Monte Carlo methods for stationary solutions of general-relativistic Vlasov systems:

Planar accretion onto a moving Schwarzschild black hole

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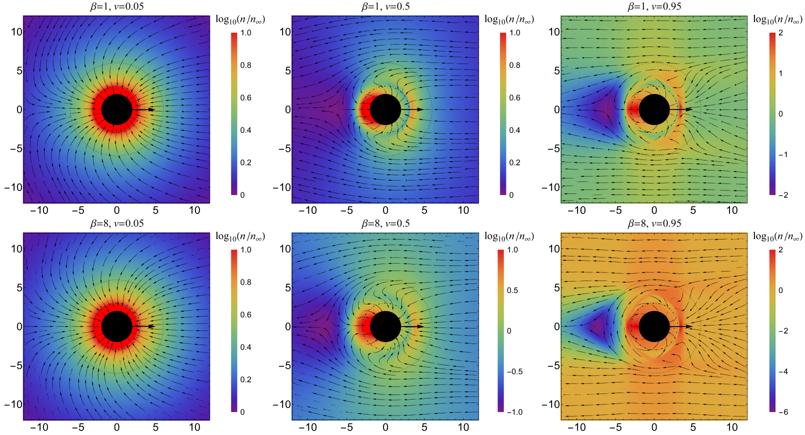
Kinetic Theory

- Aim: to provide an alternative description to hydrodynamic models when collisions between particles are rare and remain generally unimportant in comparison to characteristics of the system
- Assumptions: test particles move along timelike geodesics, there are no collisions between particles, and the description is done in phase space
- Properties: models are solutions of the general-relativistic Vlasov equation (collisionless case)
- Applications:
  - observations of black holes in the centres of galaxies
  - description of a distribution of stars around a supermassive black hole
  - modelling dark matter accretion
- Difficulties: computing observable quantities requires a good understanding of the regions in phase space available for motion



Schwarzschild black hole moving through a cloud of gas
(P. Mach and A. Odrzywołek: 2021,2022)

Relativistic phase space

Let $(\mathcal{M}, g)$ be a spacetime manifold. The cotangent bundle of $\mathcal{M}$ is defined as

$$T^*\mathcal{M} = \{(x, p)\colon x \in \mathcal{M},\, p \in T^*_x\mathcal{M}\}.$$

We consider the so-called "simple gas," i.e., the case in which the masses of all particles are the same. This limits the discussion to the mass shell Γ^+_m, defined as follows:

$$\Gamma^+_m = \{(x, p) \in T^*\mathcal{M}\colon g^{\mu\nu} p_\mu p_\nu = -m^2,\, p \text{ is future-directed}\}.$$

The mass shell condition can also be imposed by defining the distribution function $\mathcal{F}$ on $T^*\mathcal{M}$ and assuming that

$$\mathcal{F} \sim \delta(\sqrt{-p_\mu p^\mu} - m).$$

Physical observables

Let S denote a spacelike hypersurface in $\mathcal{M}$, the average number of particle trajectories whose projections on $\mathcal{M}$ intersect S can be expressed as

$$\mathcal{N}[S] = -\int_S \left[\int_{P_x^+} \mathcal{F}(x,p) p_\mu s^\mu \mathrm{dvol}_x(p) \right] \eta_S, \tag{1}$$

where

$$P_x^+ = \{ p \in T_x^* \mathcal{M} : g^{\mu\nu} p_\mu p_\nu < 0, p \text{ is future-directed} \},$$

and s is a future-directed unit vector normal to S, η_S denotes the induced volume element on S, and $\mathrm{dvol}_x(p)$ is the volume element on P_x^+. In local adapted coordinates $\mathrm{dvol}_x(p)$ is given as

$$\mathrm{dvol}_x(p) = \sqrt{-\det g^{\mu\nu}(x)} dp_0 dp_1 dp_2 dp_3.$$

By defining the particle current density as

$$\mathcal{J}_\mu(x) = \int_{P_x^+} \mathcal{F}(x,p) p_\mu \mathrm{dvol}_x(p),$$

Eq. (1) can be expressed in the form

$$\mathcal{N}[S] = -\int_S \mathcal{J}_\mu s^\mu \eta_S. \tag{2}$$

It can be demonstrated that the particle current density obeys the conservation law $\nabla_\mu \mathcal{J}^\mu = 0$, which further supports the expression (2). The particle number density can be covariantly defined as

$$n = \sqrt{-\mathcal{J}_\mu \mathcal{J}^\mu}.$$

Our case

- Stationary solution – geodesic trajectories do not depend on time
- Bondi-type accretion – a compact object travelling through the interstellar medium
- Planar accretion
- Asymptotically, the gas is assumed to be homogeneous and described by the two-dimensional Maxwell-Jüttner distribution, boosted with a constant velocity v along the x axis. In the Cartesian coordinates, the asymptotic distribution function is given by

$$\mathcal{F}(x,p) = \alpha\delta\left(\sqrt{-p_\mu p^\mu} - m_0\right)\exp\left[\frac{\beta}{m_0}\gamma(p_t + v p_x)\right],$$

and in spherical coordinates

$$\mathcal{F}(x,p) = \alpha\delta\left(\sqrt{-p_\mu p^\mu} - m_0\right)\exp\left[\frac{\beta}{m_0}\gamma\left(p_t + v\cos\varphi\, p_r\right)\right].$$

Particle current density

With the help of Hamilton's formalism and action-angle variables, it can be shown that in the vicinity of the Schwarzschild black hole, the distribution function is given by

$$f(x,p) = \alpha\delta\left(\sqrt{-p_\mu p^\mu} - m_0\right)\exp\left\{-\beta\gamma\left[\varepsilon - \epsilon_r v\sqrt{\varepsilon^2 - 1}\cos\left[\varphi + \epsilon_\varphi \epsilon_r X(\xi,\varepsilon,\lambda)\right]\right]\right\},$$

where

$$X(\xi,\varepsilon,\lambda) = \lambda\int_\xi^\infty \frac{d\xi'}{\xi'^2\sqrt{\varepsilon^2 - \left(1 - \frac{2}{\xi'}\right)\left(1 + \frac{\lambda^2}{\xi'^2}\right)}}.$$

Additionally, components of surface particle current density read

$$J_\mu(\xi,\varphi) = \sum_{\epsilon_\varphi = \pm 1}\frac{1}{\xi}\int f(\xi,\varphi,m,\varepsilon,\epsilon_\varphi,\lambda)p_\mu \frac{m^2 dm d\varepsilon d\lambda}{\sqrt{\varepsilon^2 - U_\lambda}}.$$

Monte Carlo approach

Consider a discrete distribution function

$$\mathcal{F}^{(N)}(x^\mu, p_\nu) = \sum_{i=1}^{N} \int \delta^{(4)}\left(x^\mu - x^\mu_{(i)}(\tau)\right) \delta^{(4)}\left(p_\nu - p^{(i)}_\nu(\tau)\right) d\tau$$

representing a sample of N particles moving along given trajectories $\tau \mapsto \left(x^\mu_{(i)}(\tau), p^{(i)}_\nu(\tau)\right)$, $i = 1, \ldots, N$. The particle current density associated with $\mathcal{F}^{(N)}$ is given as

$$\mathcal{J}^{(N)}_\mu(x) = \int_{P^+_x} \mathcal{F}^{(N)}(x,p) p_\mu \sqrt{-\det g^{\alpha\beta}(x)} dp_0 dp_1 dp_2 dp_3.$$

Let $\Sigma \in \mathcal{M}$ be a hypersurface, not necessarily space-like. We choose a small region $\sigma \in \Sigma$ (a numerical cell) such that $x \in \sigma$. The components of $\mathcal{J}_\mu$ are approximated by the averages

$$\langle \mathcal{J}_\mu(x) \rangle = \frac{\int_\sigma \mathcal{J}^{(N)}_\mu \eta_\Sigma}{\int_\sigma \eta_\Sigma},$$

where η_Σ denotes the volume element on Σ.

Intersections of trajectories with arcs of constant radius

For a planar stationary accretion flow in the Schwarzschild spacetime, we select surfaces of constant $r = \bar{r}$ defined by

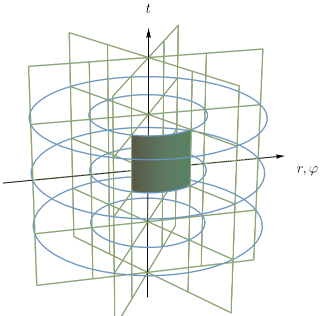
$$\tilde{\Sigma} = \{(t, r, \theta, \varphi) \colon t \in \mathbb{R},\ r = \bar{r},\ \theta = \pi/2,\ \varphi \in [0, 2\pi)\}$$

and cells

$$\tilde{\sigma} = \{(t, r, \theta, \varphi) \colon t_1 \leq t \leq t_2,\ r = \bar{r},\ \theta = \pi/2,\ \varphi_1 \leq \varphi \leq \varphi_2\}.$$

More precisely let $\Phi_\tau(x_0^i)$ denote the orbit of timelike Killing vector field $\xi^\mu = (1, 0, 0, 0)$, passing through x_0^i at $\tau = 0$, i.e., $\Phi_0(x_0^i) = x_0^i$. Then $\tilde{\sigma}$ can be expressed as the image

$$\tilde{\sigma} = \Phi_{[t_1, t_2]}(S).$$



The particle current surface density can now be approximated as

$$\langle J_\mu \rangle = \frac{\int_{\tilde{\sigma}} J_\mu \eta_{\tilde{\Sigma}}}{\int_{\tilde{\sigma}} \eta_{\tilde{\Sigma}}} = \frac{1}{Mm\bar{\xi}(t_2 - t_1)(\varphi_2 - \varphi_1)} \sum_{j=1}^{N_{\mathrm{int}}} \frac{p_\mu^{(j)}}{\sqrt{\varepsilon_{(j)}^2 - \left(1 - 2/\bar{\xi}\right)\left(1 + \lambda_{(j)}^2/\bar{\xi}^2\right)}}.$$

For stationary problems, the result should be independent of the choice of t_1 and t_2 in a sense that the number of trajectories that intersects $\tilde{\Sigma}$ should be proportional to the length $t_2 - t_1$, if the latter is sufficiently large. In practice, we omit the factor $t_2 - t_1$ and normalise the results by the number of trajectories taken into account. Moreover, instead of considering complete orbits in the four-dimensional spacetime, it is sufficient to work with projections of trajectories onto surfaces of constant t.

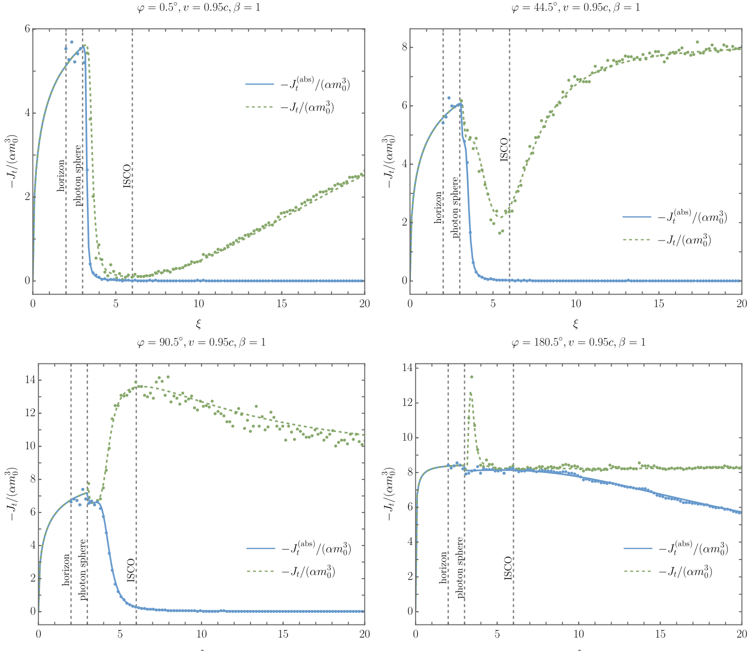
Selection of geodesic parameters

- We select the parameters $\{\xi_0, \varphi_i^{(\mathrm{init})}, \varepsilon_i, \lambda_i\}$, representing the radial and the azimuthal coordinates of the initial position, the energy, and the total angular momentum of i-th particle, respectively
- The first coordinate is the same for all trajectories—all particles start at a fixed radius $r_0 = M\xi_0$. It is important to ensure that this value is sufficiently large
- The coordinate values $\varphi_i^{(\mathrm{init})}$ and ε_i are sampled from the planar asymptotic ($\xi \to \infty$) distribution function:

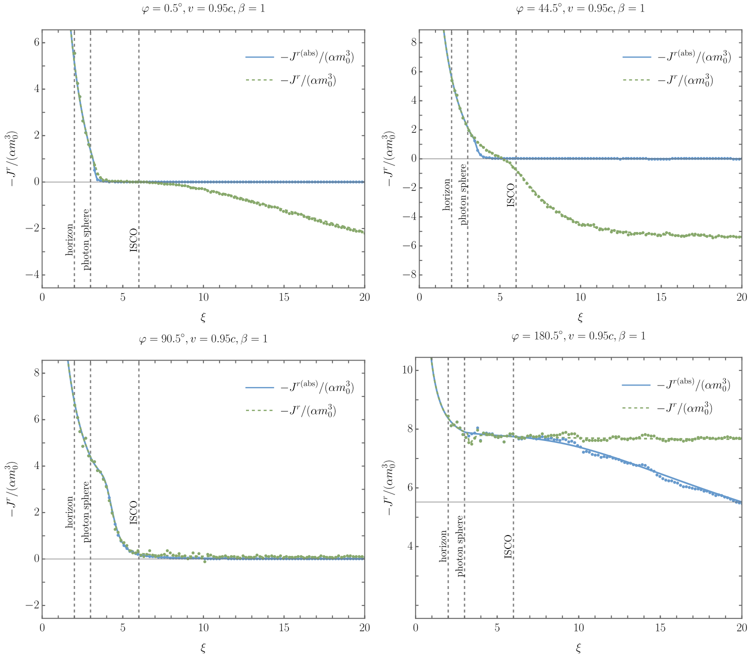
$$f(x,p) = \alpha\delta\left(\sqrt{-p_\mu p^\mu} - m\right)\exp\left[-\beta\gamma\left(\varepsilon - \epsilon_r v\sqrt{\varepsilon^2 - 1}\cos\varphi\right)\right] \quad (3)$$

- To randomise the parameters $\varphi_i^{(\mathrm{init})}, \varepsilon_i$ according to the distribution function (3), we use the Markov Chain Monte Carlo (MCMC) method, implemented in the *Wolfram Mathematica.*
- Values of λ_i are distributed uniformly.
- From the selected parameters, we choose those corresponding to the unbound trajectories and then divide them into absorbed and scattered

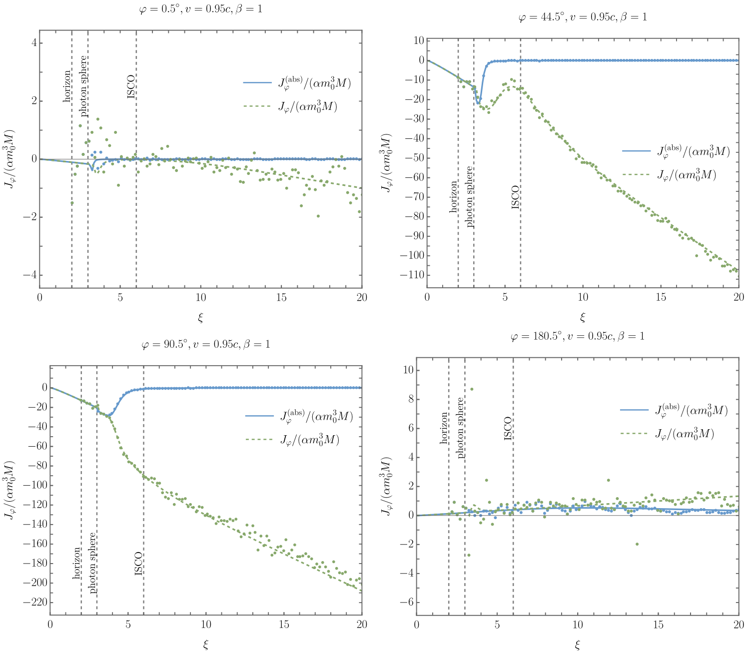
Results

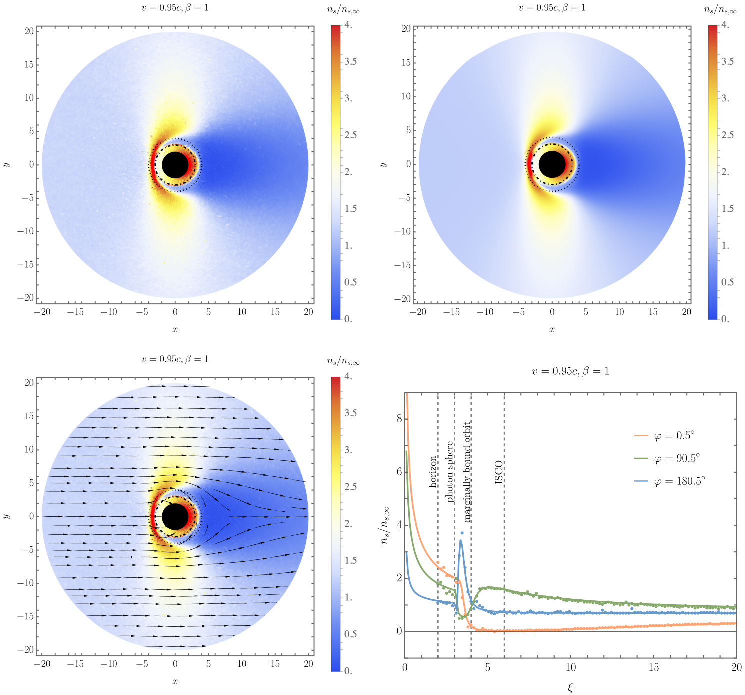


$\varphi = 0.5^\circ, v = 0.95c, \beta = 1$
$\varphi = 44.5^\circ, v = 0.95c, \beta = 1$
$\varphi = 90.5^\circ, v = 0.95c, \beta = 1$
$\varphi = 180.5^\circ, v = 0.95c, \beta = 1$
$-J_t/(\alpha m_0^3)$
ξ
horizon
photon sphere
ISCO
$-J_t^{(\text{abs})}/(\alpha m_0^3)$
$-J_t/(\alpha m_0^3)$



$\varphi = 0.5^\circ, v = 0.95c, \beta = 1$
$-J^{r(\mathrm{abs})}/(\alpha m_0^3)$
$-J^r/(\alpha m_0^3)$
horizon
photon sphere
ISCO
$-J^r/(\alpha m_0^3)$
ξ
$\varphi = 44.5^\circ, v = 0.95c, \beta = 1$
$-J^{r(\mathrm{abs})}/(\alpha m_0^3)$
$-J^r/(\alpha m_0^3)$
horizon
photon sphere
ISCO
$-J^r/(\alpha m_0^3)$
ξ
$\varphi = 90.5^\circ, v = 0.95c, \beta = 1$
$-J^{r(\mathrm{abs})}/(\alpha m_0^3)$
$-J^r/(\alpha m_0^3)$
horizon
photon sphere
ISCO
$-J^r/(\alpha m_0^3)$
ξ
$\varphi = 180.5^\circ, v = 0.95c, \beta = 1$
$-J^{r(\mathrm{abs})}/(\alpha m_0^3)$
$-J^r/(\alpha m_0^3)$
horizon
photon sphere
ISCO
$-J^r/(\alpha m_0^3)$
ξ


$\varphi = 0.5°, v = 0.95c, \beta = 1$
$\varphi = 44.5°, v = 0.95c, \beta = 1$
$\varphi = 90.5°, v = 0.95c, \beta = 1$
$\varphi = 180.5°, v = 0.95c, \beta = 1$
horizon
photon sphere
ISCO
$J_\varphi^{(\mathrm{abs})}/(\alpha m_0^3 M)$
$J_\varphi/(\alpha m_0^3 M)$
ξ



v = 0.95c, β = 1
n_s/n_{s,∞}
x
y
horizon
photon sphere
marginally bound orbit
ISCO
φ = 0.5°
φ = 90.5°
φ = 180.5°
ξ

Conclusions

- We confirmed the analytical results in the case of planar accretion
- We demonstrated that the developed Monte Carlo simulation method can be used for cases that do not have spherical symmetry
- Outlook:
  - Preparation of a three-dimensional simulation (P. Mach and A. Odrzywołek: 2021, 2022)
  - Preparing a simulation of Vlasov gas accretion in Kerr spacetime (A. Cieślik, P. Mach, A. Odrzywołek: 2022; P. Rioseco, O. Sarbach: 2018, 2023)
  - Generalisation to general-relativistic Vlasov systems coupled with the electromagnetic field (M. Thaller: 2023)