Phenomenological Impact of Non-Extensivity on Black Hole Thermodynamics

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What am I working on

- Constructing consistent thermodynamics of black holes for various non-extensive entropies.
  - Tsallis-q Renyi
  - Sharma-Mital
  - Barrow
  - Tsallis-Cirto
  - Kaniadakis
- Finding the effect of these entropies on the sparsity of the radiation under generalised uncertinity principle.
- Bounding the variables introduced by these variables through cosmological data, mainly through the idea of holographic dark energy,.

- The laws of black hole thermodynamics are analogous to the laws of classical thermodynamics.
- Entropy scales with area, as well known from Bekenstein's Area Law (Bekenstein 73) whereas temperature scales with surface gravity (Hawking 74).

$$S_{bh} = \frac{k_B A c^3}{4G\hbar}, \; T_{bh} = \frac{\hbar\kappa}{2\pi k_b c}.$$

$$A = 4\pi r_+^2, \; r_+ = 2GM/c^2, \; \kappa = \frac{1}{2}\partial_r g_{tt} = c^4/4GM \tag{1}$$

4-Laws of Black Hole Thermodynamics

- The surface gravity is constant on the event horizon of stationary black holes. It is proportional to the black hole's temperature.

$$dE = \frac{\kappa}{8\pi} dA + \Omega_H dJ + VdQ \tag{2}$$

- Horizon Area is strictly increasing with time
- One can not reach to zero temperature in a physical process.

Postulate 1

There exist equilibrium states characterized by state functions X_i; time independence is our equilibrium creteria.

Postulate 2

There exists a first-order homogeneous function of the extensive macroscopic variables called entropy S. States for which $(\partial E/\partial S)_{N,V} = 0$ have zero entropy.

Postulate 3

S is additive over subsystems. It is a continuous, differentiable, and increasing function of total internal energy E.

In Gibbs statistical mechanics, entropy is defined as

$$S = -k_B \sum_i p_i \log p_i \quad \to S = k_B \log W \tag{3}$$

The additivity of subsystems is only valid when one has shot-range interactions, i.e.

$$E = E_1 + E_2, \text{ so that } S(E, V, N) = S_1(E_1, V_1, N_1) + S_2(E_2, V_2, N_2) \quad (4)$$

But more generally

$$E = E_1 + E_2 + E_{12}, \quad E_{12} \rightarrow \text{Interraction Term} \quad (5)$$

but $E_i \approx V_i$ volume of the subsystem, and if interaction is short range, $E_{12} \approx A$, area seperating subsystems, as $V \rightarrow \infty$, $A/V \rightarrow 0$, so the energy is additive.

Thus In classical Gibbs SM

When this assumption fails (such as gravitational long-range interactions), the thermodynamic limit ($V \rightarrow \infty$) may depend on the shape of the system.

Definition: Extensivity

Let us define a function f, the fundamental relation of thermodynamic variables (X_0, X_1,X_2,...,X_k) such that $X_0 = f(X_1, X_2, ..., X_k)$. Here, f is homogeneous first order function of X_1,X_2,...,X_k. If $f(aX_1, aX_2, ..., aX_k) = af(X_1, X_2, ..., X_k)$ for every positive real numbers a for all X_1, X_2, ...X_k then X_0 is extensive. The thermodynamic variables X_i can be the energy U, entropy S and mole number N and expressing f in differential form will give the first law of thermodynamics.

In Gibbs thermodynamics, the fundamental relation f for the entropy S can be written as $S = f(U, V, N)$ for an ideal case and and $f(aU, aV, aN) = af(U, V, N)$, hence S is extensive.

Factorasibility of Non-Extensivity

Non-extensive statistical mechanics is the given names to formalisms that aims to generalise the Gibbs entropic form to account for long range interactions.
These in general deforms the composition rule

$$S_T = S_1 + S_2 + f(S_1, S_2) \tag{6}$$

However, in order to be able to define a physical temperature scale, the first law has to be factorizable. [1] This strictly limits the form of the functional $f(S_1, S_2)$.

Ex: For a power law $f(S_1, S_2) = \lambda_1 (S_1\, S_2)^{\lambda_2}$, possible functionals;

- $\lambda_1 = 0 \rightarrow \beta = \partial S / \partial U$
- $\lambda_2 = 1/\lambda_1 = 1/2 \rightarrow \beta^* = \partial S / \partial U \times 1/\sqrt{S}$
- $\lambda_1 \neq 0, \lambda_2 = 1 \rightarrow \partial S / \partial U \times 1/(1 + \lambda_1 S)$

[1] In other words, while minimizing the composition rule via the Lagrange multiplier method, the thermodynamic quantities of different bodies need to be seperable. (Biró et al. 2011), (2301.00609).

$W(N) \approx A\xi^N$	$W(N) \approx BN^\tau$	$W(N) \approx C\nu^{N^\gamma}$
$S = -k_B \sum_i p_i \log p_i$	$S_q = k_B \frac{1-\sum_i p_i^q}{q-1}$ $\overline{S} = \frac{1}{1-q} \log \sum_i (p_i)^q$ $S_{SM} = \frac{1}{1-r}\left(\left(\sum_i p_i^q\right)^{\frac{1-r}{1-q}} - 1\right)$	$S_\delta = k_B \sum_i^W p_i (\log \frac{1}{p_i})^\delta$ where $(\delta > 0)$

If a system is composed of N elements and these elements are independent, we have

$$W(N) \approx A\xi^N (A > 0, \xi > 1; N \to \infty) \quad \to \quad S_{BG} \approx \log W(N) \approx N \quad (7)$$

$$W(N) \approx BN^\tau (B > 0, \tau > 0; N \to \infty) \quad \to \quad S_q \approx \log_q W(N) \approx N \quad (\text{for} \;\; q = 1 - \frac{1}{\tau}) \quad (8)$$

Tsallis and Rènyi

C. Tsallis, in his 98 paper [2] proposed the following generalization to Gibbs entropy as

$$S_q = k_B \frac{1 - \sum_i p_i^q}{q-1} = k_B \sum_i p_i \frac{1 - p_i^{q-1}}{q-1} \qquad (9)$$

where the new form of q-entropy scales with p_i^q instead of p_i as $q \in \mathbb{R}$. The parameter q determines the degree of nonextensivity. Extremizing S_q under $\sum_i p_i = 1$ with the Lagrange multiplier method under equiprobability (i.e., $p_i = 1/W$) yields the micro-canonical entropy.

$$S_q = k_B \frac{W^{q-1} - 1}{q-1} \quad \text{where} \quad \lim_{q \to 1} S_q = S \qquad (10)$$

[2] for a brief review check (cond-mat/0406178v5)

In general, we can define the following q-functions as

$$\log_q \chi = \frac{\chi^\lambda - 1}{\lambda} \qquad \exp_q \chi = (1 + \lambda\chi)^{1/\lambda} \tag{11}$$

where $\chi > 0$ and $\lambda, \chi \in \mathbb{R}$. These q-logarithm and q-exponential are generalizations of the standard logarithm and exponential functions, where in the $\lambda \to 0$ limit; they reduce to their standard forms.[3]
Since

$$\log_q W_1 W_2 = \log_q W_1 + \log_q W_2 + \lambda \log_q W_1 \log_q W_2 \tag{12}$$

We end up with the following modified first law.

$$S_q(W_{Tot}) = S_q(W_1) + S_q(W_2) + \frac{\lambda}{k_B} S_q(W_1) S_q(W_2) \tag{13}$$

[3]We also use $q - 1 \to \lambda$ interchangeably, $q \to 1$ implies $\lambda \to 0$.

Now assume two independent systems A and B with configurations $\Omega^A = \{1, ..., W_A\}$ and $\Omega^B = \{1, ..., W_B\}$. Let $p_{ij}^{A\cup B}$ be joint probability, independence means $p_{ij}^{A\cup B} = p_i^A p_j^B$, $\forall(i,j)$ most generally one can write

$$\sum_{i,j}^{W_A,W_B} (p_{ij}^{A\cup B})^q = \sum_1^{W_A}(p_i^A)^q \times \sum_1^{W_B}(p_i^B)^q \to \overline{S}_q^{A\cup B} = \overline{S}_q^A + \overline{S}_q^B \quad \textit{(additivity)} \tag{14}$$

For arbitraty q parameter, $\overline{S}_q$ reproduces Rényi entropy (A.Rényi 70),(Tsallis 98)

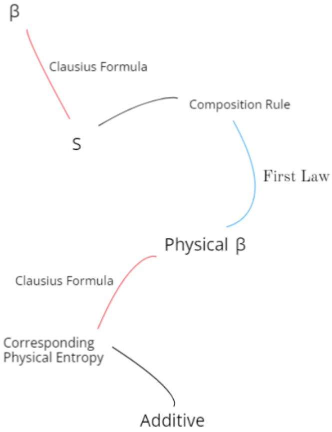
.

$$\overline{S}_q = \frac{1}{1-q}\log\sum_i (p_i)^q \quad \to \quad \overline{S}_q = k\frac{\log(1+(1-q)S_q/k)}{1-q} \tag{15}$$

Rényi entropy and Tsallis q entropy are related to each other, and by minimizing their respective first laws we end up with the same physical temperature (2301.00609),(cond-mat/0011012).

Renyi	additive	Systems are independent
Tsallis q	non-additive	Interraction dependent

Flowchart


β
Clausius Formula
Composition Rule
S
First Law
Physical β
Clausius Formula
Corresponding
Physical Entropy
Additive

Application of q-Entropy

- HEP → (PRD 87,114007), (PRL 105,022002), (EPJC 71,1655), (PRD 83,052004)
- Spin Glasses (PRL 102, 097202)
- Cosmic Ray Fluxes (PLA 310, 372)
- Anamolous Diffusion (PRL 105, 260601), (PRL 107, 088901)
- Cold atom in optical latices (PRL 96, 110601)
- Solar Physics (Astrophys J. 644, L83), (Astrophys J. 737, 35)

Why and where people are using it

- Physically, q is the result of the introduction of a bias in the microstates due to the presence of long-range interraction or strong coupling in the system. (Due to gravity, for example.) (cond-mat/0612032)
- Blackhole entropy is not extensive (1202.2154)
- In the cosmological context this statistics is used via the assumption that $S_{BH} \approx S_q$.
- Mainly it is used in Cosmological models such as holographic dark energy, for tensions.(1804.02983), (2312.16901)
- In LQG, with the help of S_q, area law can also be calculated from the available microstates, $\gamma = (\log 2/\pi\sqrt{3}) \times (1-q)A/4 \log[1 + (1-q)A/4]$ [4](1703.09355).

[4] $\lim q \to 1 = \log 2/\pi\sqrt{3}$

We investigate the aforementioned entropies tied to the Tsallis-q entropy [5] near the evaporation phase with the help of the Generalized Uncertinity Principle (GUP).[6]

$$\Delta x \Delta p \geq \hbar[1 + \alpha_0 \frac{l_p^2}{\hbar^2} (\Delta p)^2] \tag{16}$$

The thermodynamic quantities with GUP correction are:

$$T_{Gup} = T_H \mathcal{K}(\alpha, M) \ \text{ where, } \ \mathcal{K}(\alpha, M) = \frac{2}{1 + \sqrt{1 - \frac{\alpha}{4GM^2}}} \tag{17}$$

$$S_{Gup} = \frac{S_B}{\mathcal{K}} - \frac{\alpha\pi}{2} \log\left[\frac{4M}{m_0 \mathcal{K}}\right] \quad \text{and} \quad C_{Gup} = C_{Sc}\left[\frac{2 - \mathcal{K}}{\mathcal{K}^2}\right] \tag{18}$$

[5] And the other non-extensive entropic forms as well

[6] For a nice rewiev (2305.16193v2), also (Scardigli 99), (1203.6191) and (hep-th/9309034).

Sparsity of Radiation

Another important aspect of the Hawking radiation is that it is extremely sparse (1506.03975). The sparsity can be found by integrating the flux of radiated particles through the cross-sectional area.

We can define sparsity as:

$$\eta = \frac{C}{g}\left(\frac{\lambda_{thermal}^2}{A_{eff}}\right) \tag{19}$$

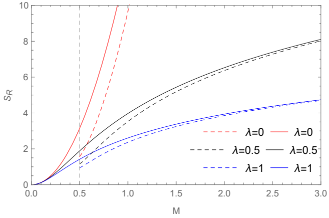
where C is a dimensionless constant, g is the spin degeneracy factor. A_{eff} is not the horizon, but it is related to it as $A_{eff} = 27A/4$.

For massless bosons, $\lambda_{thermal}/A_{eff} = 64\pi^3/27$ is much larger than the classical range, which is generally unity.

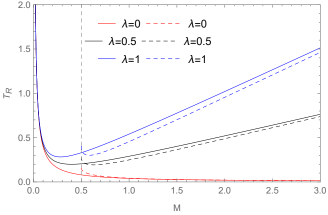
$$S_{RGUP} = \frac{k_B}{\lambda} \log \left(\pi\lambda \frac{M^2}{m_p^2} \mathcal{H}(M) - \frac{1}{2}\pi\alpha\lambda \log \left(\frac{M}{M_0} \left(\sqrt{4 - \alpha m_p^2} + 2 \right) \right) + 1 \right) \quad \text{where} \quad \mathcal{H}(M) = \left(\sqrt{4 - \frac{\alpha m_p^2}{M^2}} + 2 \right)$$

$$T_{RGUP} = \frac{c^2 \left(2 \left(\pi\lambda M^2 \mathcal{H}(M) + m_p^2 \right) - \pi\alpha\lambda m_p^2 \log \left(\frac{M}{M_0} \mathcal{H}(M) \right) \right)}{4\pi k_B M \mathcal{H}(M)} \tag{21}$$

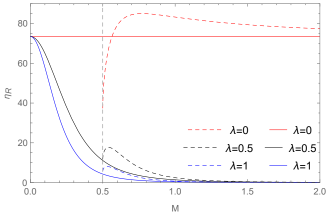
$$\eta_{RGUP} = \frac{128\pi^3 C m_p^4 M^2 \mathcal{H}(M)^2}{27g \left(2M^2 \mathcal{H}(M) - \alpha m_p^2 \log \left(\frac{M}{M_0} \mathcal{H}(M) \right) \right) \left(\pi\alpha\lambda m_p^2 \log \left(\frac{M}{M_0} \mathcal{H}(M) \right) - 2 \left(\pi\lambda M^2 \mathcal{H}(M) + m_p^2 \right) \right)^2} \tag{22}$$



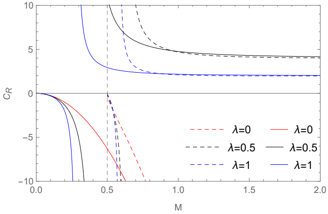
(a) Entopy



(b) Temprature



(a) Sparsity



(b) Heat Capacity

Implications

- For all analytic solutions $\lambda \to 0$ limit yields Bekenstein entropy with GUP corrections, while $\alpha \to 0$ limit gives the classical version of the entropy.
- Entropy suppressed logarithmically (mimics an AdS blackhole) for $0 < \lambda < 1$.
- Like in the GUP corrected Bekenstein case around $M = \alpha m_p^2/4$ there is a cut-off to the approach of GUP.
  - The solutions yield complex values behind this value. This "object" is called a black hole remnant (gr-qc/0511054),(1908.03498). There are a lot of hypotheses about their nature and existence, but for us, it is a relic of the cutoff we use. The object after this point is up to QG to solve.
- There is a characteristic mass scale $\mathcal{M}_{RGUP}$ where the black hole becomes thermodynamically stable. It is the solution to the following identity for the GUP case:.

$$\lambda \approx \frac{m_p^2}{4\pi M^2} + \frac{3\alpha m_p^4}{64\pi M^4} + \frac{\alpha m_p^4 \log(\frac{4M}{m_p})}{32\pi M^4} \tag{23}$$

- The mass scale (or energy in a general sense) is a property of Rényi entropy; however, Gup modifies this length scale.
- We see that introducing the non-extensivity parameter λ decreases the sparsity profile of the black hole.
- That is the time frame between each successive photon in lower, The thermal wavelength of associated photons is much smaller, and black hole radiation is more classical (like a black body).

$$\eta_{RGUP} = \frac{128\pi^3 C m_p^4 M^2 \mathcal{H}(M)^2}{27g\left(2M^2\mathcal{H}(M) - \alpha m_p^2 \log\left(\frac{M}{M_0}\mathcal{H}(M)\right)\right)\left(\pi\alpha\lambda m_p^2 \log\left(\frac{M}{M_0}\mathcal{H}(M)\right) - 2\left(\pi\lambda M^2\mathcal{H}(M) + m_p^2\right)\right)^2} \tag{24}$$

Conclusion

What have we done so far:

I. Çimdiker, M. P. Dabrowski and H. Gohar, "Equilibrium temperature for black holes with nonextensive entropy," Eur. Phys. J. C **83** (2023) no.2, 169 doi:10.1140/epjc/s10052-023-11317-0 [arXiv:2208.04473 [gr-qc]].

There were inconsistencies in the non-extensive thermodynamics of the black hole framework in many papers in the field.

- The main approach in the field was generalizing entropy while using Hawking temperature to arrive at thermodynamic quantities.
- By not using consistent thermodynamic quantities, many authors would arrive at non-physical internal energies and mass values, which are rich (!) from the physics side but inconsistent in their foundations.

In our second paper

I. Cimidiker, M. P. Dabrowski and H. Gohar, “Generalized uncertainty principle impact on nonextensive black hole thermodynamics,”
Class. Quant. Grav. **40** (2023) no.14, 145001
doi:10.1088/1361-6382/acdb40
[arXiv:2301.00609 [gr-qc]].
We found and analyzed thermodynamic quantities with GUP correction for all entropies that I (could not) mentioned in this talk.

Which are

- Tsallis-q Renyi
- Sharma-Mital
- Barrow
- Tsallis-Cirto
- Kaniadakis

Thank you for listening.

Heuristic approach to Thermodynamics

We can heuristically deduce the Hawking temperature through the HUP.

$$E = c\Delta p = k_B T \tag{25}$$

We can find the corresponding temperature around $\Delta x = 2GM/c^2$ as

$$T = \frac{c^2 m_p^2}{8\pi M k_B} \quad \textit{where,} \quad m_p = \sqrt{\frac{\hbar c}{G}} \tag{26}$$

We can also find the associated entropy by following the Clausius formula.

$$S = \int \frac{c^2}{T(M)} dM \rightarrow S_B = \frac{4\pi M^2 k_B}{m_p^2} \tag{27}$$

From here, using usual thermodynamic relations, we can find heat capacity as

$$C(M) = -\frac{S'^2(M)}{S''(M)} \quad \rightarrow \quad -\frac{8 k_B M^2 \pi}{m_p^2} \tag{28}$$

Furthermore, we can define thermal wavelength and sparsity as the separation between the successive quantas as

$$\lambda_t = 2\pi \left(\frac{\hbar c}{k_B T} \right) \quad , \quad \eta = C \left(\frac{\lambda_t^2}{g A_{eff}} \right) \quad \eta_H = \frac{64\pi^3 c^4}{27} \tag{29}$$

For a Schwarzchild BH we got η_H, Which is constant and larger than a classical body sparsity, i.e. the sparsity of Hawking radiation do not fit to a classical description.

GUP Corrected Shwarzchild Case

$$\Delta p = \Delta x \frac{\hbar}{\alpha l_p^2}\left[1 \pm \sqrt{1 - \frac{\alpha l_p^2}{(\Delta x)^2}}\right] \quad \rightarrow$$

$$T_{Gup} = T_H \mathcal{K}(\alpha, M) \quad \textit{where,} \quad \mathcal{K}(\alpha, M) = \frac{2}{1 + \sqrt{1 - \frac{\alpha}{4GM^2}}} \tag{30}$$

$$S_{Gup} = \frac{S_B}{\mathcal{K}} - \frac{\alpha\pi}{2} \log\left[\frac{4M}{m_0 \mathcal{K}}\right] \tag{31}$$

$$C_{Gup} = C_{Sc}\left[\frac{2 - \mathcal{K}}{\mathcal{K}^2}\right] \quad , \quad \eta_{Gup} = \frac{\eta_H}{\mathcal{K}^2}\left[\frac{A}{A_{Gup}}\right] \tag{32}$$

Generalised Uncertinity Principle

The generalized uncertainty principle (GUP) is a low-energy approach to Quantum Gravity. QM revolves around non-commutative algebra of the position and momentum operators.

$$[\hat{x}_i, \hat{p}_k] = i\hbar\delta_{ik} \tag{33}$$

With the help of the deviation of the expectation values of two operators as

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle| \tag{34}$$

one can deduct the Heisenberg uncertainty relation (HUP).

$$\Delta p \Delta x \geq \frac{\hbar}{2} \tag{35}$$

GUP can be invoked by deforming the regular QM algebra with a deformation parameter κ as

$$[\hat{x}_i, \hat{x}_j] = -\left(\frac{\hbar}{\kappa c}\right)^2 i\epsilon_{ijk}J_k \ , \ [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\sqrt{1 + \frac{p^2}{\kappa^2 c^2}} \tag{36}$$

$$\Delta x \Delta p \geq \hbar[1 + \alpha_0 \frac{l_p^2}{\hbar^2}(\Delta p)^2] \tag{37}$$

At the $\alpha_0 \to 0$ limit, and also at the low-energy levels where $\mathcal{O}(\Delta p)^2 \approx 0$, we obtain HUP.

What q stands for → Bias

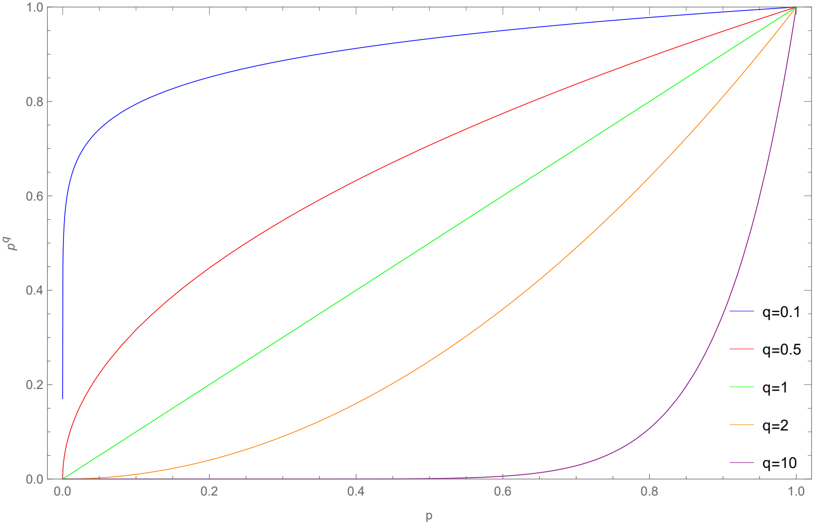


Figure: p^q for several values of q. A larger value of q biases the probability, while smaller value of q unbiases it.

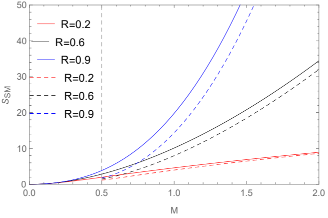
Sharma-Mittal Entropy

$$S_{SMGUP} = \frac{k_B}{R}\left(\left(\frac{\pi\lambda M^2\mathcal{H}(M)}{m_p^2} - \frac{1}{2}\pi\alpha\lambda\log\left(\frac{M}{M_0}\mathcal{H}(M)\right) + 1\right)^{R/\lambda} - 1\right) \quad (38)$$

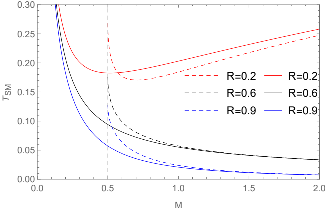
$$T_{SMGUP} = \frac{c^2 m_p^2\left(\frac{\pi\lambda M^2\mathcal{H}(M)}{m_p^2} - \frac{1}{2}\pi\alpha\lambda\log\left(\frac{M}{M_0}\mathcal{H}(M)\right) + 1\right)^{1-\frac{R}{\lambda}}}{2\pi k_B M\mathcal{H}(M)} \quad (39)$$

$$\eta_{SMGUP} = \frac{16\pi^4 C k_B M^2\mathcal{H}(M)^2\left(\frac{\pi\lambda M^2\mathcal{H}(M)}{m_p^2} - \frac{1}{2}\pi\alpha\lambda\log\left(\frac{M}{M_0}\mathcal{H}(M)\right) + 1\right)^{\frac{2R}{\lambda}-2}}{27 g m_p^2\left(\frac{\pi k_B M^2\mathcal{H}(M)}{m_p^2} - \frac{1}{2}\pi\alpha k_B\log\left(\frac{M}{M_0}\mathcal{H}(M)\right)\right)} \quad (40)$$

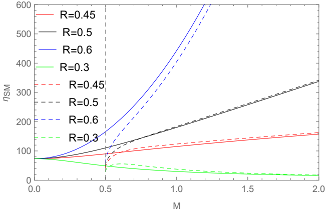
Sharma-Mittal Entropy



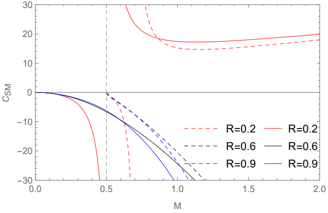
(a) Entopy



(b) Temprature



(a) Sparsity



(b) Heat Capacity

Sharma-Mittal Entropy

- This time in $R \to 0$ is limit Rènyi entropy, while in $R \to \lambda$ is Tsallis-q entropy (which was assumed to be in the form of Bekenstein entropy).
- Depending on the parameters,
  - $\lambda - 2R < 0$ leads to an exponential decrease in temperature as a function of M.
  - $\lambda - 2R > 0$ mimics Renyi temperature. In this case, there is a turning point in temperature $\mathcal{L}_{SM*}$ where it differentiates two different regions. Above $\mathcal{L}_{SM*}$ the temperature increases (exponentially or sub linearly depending on the parameters), and below it decreases exponentially with energy.
- Depending on the relation between R and λ, the sparsity profile either mimics Rènyi or Tsallis-q.

Tsallis-Cirto and Barrow

Tsallis and Cirto in (81202.2154) argued for another form of generalization in order to sustain the Legendre structure of thermodynamics. We are interested in the microstate dependence of

$$W(N) \approx C\nu^{N^\gamma}(C > 0, \nu > 1; 0 < \gamma < 1) \tag{41}$$

where the following holds

$$BN^\tau << C\nu^{N^\gamma} << A\xi^N \tag{42}$$

where entropy associated with $\gamma \to 1$ is associated with Boltzman Gibbs entropy. For $0 < \gamma < 1$ following entropy can be written as:

$$S_\delta = k_B \sum_i^W p_i (\log \frac{1}{p_i})^\delta \text{ where } (\delta > 0) \tag{43}$$

Which results under equiprobability in

$$S_\delta = k_B \log^\delta W \tag{44}$$

Which has the following composition rule:

$$\frac{1}{k_B} S_\delta(A+B) = ([\frac{S_\delta}{k_B}]^{1/\delta} + [\frac{S_\delta(B)}{k_B}]^{1/\delta})^\delta \tag{45}$$

Notice that one can get the physical temperature by minimizing the composition rule (equilibrium happens at maximized entropy). Thus, the dependence of the entropy on microstates is different.

Also, Barrow in (2004.09444) creates a three-dimensional analog of a "Koch Swowflake" to fractalize the surface of the black hole. Consider that the black hole surface would contain bubbles like a fractal. In each iteration, we would have N new black hole bubbles with radius scales with a factor of λ. The total area would yield.

$$A_T = 4\pi r_+^2 \sum_0^{\infty} (N\lambda^2)^n \quad \rightarrow \quad \frac{4\pi r_+^2}{1 - N\lambda^2} \tag{46}$$

If the surface is a pure fractal, the area will vary with the radius as $r^{\Delta+2}$, which yields an entropy in the form.

$$S_B = (A/A_p)^{1+\frac{\Delta}{2}} \tag{47}$$

Which has the same form if the former one is Bekenstein entropy.

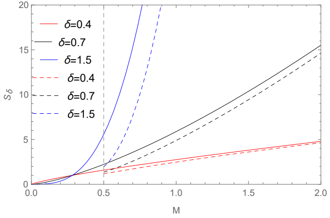
Tsallis Cirto and Barrow ($\Delta = 1 + \delta/2$)

$$S_{\delta GUP} = k_B \left(\frac{\pi M^2 k_B \mathcal{H}(M)}{m_p^2} - \frac{1}{2}\pi\alpha \log\left(\frac{M k_B \mathcal{H}(M)}{M_0}\right)\right)^{\delta} \tag{48}$$

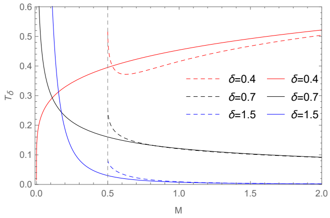
$$T_{\delta GUP} = \frac{c^2 m_p^2 \left(\frac{\pi M^2 k_B \mathcal{H}(M)}{m_p^2} - \frac{1}{2}\pi\alpha \log\left(\frac{M k_B \mathcal{H}(M)}{M_0}\right)\right)^{1-\delta}}{2\pi\delta k_B M k_B \mathcal{H}(M)} \tag{49}$$

$$\eta_{\delta GUP} = \frac{64\pi^3 C\delta^2 \left(\frac{\pi M^2 k_B \mathcal{H}(M)}{m_p^2} - \frac{1}{2}\pi\alpha \log\left(\frac{M k_B \mathcal{H}(M)}{M_0}\right)\right)^{2(\delta-1)}}{27g} \tag{50}$$

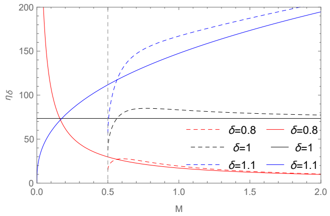
Tsallis Cirto and Barrow ($\Delta = 1 + \delta/2$)



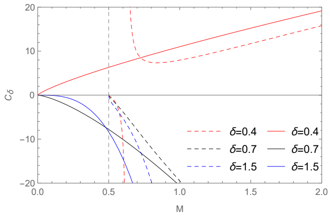
(a) Entopy



(b) Temprature



(a) sparsity



(b) Heat Capacity

Tsallis Cirto and Barrow ($\Delta = 1 + \delta/2$)

- $\delta \to 1$ limit yields the Tsallis-q entropy.
- δ characterizes the profile as follows,
  - $\delta > 1/2 \to$ Quadratic
  - $\delta < 1/2 \to$ Sublinear
- Altought there is no characteristic length scale in Tsallis-Cirto type entropy in the classical case, there is one in the GUP corrected case.
- It can be found for the energy solution for $\delta = S_{\delta GUP}/C_{\delta GUP}$ where C is the heat capacity.
- It is interesting that a small deviation in δ below the natural value of 1 can lead to less sparse radiation for classical black holes.