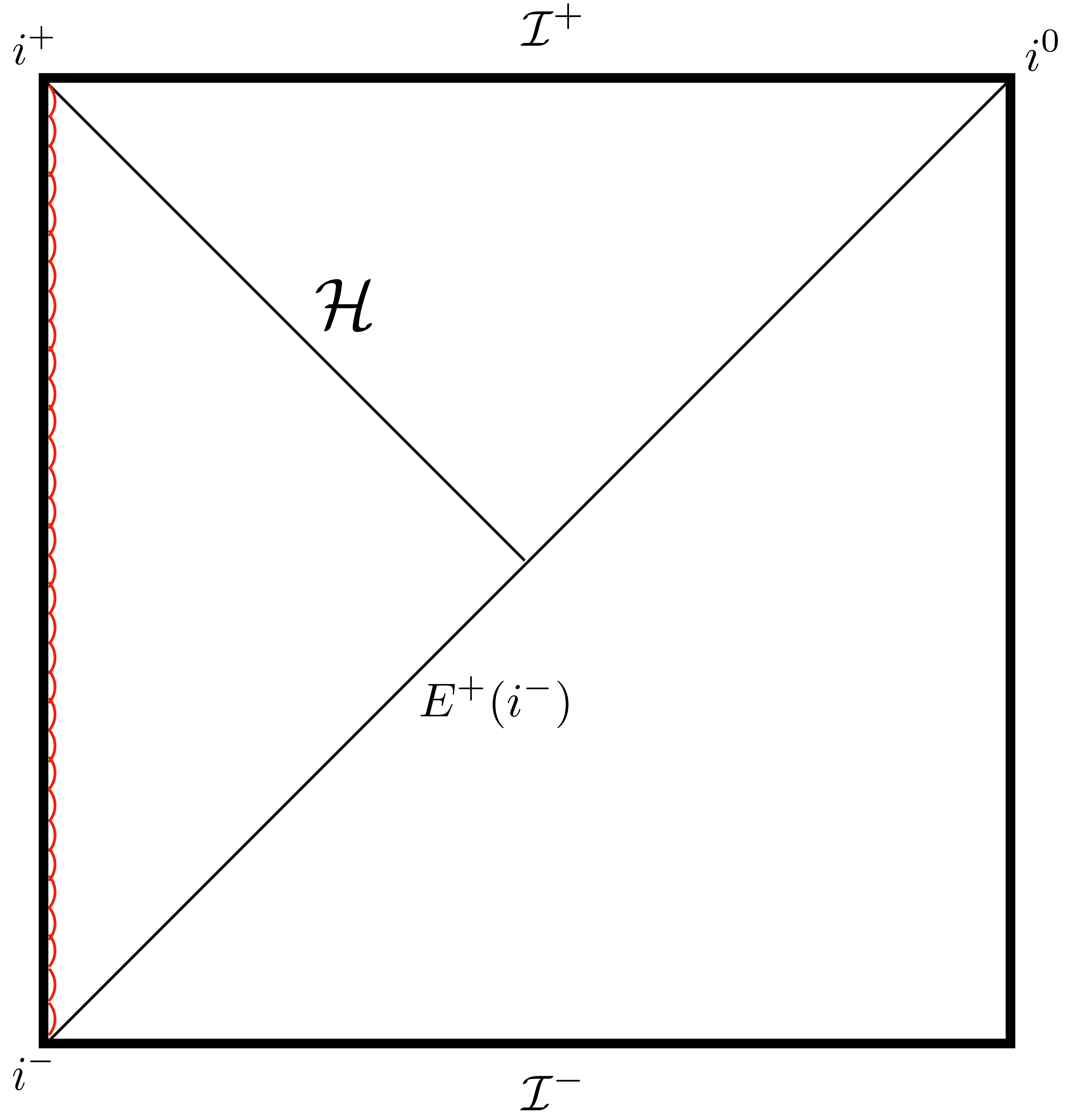

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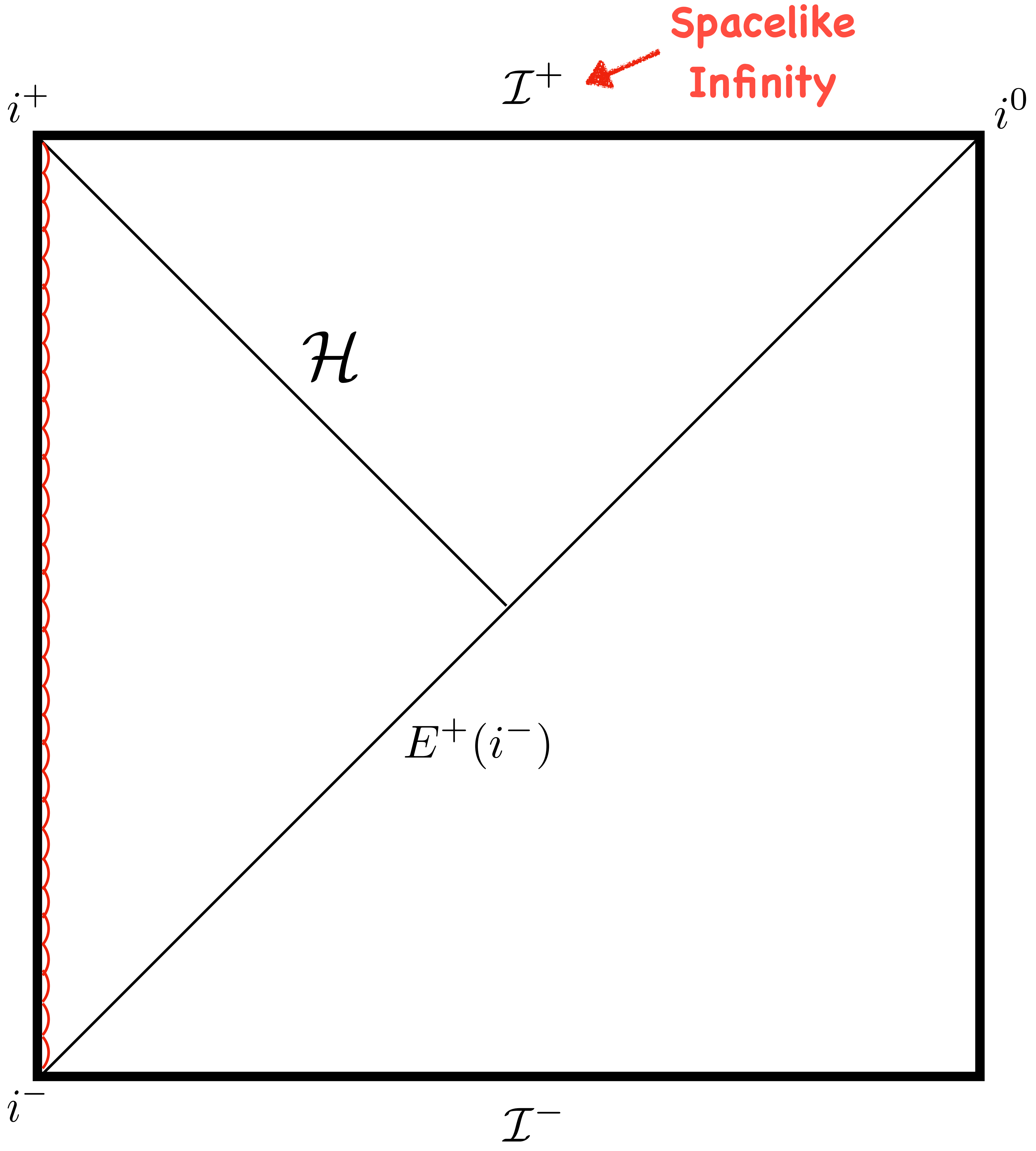

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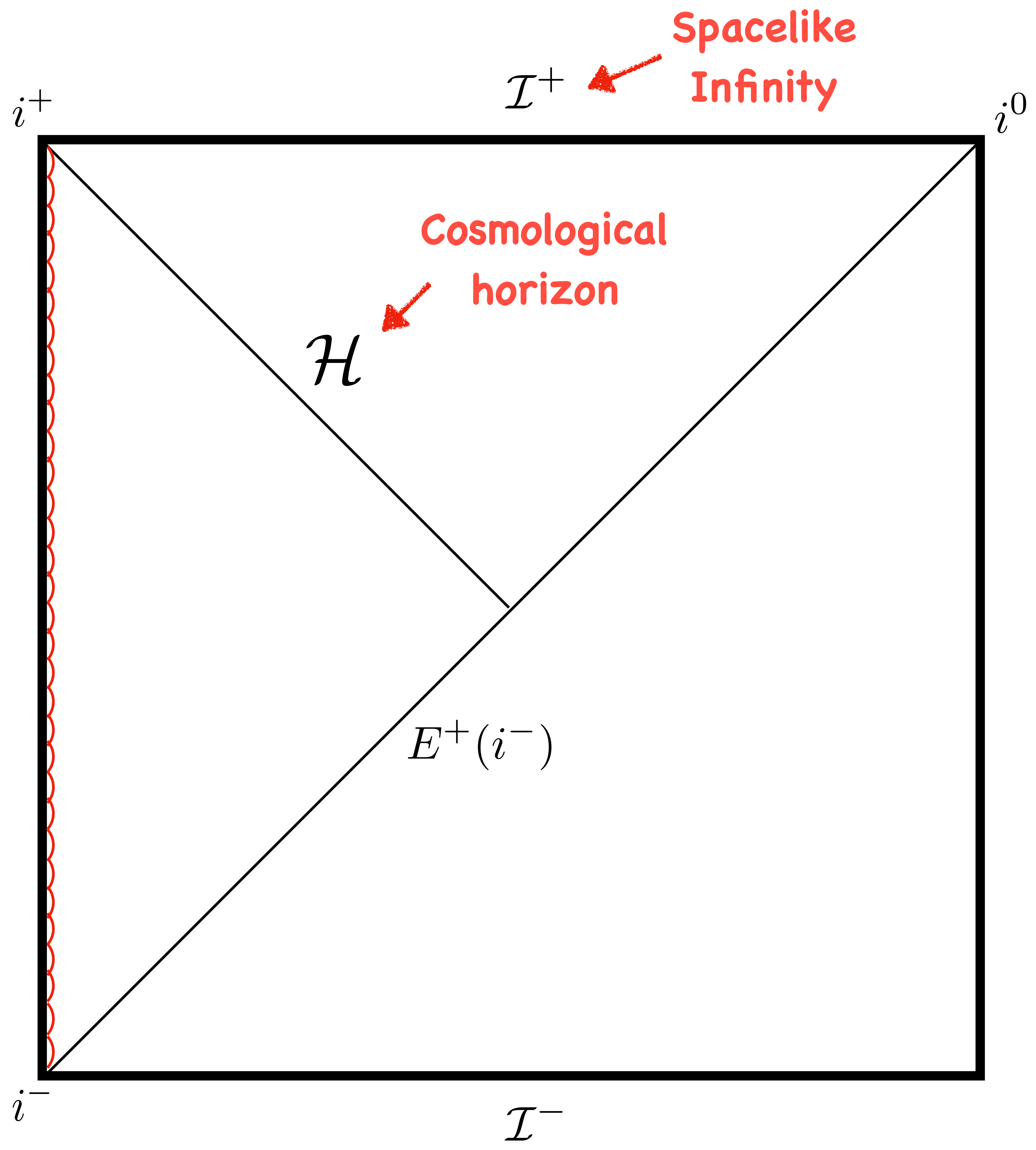

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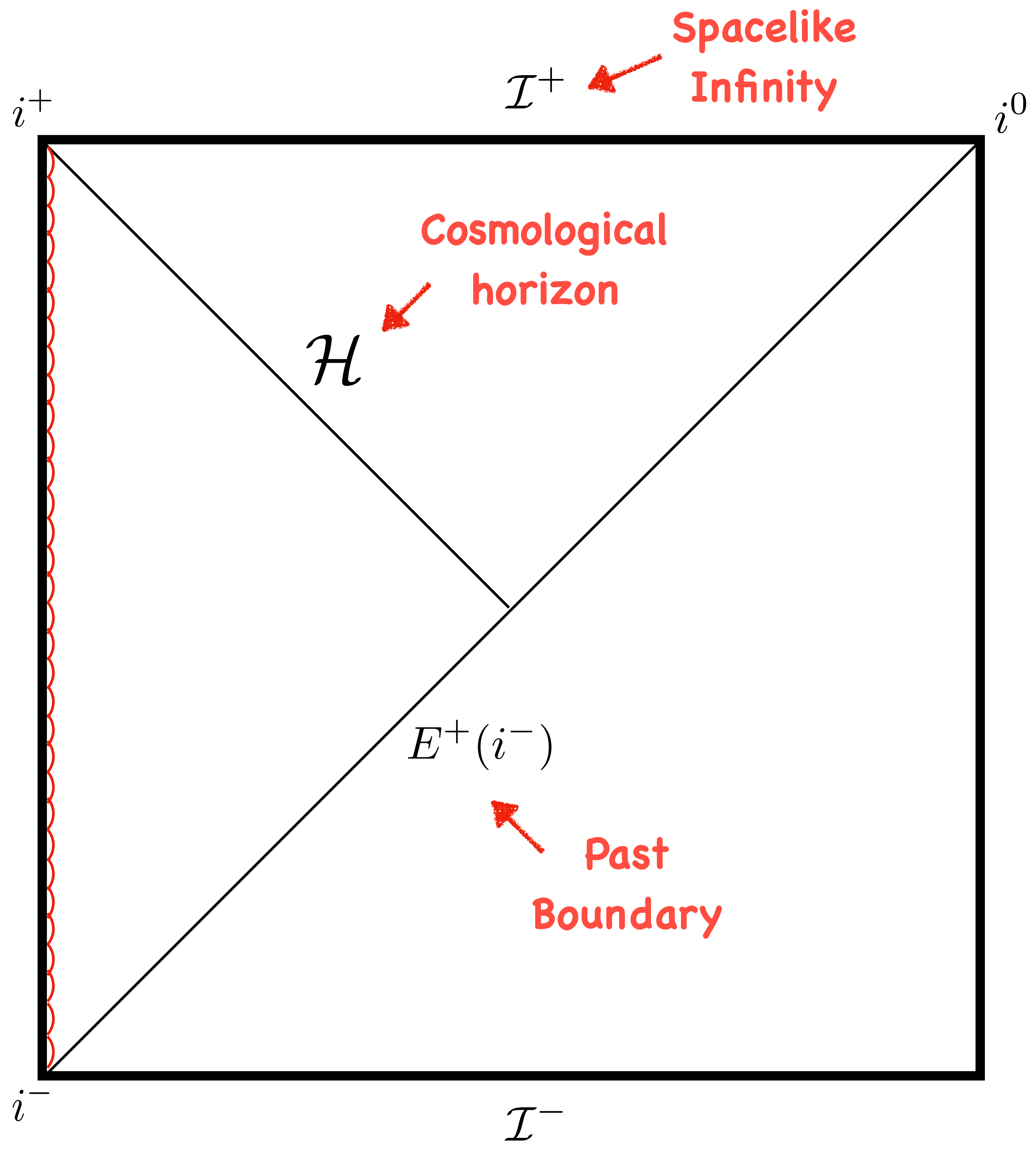

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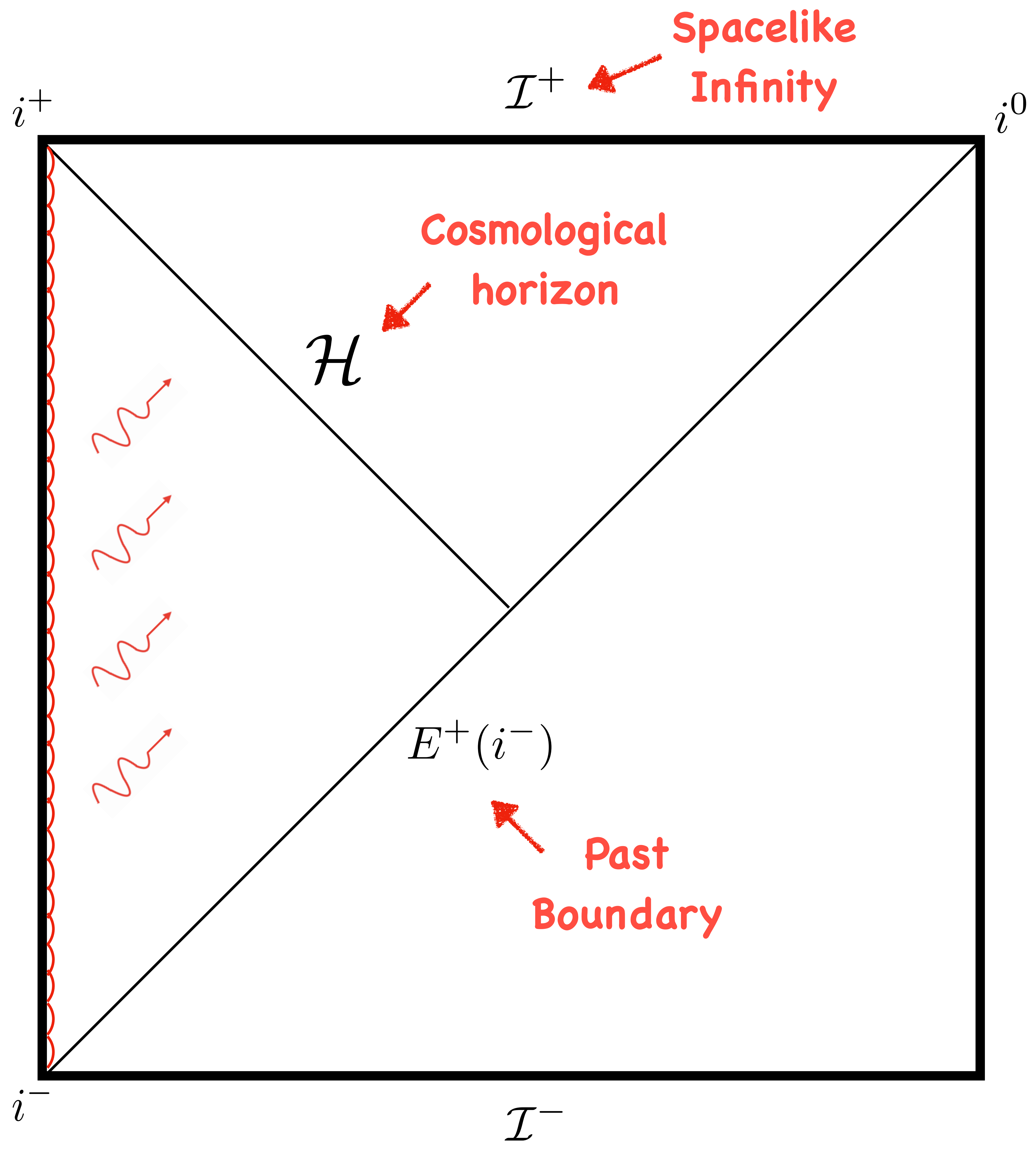

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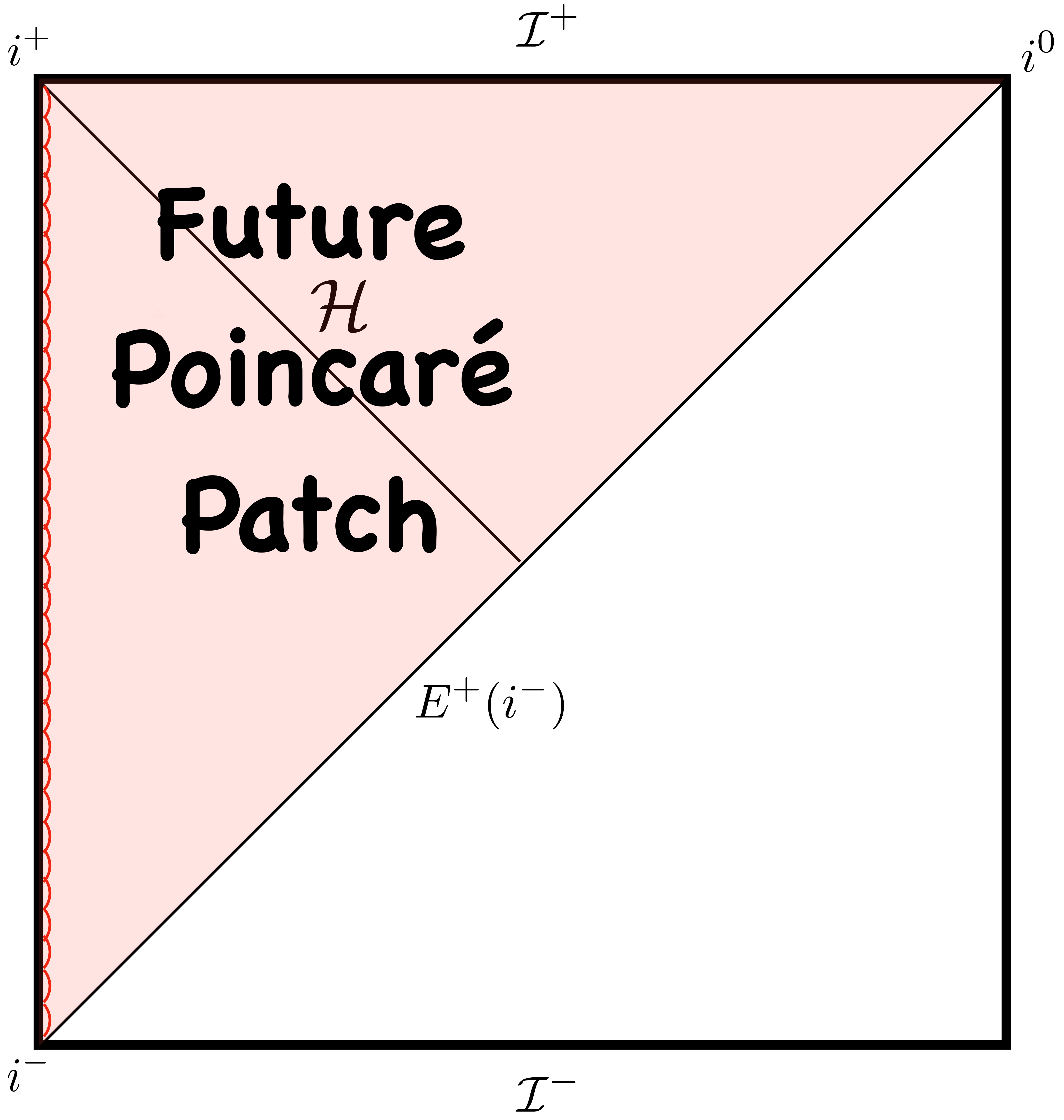

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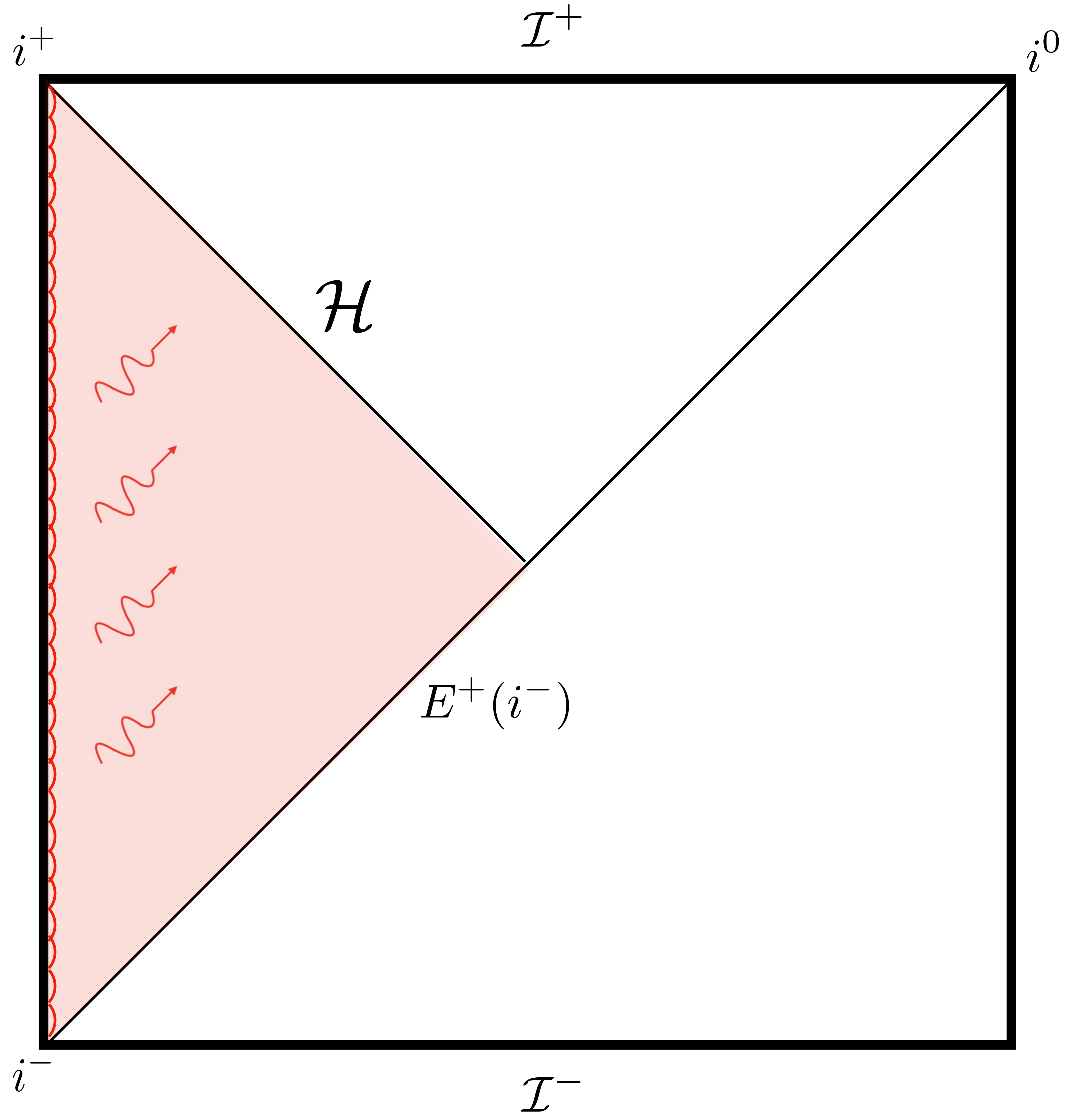

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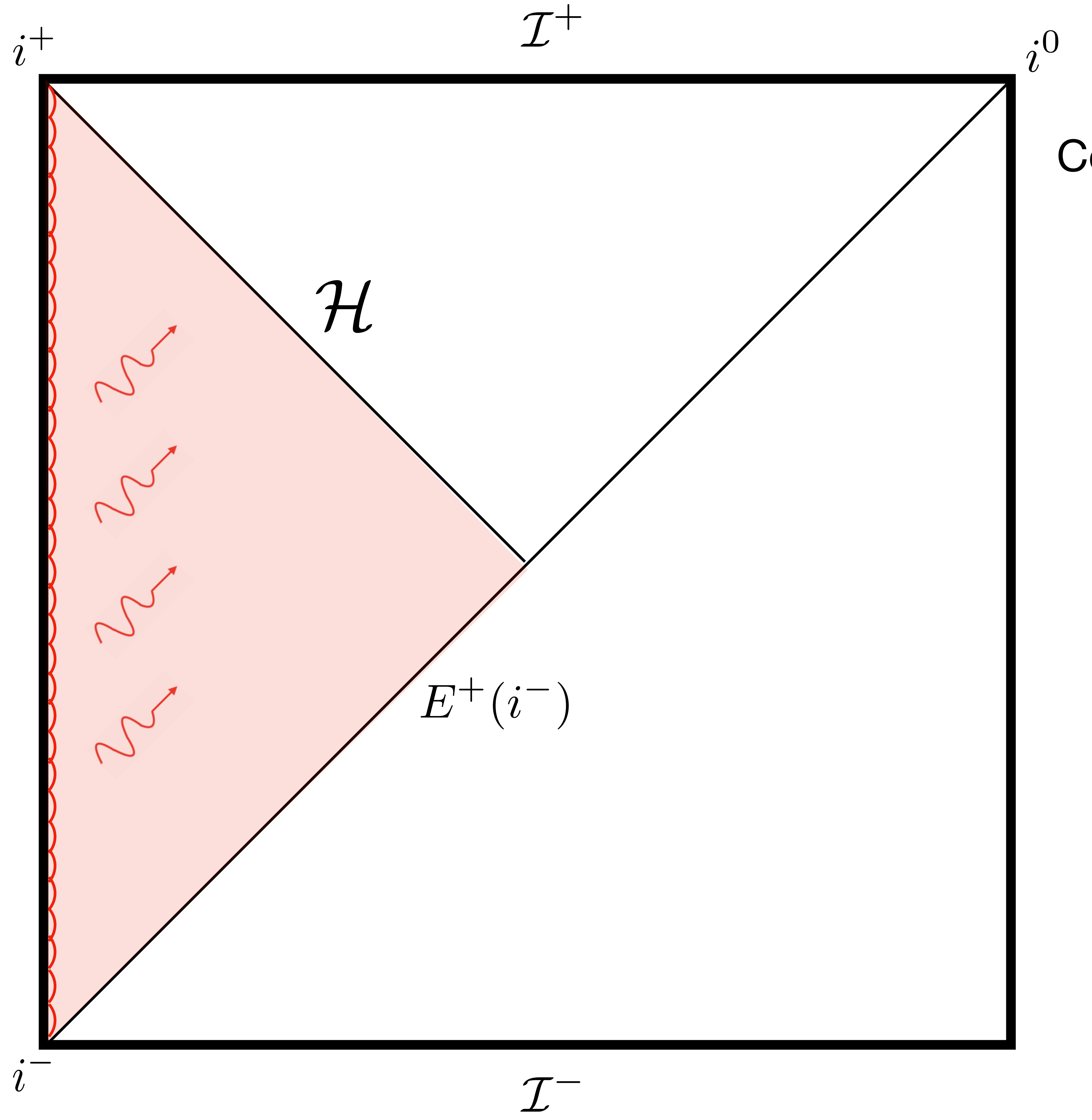

Consider the perturbed metric tensor $g_{\alpha\beta}$ of the form: $g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \epsilon\gamma_{\alpha\beta}$

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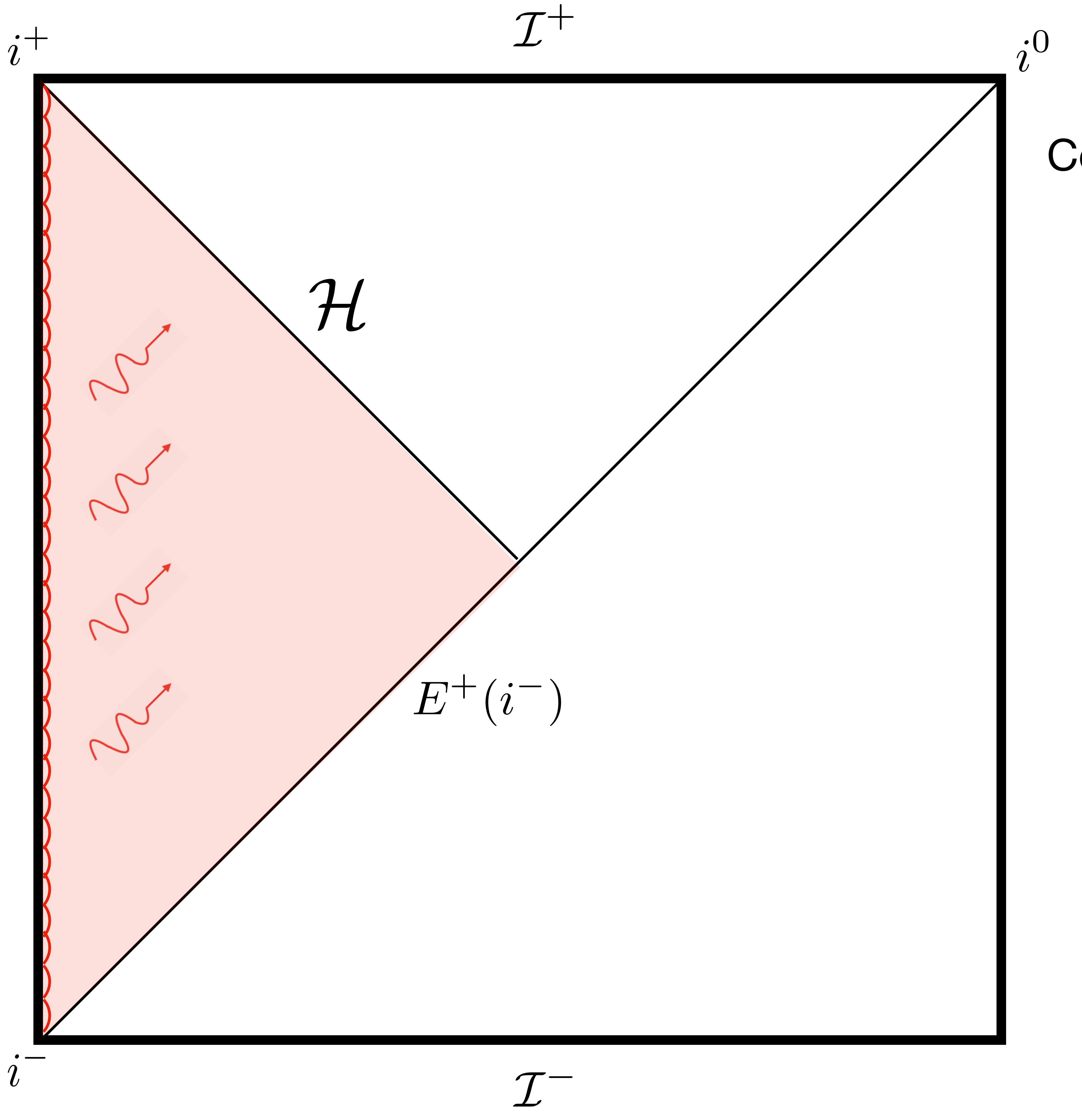

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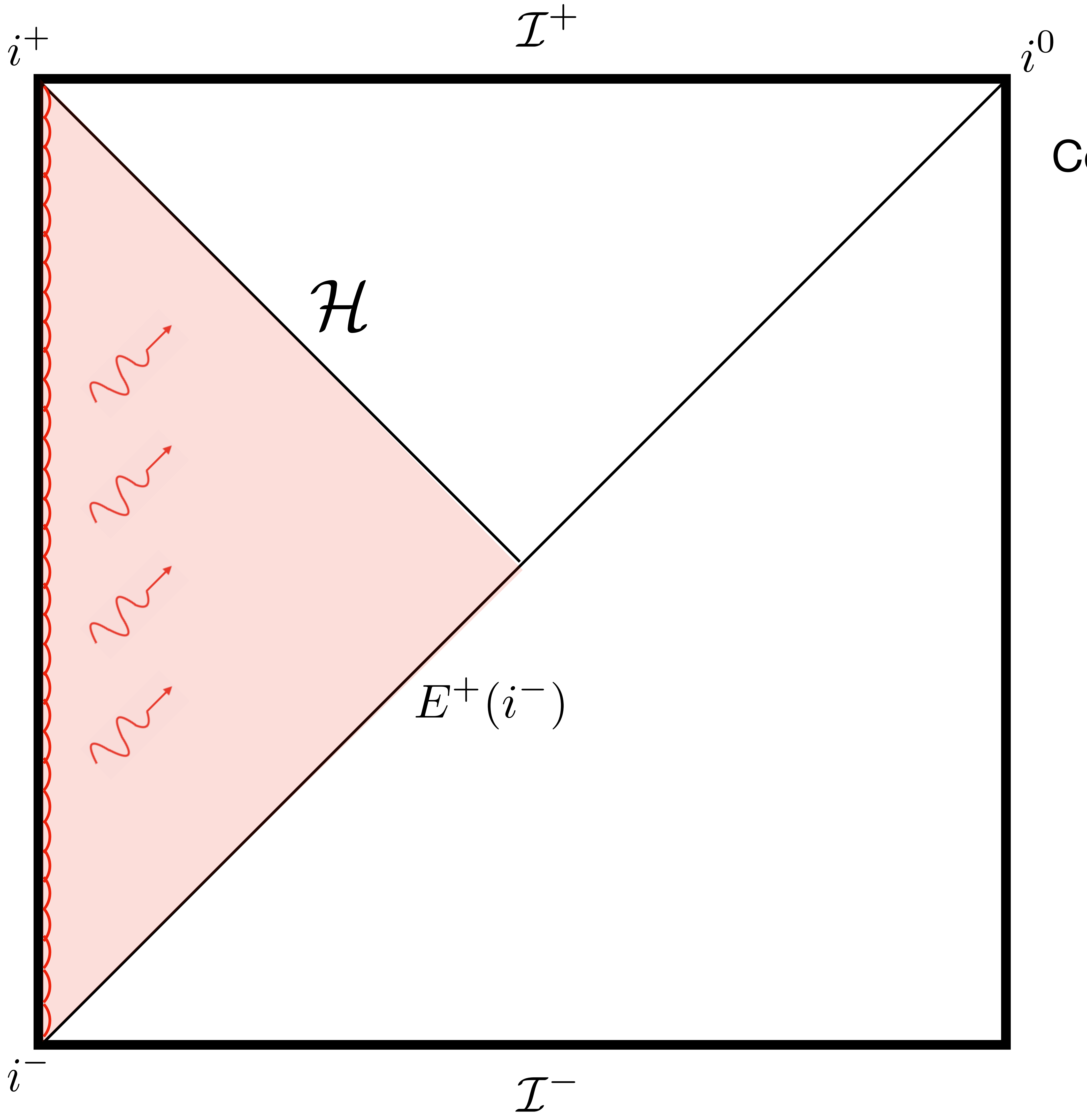
smallness parameter

background de Sitter metric

perturbation

Time changing matter source emitting gravitational radiation in de Sitter spacetime



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$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{H^2\eta^2}\mathring{g}_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{H^2\eta^2}(-d\eta^2 + dx^2 + dy^2 + dz^2)$$

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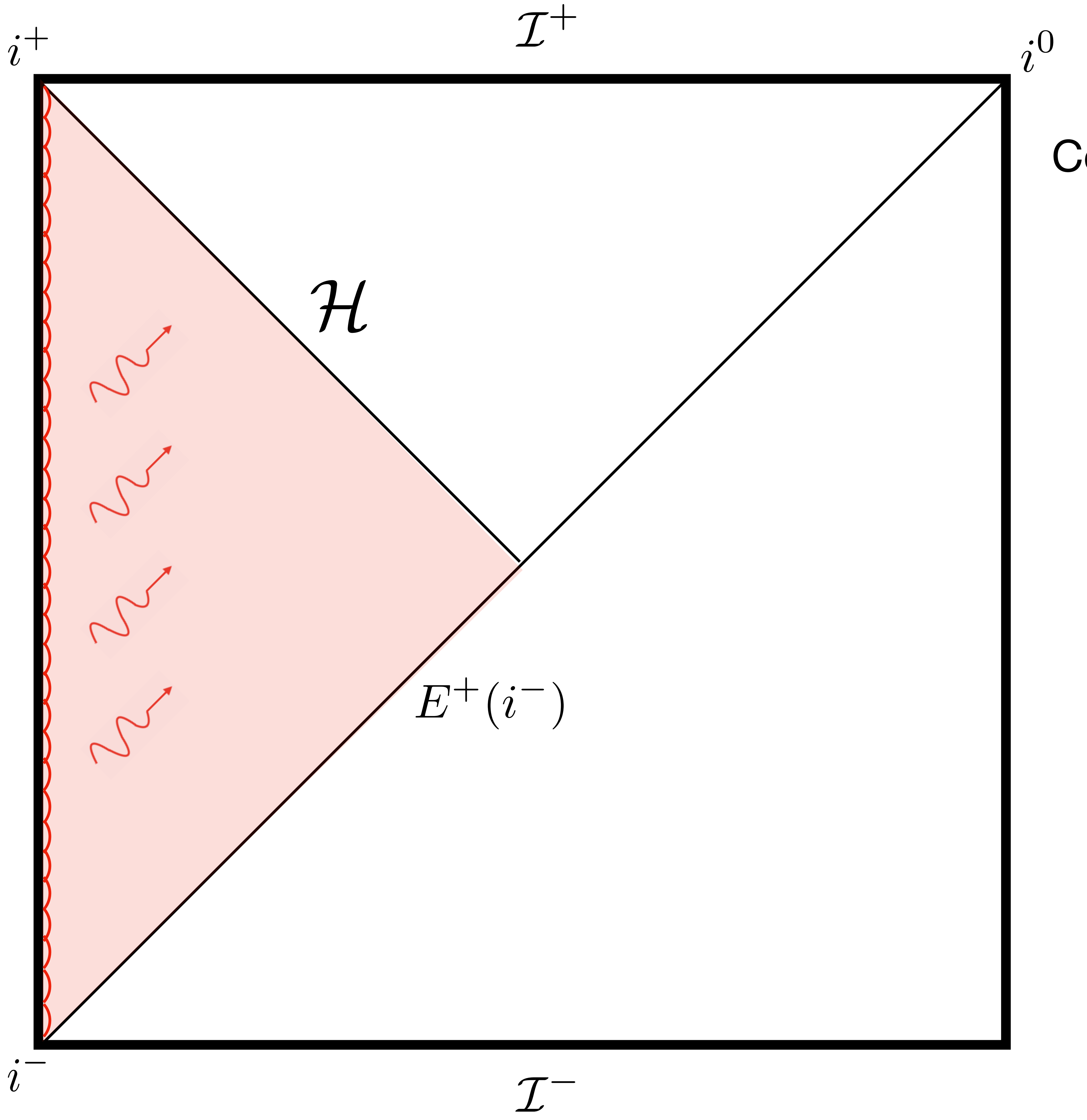

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Hubble parameter

$H := \sqrt{\Lambda/3}$

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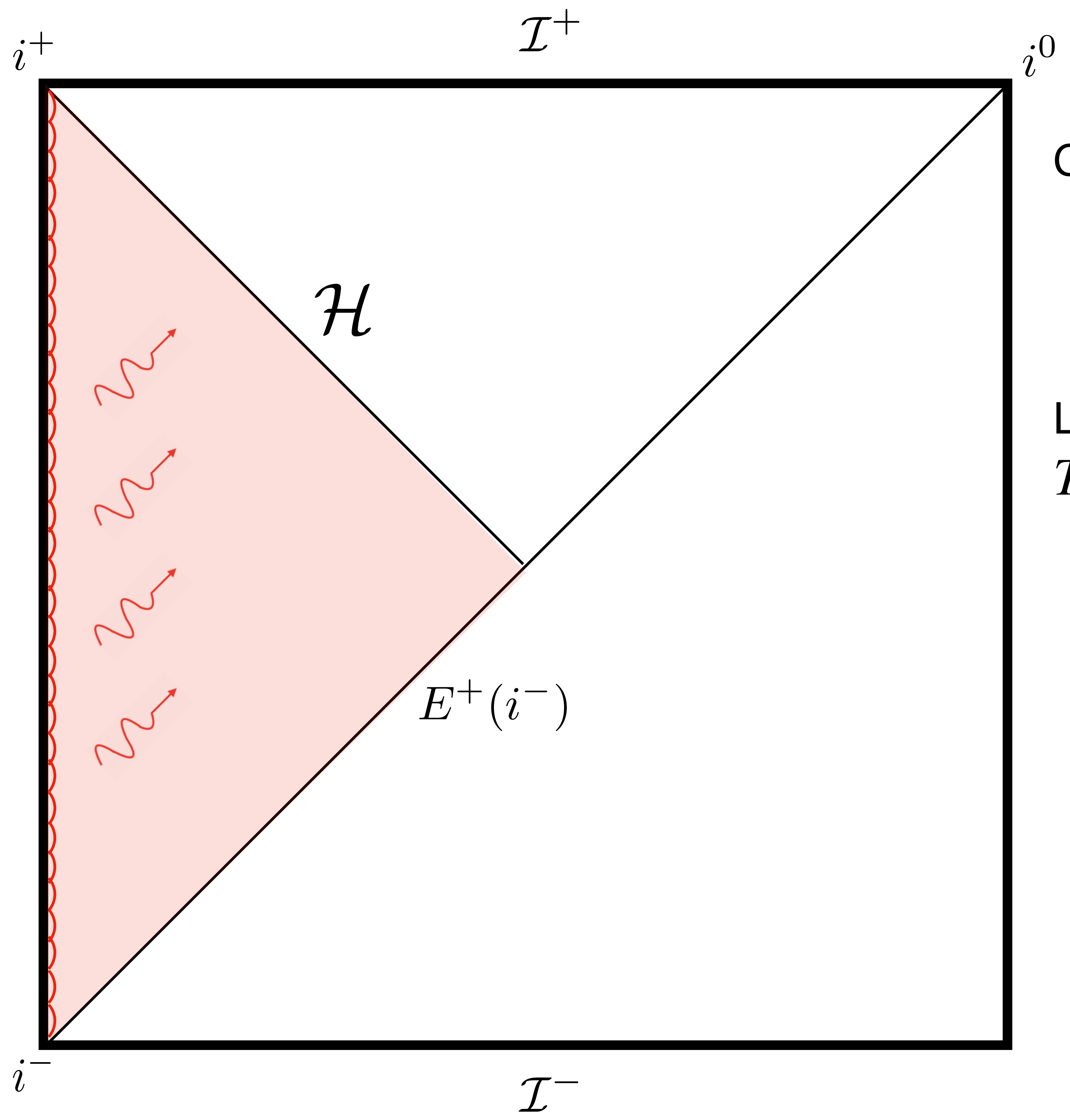

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Linearized field equation in the presence of the first order linearized source $T_{\alpha\beta}$:

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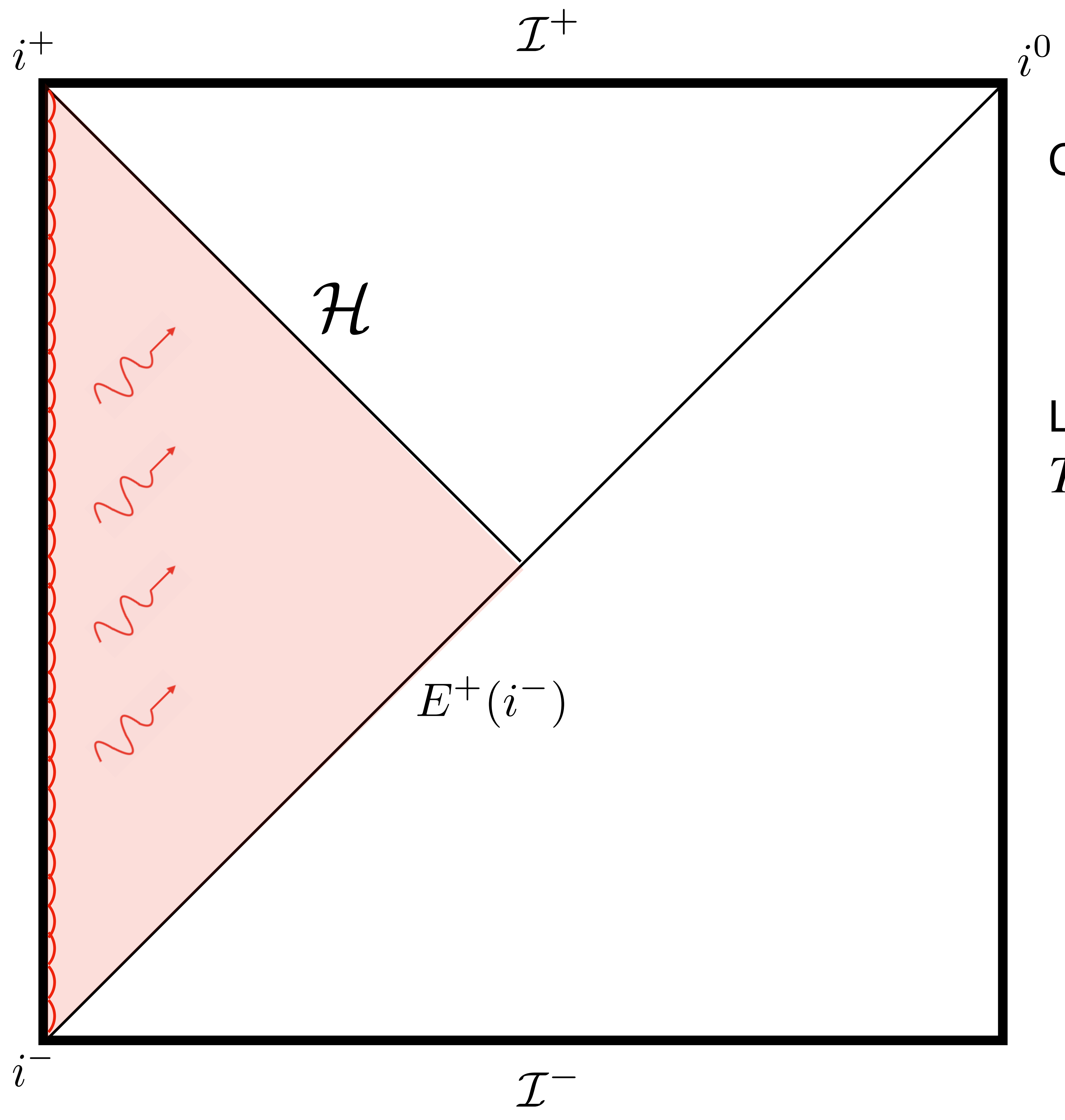

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$$\bar{\Box}\,\bar{\gamma}_{\alpha\beta} - 2\,\bar{\nabla}_{(\alpha}\,\bar{\nabla}^\mu\bar{\gamma}_{\beta)\mu} + \bar{g}_{\alpha\beta}\,\bar{\nabla}^\mu\,\bar{\nabla}^\nu\bar{\gamma}_{\mu\nu} - \frac{2}{3}\Lambda(\bar{\gamma}_{\alpha\beta} - \bar{g}_{\alpha\beta}\bar{\gamma}) = -\,16\pi T_{\alpha\beta}$$

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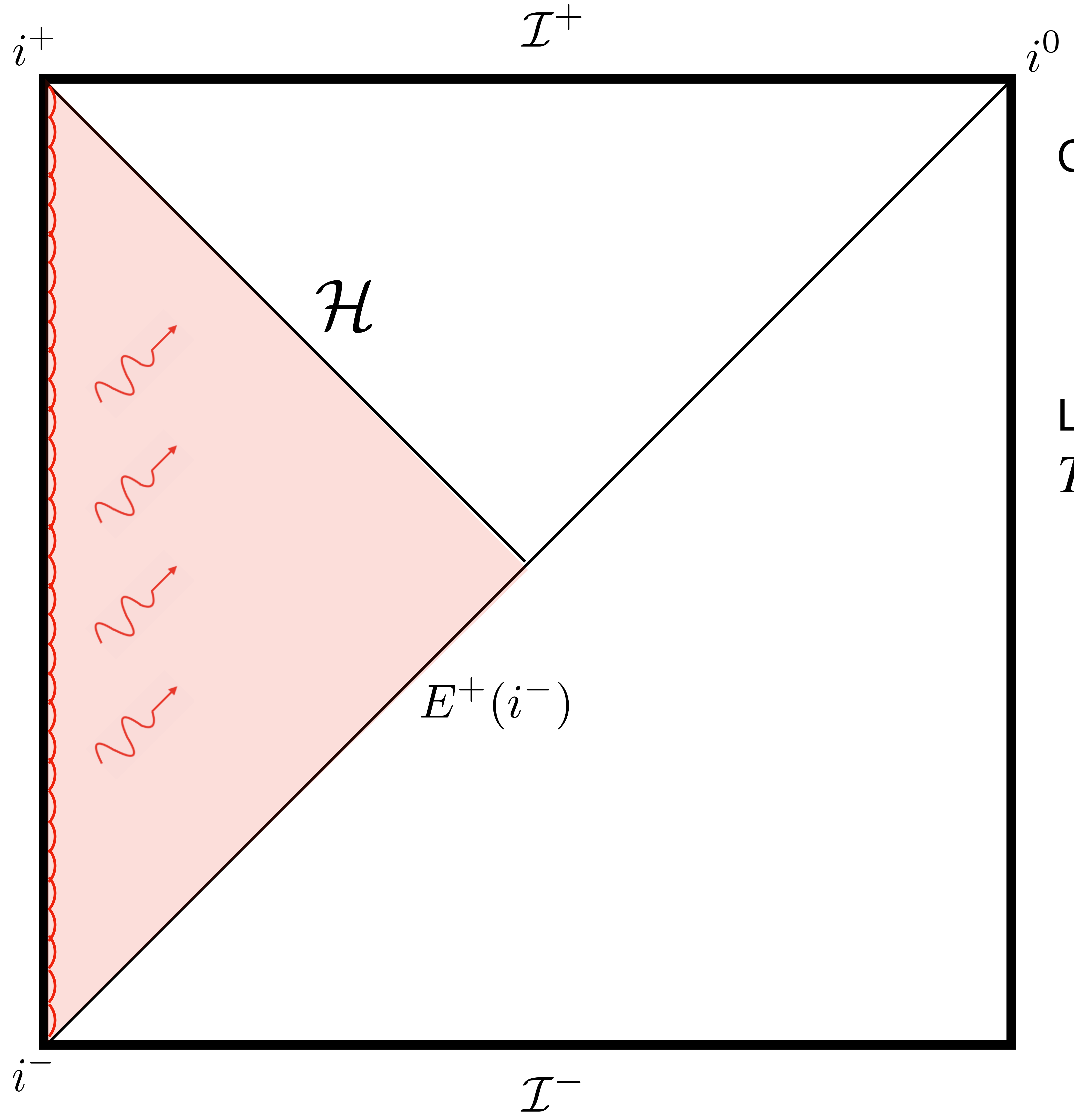

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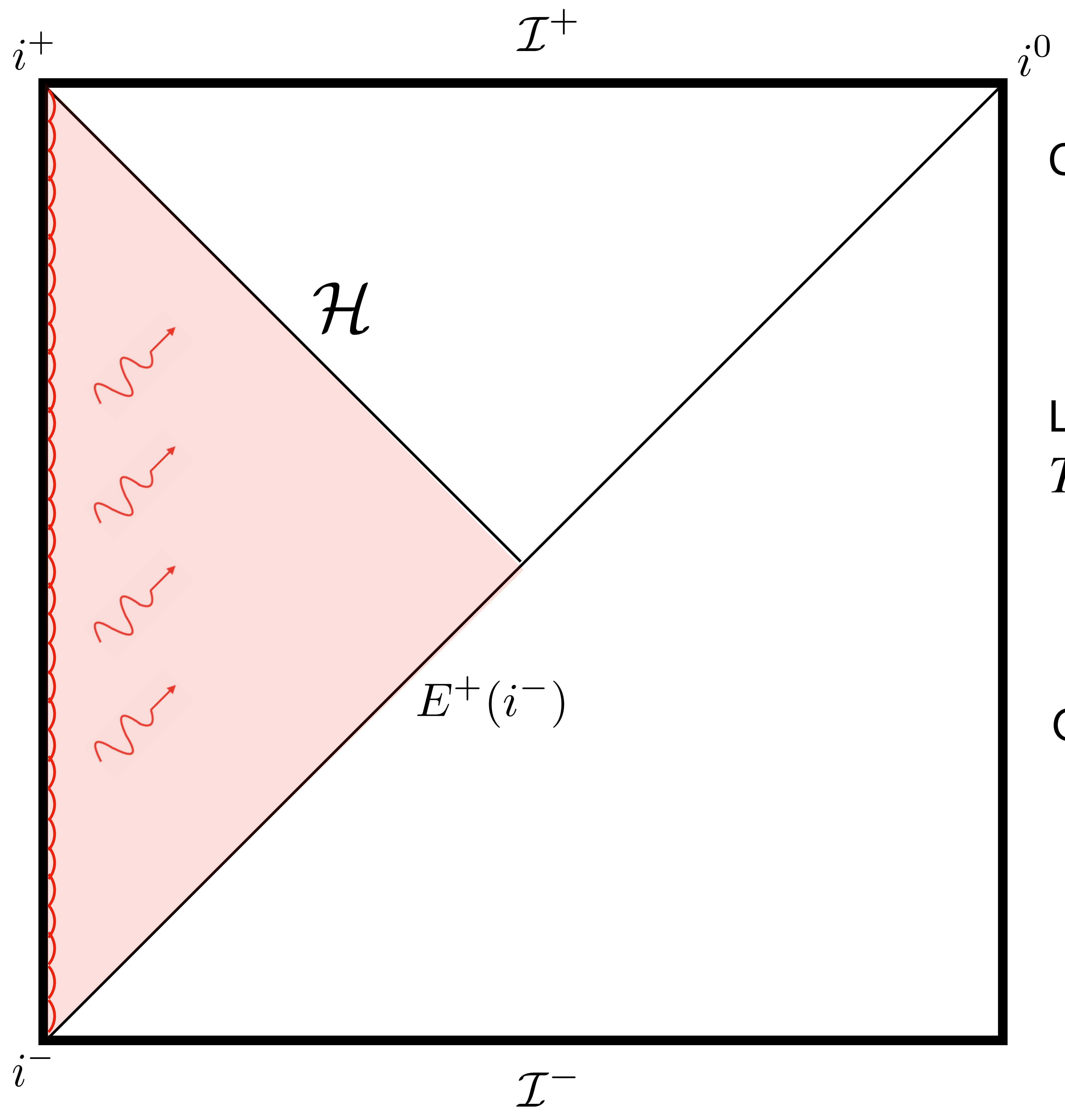
Trace-reversed metric perturbation:

$$\bar{\gamma}_{\alpha\beta} := \gamma_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}\,\gamma$$

Time changing matter source emitting gravitational radiation in de Sitter spacetime



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Gauge condition: $\bar{\nabla}^{\alpha}\bar{\gamma}_{\alpha\beta} = 2Hn^{\alpha}\bar{\gamma}_{\alpha\beta}$, where $n^{\alpha}\partial_{\alpha} = -H\eta\partial_{\eta}$

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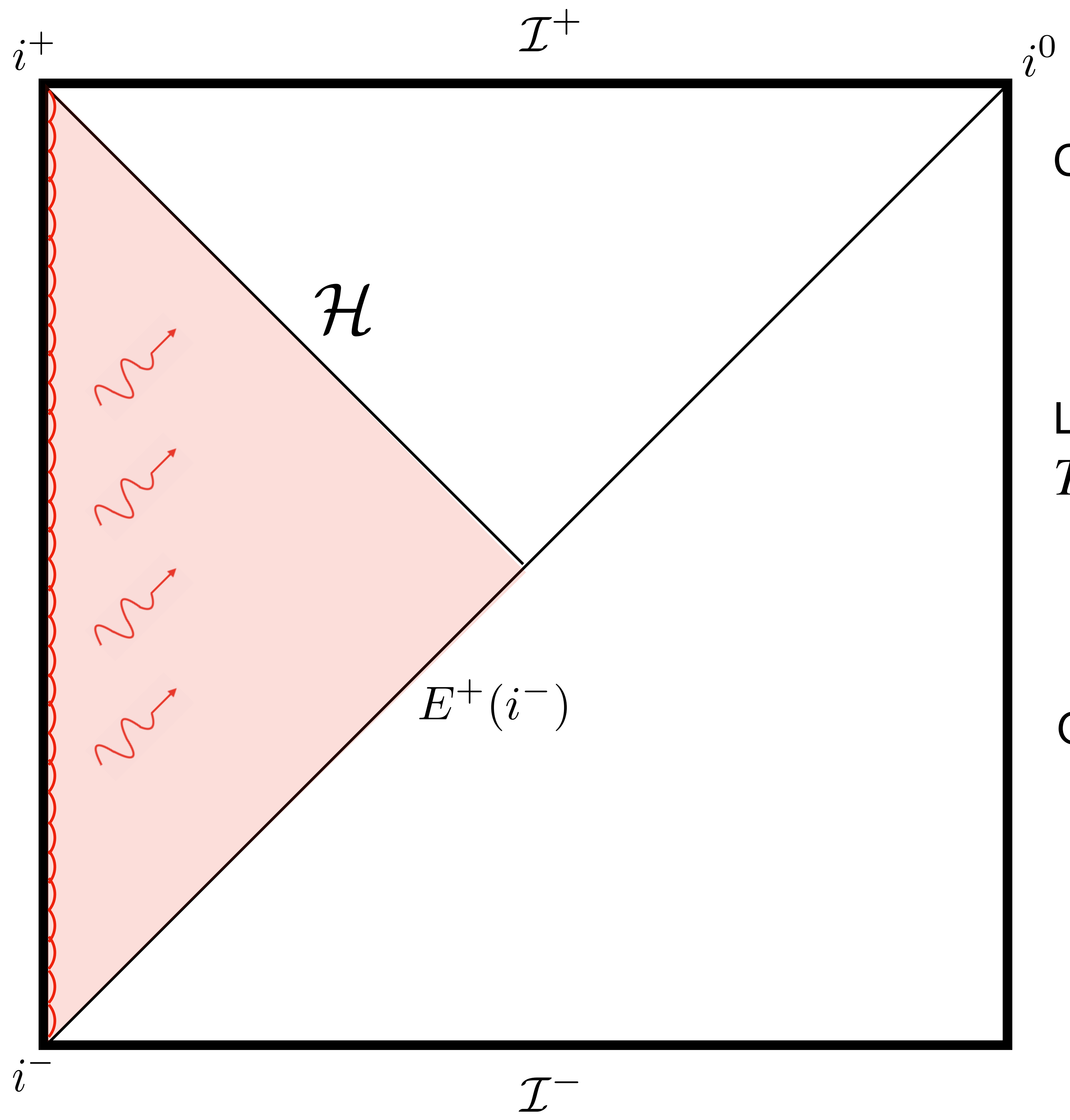

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future pointing unit normal to the cosmological slices $\eta = \text{const}$

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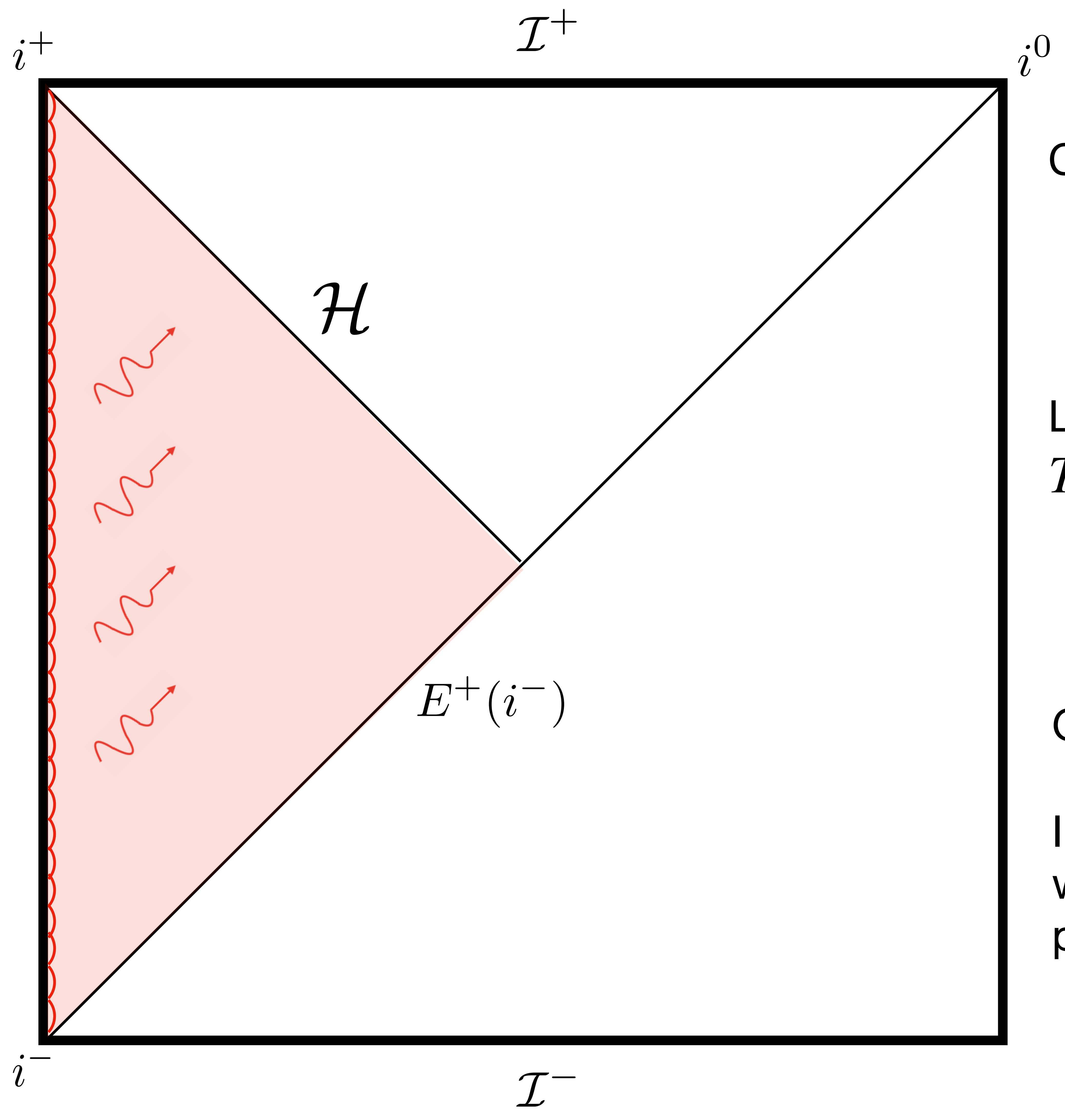

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In this gauge the linearized Einstein's equations simplify even more when rewritten in terms of the rescaled trace reverse metric perturbation:

$$\bar{\chi}_{ab} := H^2\eta^2\bar{\gamma}_{ab}$$

Solution to the linearized Einstein's equations with Λ

To solve the linearized Einstein equations in the presence of a linearized source it is convenient to perform decomposition of $\bar{\chi}_{\alpha\beta}$ and $T_{\alpha\beta}$:

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where $\mathring{q}^{\alpha\beta}$ is the spatial metric on cosmological slice $\eta = \text{const}$.

Einstein equations split into three equations:

$$\mathring{\Box}\left(\frac{1}{\eta}\tilde{\chi}\right) = -\frac{16\pi}{\eta}\tilde{\mathscr{T}}$$

$$\mathring{\Box}\left(\frac{1}{\eta}\chi_a\right) = -\frac{16\pi}{\eta}\mathscr{T}_a$$

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$$+4\int d^3\vec{x}' \int_{-\infty}^{\eta_{Ret}} d\eta' \frac{1}{\eta'}\partial_{\eta'}\mathscr{T}_{ab}(\eta', \vec{x}')$$

$$\eta_{Ret} := \eta - |\vec{x} - \vec{x}'|$$

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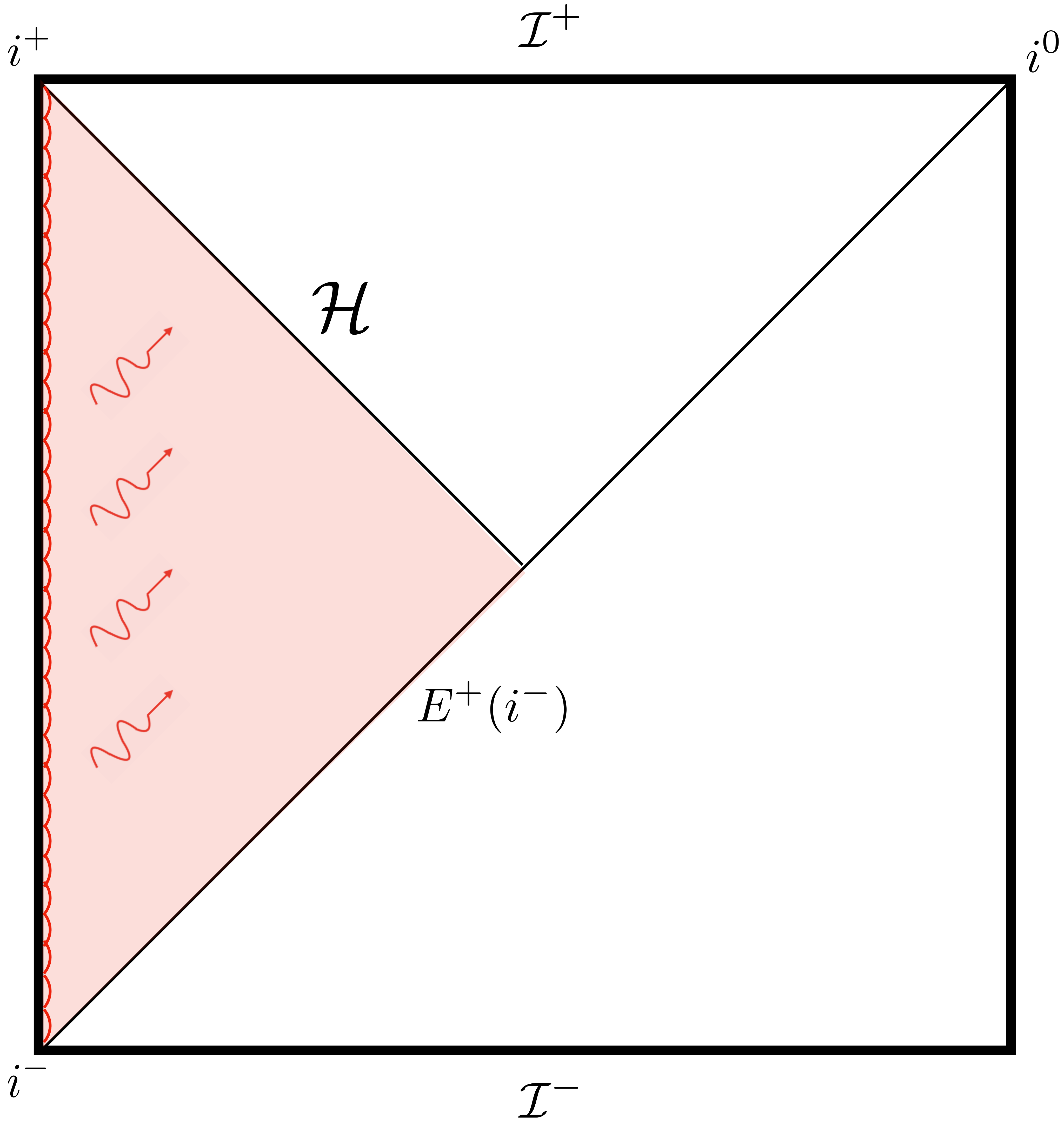
de Vega, Ramirez, Sanchez 1999

Physical set-up — assumptions & approximations



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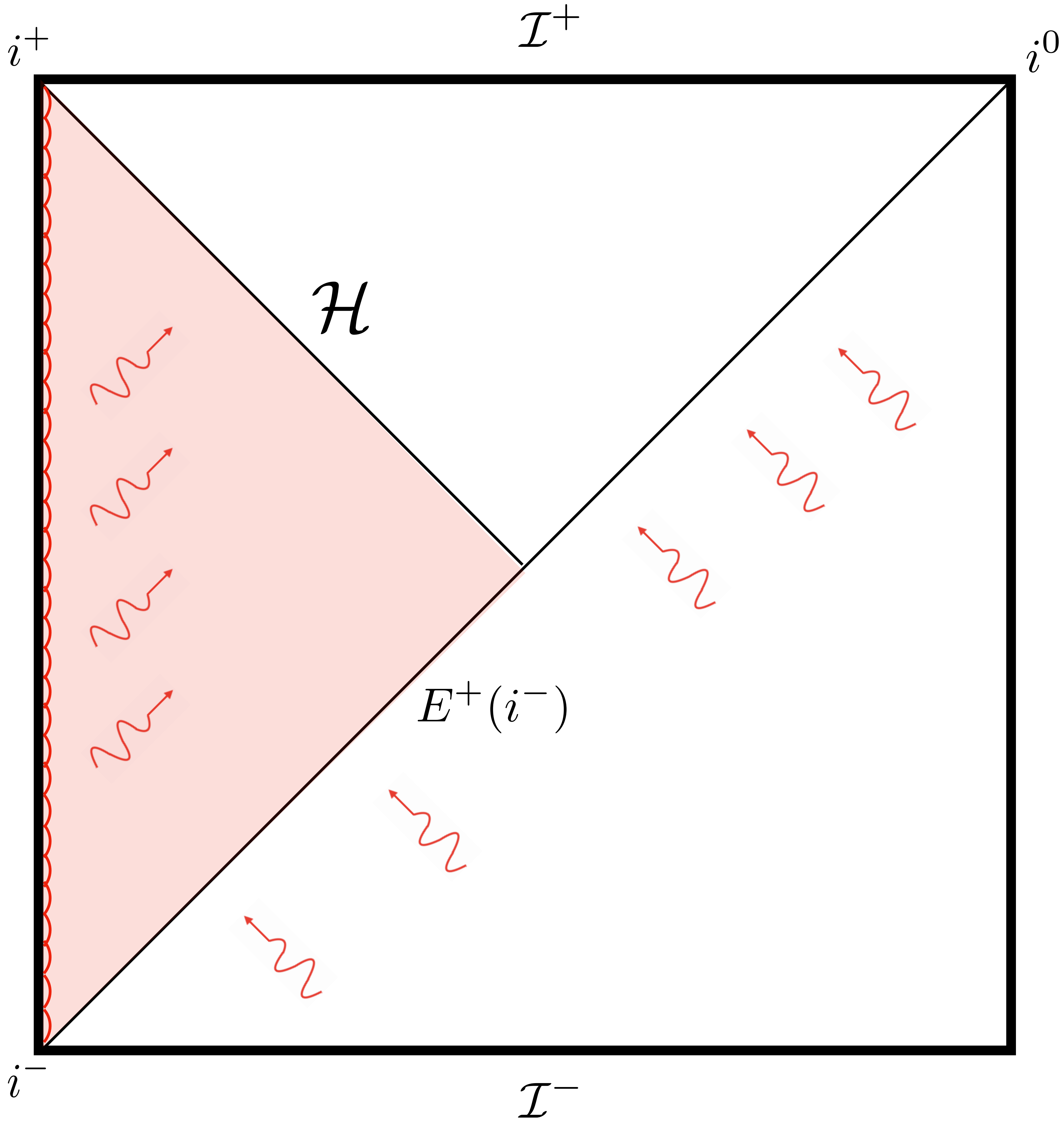

1. No incoming radiation from the past Poincaré patch

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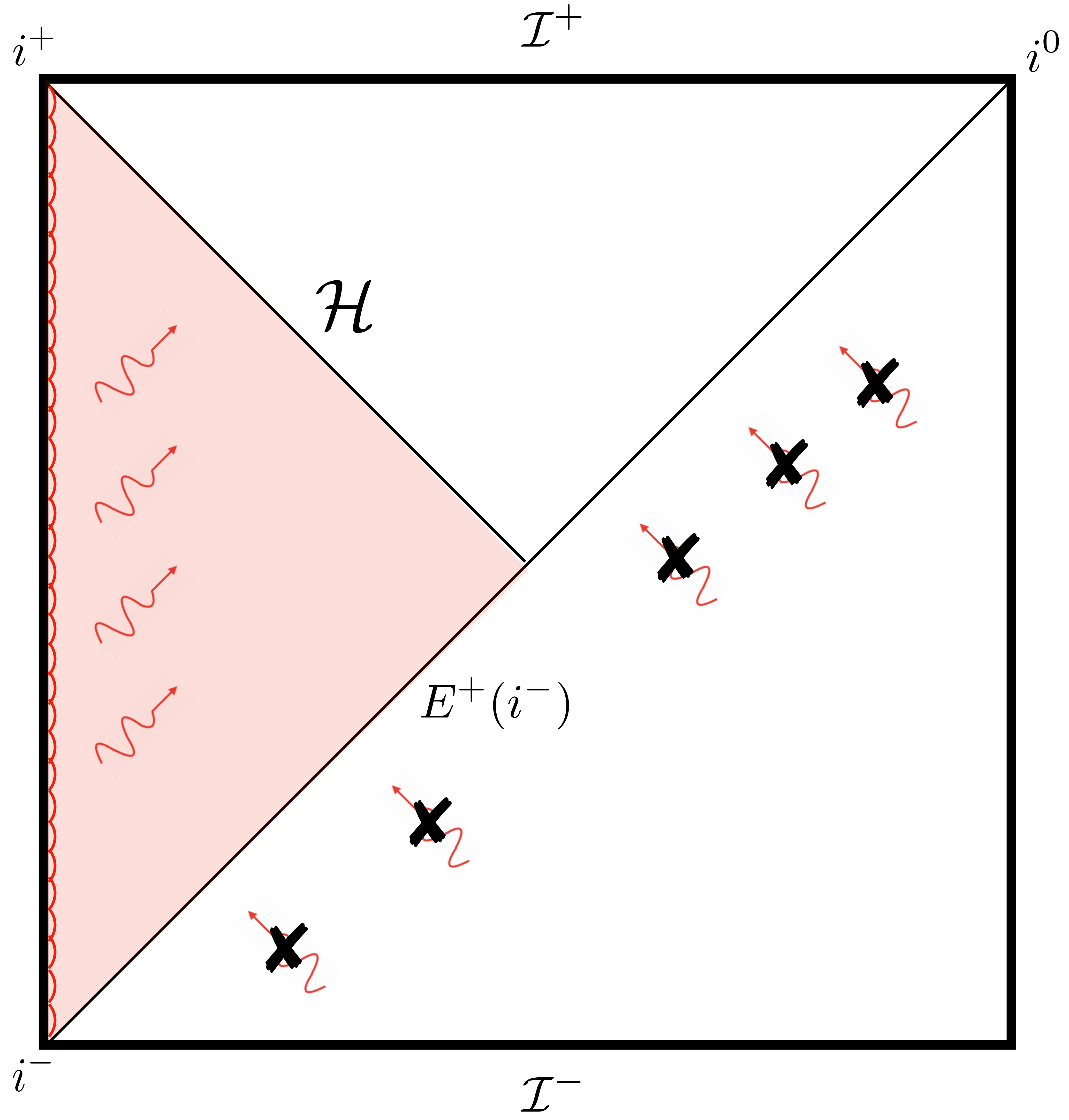

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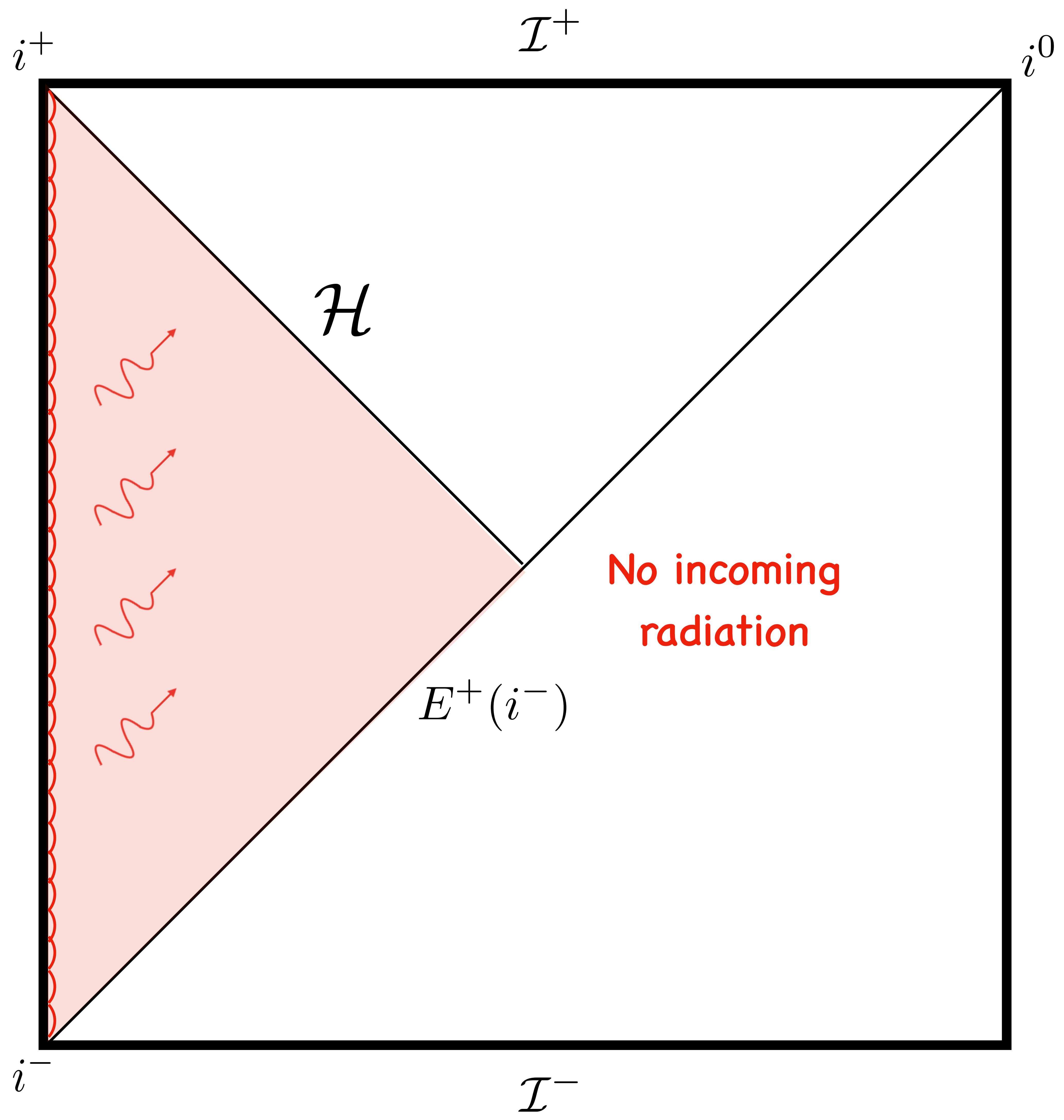

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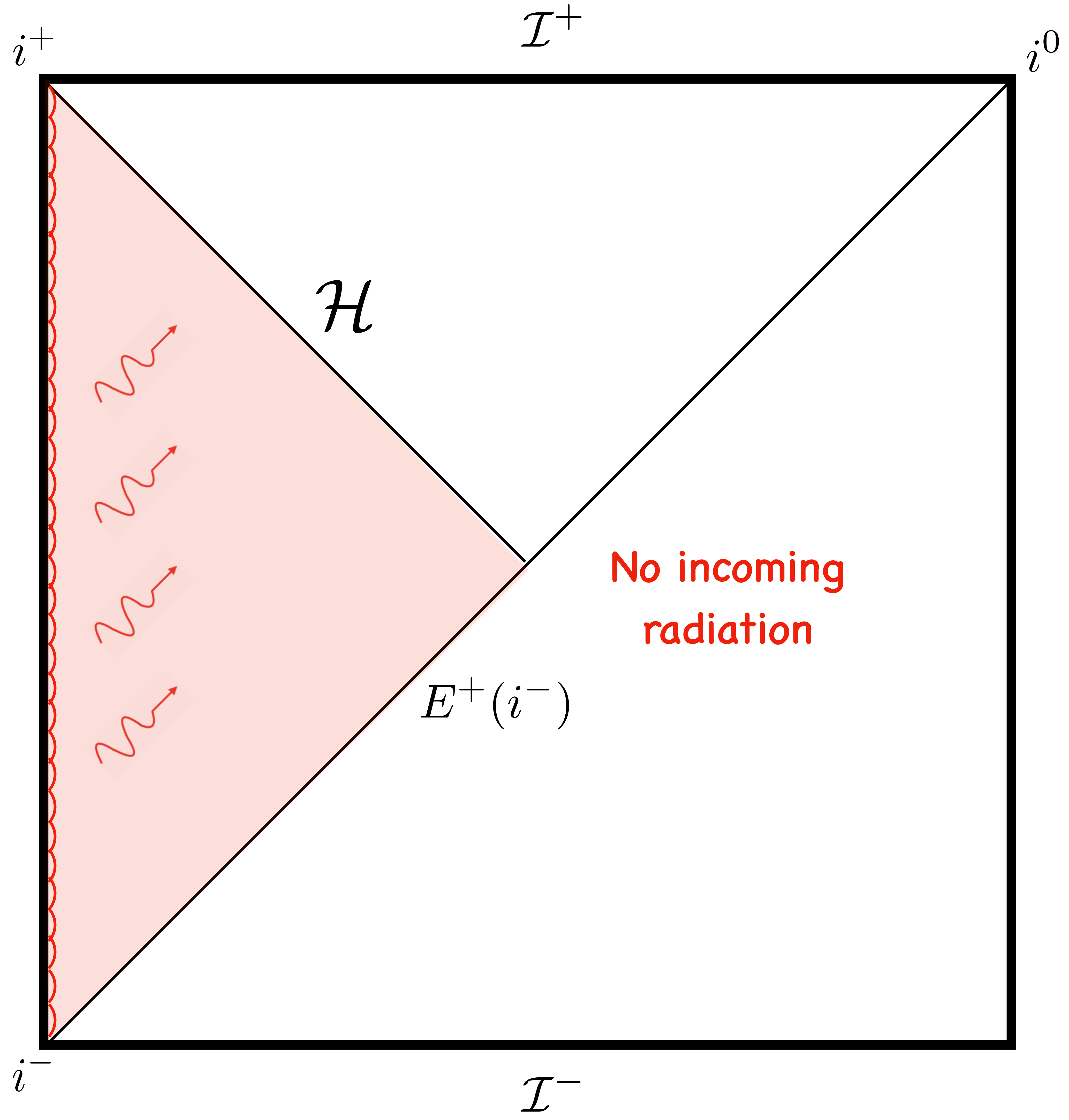

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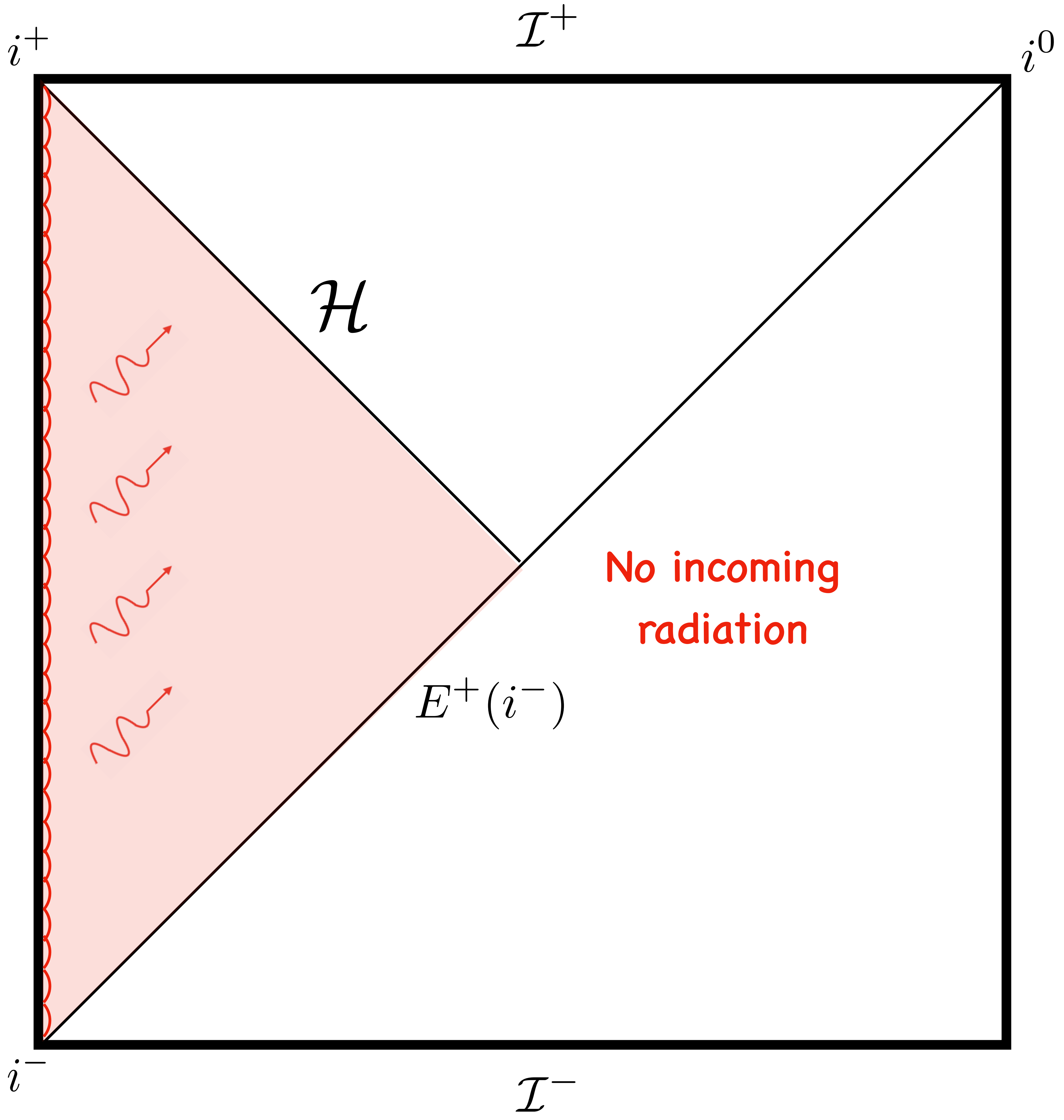

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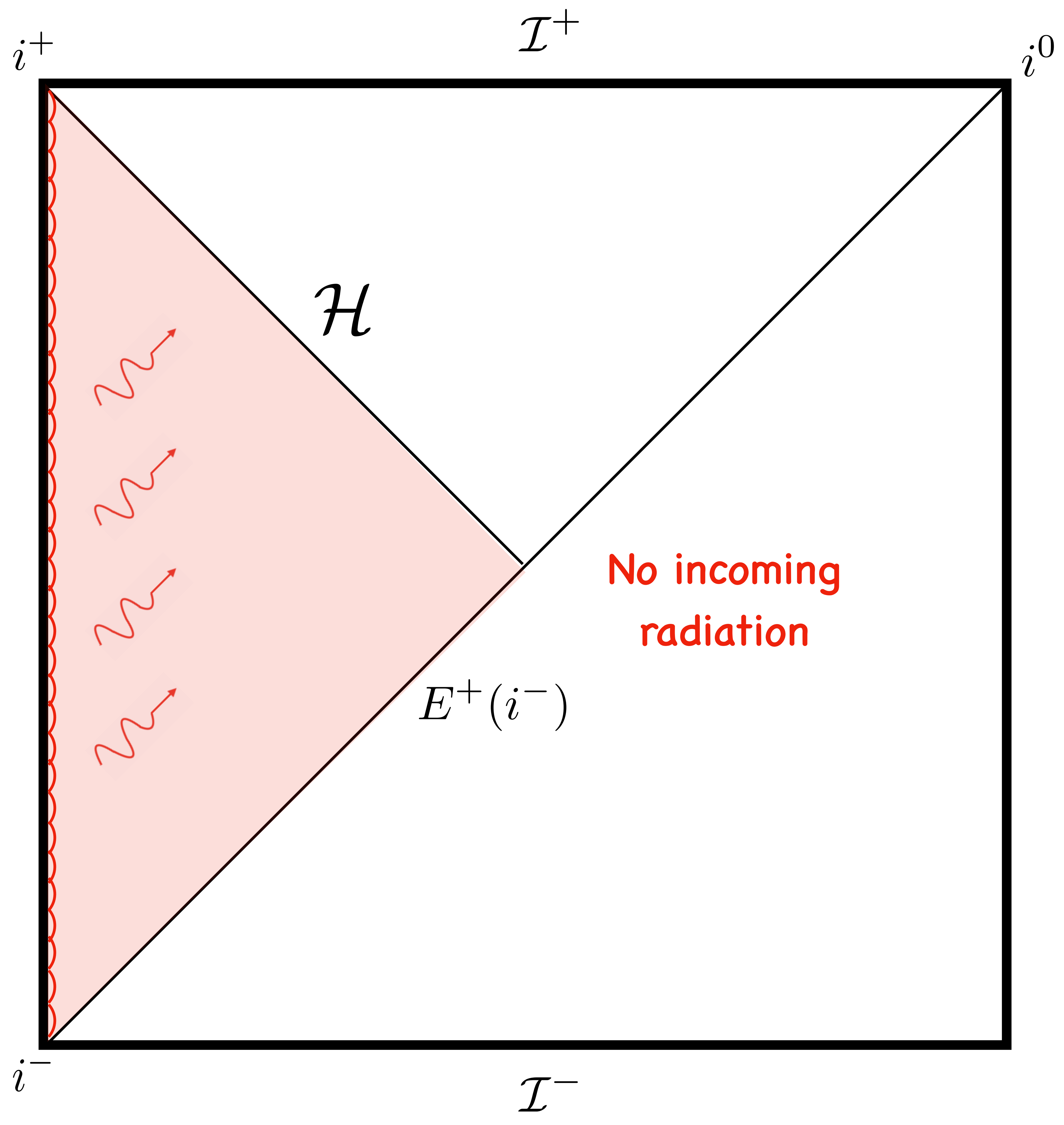

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5. Slow motion approximation - velocity is small compared to the speed of light: $v \ll 1$

Retarded solution χ_{ab} expressed in terms of quadrupole moments

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Notice that coordinates $(\eta, \overrightarrow{x})$ are not compatible for taking the limit $\Lambda \to 0$:

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$\rho = T_{ab}n^a n^b$ $\quad$ $\bar{x}_a := e^{Ht}x_a$ $\quad$ $p_i = T^{ab}\partial_a x_i \partial_b x_i$

Generalization of the quadrupole formula in de Sitter spacetime

— DDR & J. Lewandowski (2022)

General formula for the energy flux through a null surface

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The energy flux formula derived by Chandrasekaran et al. (2018) using Wald-Zoupas formalism (2000) is of the form:

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- E_ℓ → time translation → $T = \partial_t - Hr\partial_r$
- $\mathcal{N}$ → null surface → cosmological horizon $r = \frac{1}{H}e^{-Ht}$
- σ → shear of ℓ → shear of T
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Notice that the perturbed horizon $\mathscr{H}$ generically is not null and the above formula may not be applied.

To sustain its null character with respect to the perturbed geometry a suitable gauge is applied:

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It may be interpreted as a deformation procedure for $\mathscr{H}$ such that given the original perturbation of spacetime it remains null.

A generalization of the Einstein's quadrupole formula for positive cosmological constant

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Using the obtained retarded solution χ_{ab} expressed in terms of the quadruple moments we calculate the shear and expansion of the time translation T, to find:

$$E_T = \frac{1}{45}\int dt \sum_{i,j=1}^{3}\left[\left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3}\right)^2 + 2H\left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3}\left(\frac{d^2 q_{ij}^{(p)}}{dt^2} + 7\frac{d^2 q_{ij}^{(\rho)}}{dt^2}\right)\right)\right](t_{ret}) + \mathcal{O}(H^2)$$

where $$q_{ab}^{(i)} := Q_{ab}^{(i)} - \frac{1}{3}\mathring{q}_{ab}Q^{(i)}$$

The limit for $\Lambda \to 0$, or equivalently $H \to 0$, recovers the famous Einstein quadrupole formula:

Wald, 1984

$$E_T = \frac{1}{45}\int dt \sum_{i,j=1}^{3}\left[\left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3}\right)^2\right](t_{ret})$$

Generalization of the quadrupole formula in de Sitter spacetime

— A. Ashtekar, B. Bonga, A. Kesavan (2015)

Energy flux of the time-varying quadrupole moment across $\mathscr{I}^+$

The total energy flux across $\mathscr{I}^+$ is given by the Hamiltonian generating the time translation T^α on the covariant phase space, the result can be expressed in terms of the electric part of the Weyl tensor $-\ \mathscr{E}_{\alpha\beta}$ and the Lie derivative of the metric perturbation w.r.t. T^α :

$$E_T = \frac{1}{16\pi H}\int_{\mathscr{I}^+} d^3x \mathscr{E}_{cd}\Big(\mathscr{L}_T \chi_{ab} + 2H\chi_{ab}\Big)\mathring{q}^{ac}\mathring{q}^{bd}$$

Expressing the above formula in terms of the quadrupole moments and extracting just the transverse-traceless part of the term in the bracket (as $\mathscr{E}_{\alpha\beta}$ is TT) yields:

$$E_T = \frac{1}{8\pi}\int_{\mathscr{I}^+} dTd\Omega \ R_{ij}^{TT} \ R_{kl}^{TT} \ \mathring{q}^{ik}\mathring{q}^{jl} \ ,$$

where $\quad R_{ij} := \dddot{Q}_{ij}^{(\rho)} + 3H\ddot{Q}_{ij}^{(\rho)} + H\ddot{Q}_{ij}^{(p)} + \mathscr{O}(H^2)$, and the dot indicated the Lie derivative.

Generalization of the quadrupole formula in de Sitter spacetime

— S.J. Hoque & A. Virmani (2018)

Generalization of the quadrupole formula in the tt-projection on the cosmological horizon

Authors computed the flux integral of energy using the symplectic current density of the covariant phase space on the cosmological horizon:

$$E_T = \frac{1}{8\pi} \int_{\mathscr{H}} dt \; d\Omega \; R^{tt}_{ij} \; R^{tt}_{kl} \; \delta^{ik} \delta^{jl}$$

where again: $$R_{ij} := \dddot{Q}^{(\rho)}_{ij} + 3H\ddot{Q}^{(\rho)}_{ij} + H\ddot{Q}^{(p)}_{ij} + \mathscr{O}(H^2)$$

Generalized quadrupole formula - simplification for a source dynamically active for a finite time or a periodic source

Our quadrupole formula simplifies for the sources of compact support or periodic nature:

$$E_T = \frac{1}{45}\int_{t_0}^{t_1} dt \sum_{i,j=1}^{3} \left[\left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3} \right)^2 + 2H \left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3} \left(\frac{d^2 q_{ij}^{(p)}}{dt^2} + 7 \frac{d^2 q_{ij}^{(\rho)}}{dt^2} \right) \right) \right] (t_{ret}) + \mathcal{O}(H^2)$$

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where for a source of compact support: $t_0 = -\infty$ and $t_1 = \infty$

whereas for a periodic source: t_0 and $t_1 = t_0 + 2\pi k$

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We checked that <u>all of the generalizations of the quadrupole formula coincide</u> at least up to the linear terms in Hubble parameter H.

Example: power radiated by a binary system in a circular orbit on de Sitter spacetime (B. Bonga, J.S. Hazboun, 2018)

Power radiated by a binary system on de Sitter spacetime in terms of the reduced mass and angular velocity reads:

$$P = \frac{32}{5}\mu^2 R_*^4 \Omega^6 \left(1 + \frac{5}{12}\frac{\Lambda}{\Omega^2} + \frac{1}{36}\frac{\Lambda^2}{\Omega^4}\right),$$

where R_* is the relative physical separation between the two bodies, the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ and Ω is the physical angular velocity.

In high-frequency limit (expansion of the Universe can be neglected during the orbital cycles) this expression reduces to:

$$P = \frac{32}{5}\mu^2 R_*^4 \Omega^6 \left(1 + \mathcal{O}\left(\sqrt{\Lambda}\Omega^{-1}\right) + \mathcal{O}\left(\frac{m_1 + m_2}{R_*}\right) + \mathcal{O}\left(\frac{\mu}{R_*}\right)\right)$$

Therefore, standard expression for power radiated in Minkowski spacetimes also applies to de Sitter spacetime in high-frequency approximation. Note that high-frequency limit is critical for this equivalence.

Summary

- There are various approaches (Ashtekar et. al., Hoque & Virmani, DDR & Lewandowski) to obtaining a generalization of the quadrupole formula for radiated energy in de Sitter spacetime (energy flux across cosmological horizon vs future infinity, tt-projection vs TT-gauge).

- All of the formulae coincide up to the terms linear in the Hubble parameter $H = \sqrt{\Lambda/3}$. The zeroth order term recovers the famous Einstein's quadrupole formula obtained for the perturbed Minkowski spacetime.

- Bonga & Hazboun: in order to probe the cosmological constant by measuring the power, one needs to go beyond the high-frequency approximation and could calculate the corrections, due to the background curvature, on the power. It would then allow one to observe the cosmological constant through the power emitted by gravitational waves.

- Hope: NANOGrav exploration of the low-frequency gravitational waves via radio pulsar timing.