UNIVERSITY OF WARSAW

Status of Birkhoff 's theorem in polymerized semiclassical regime of Loop Quantum Gravity

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LC, Jerzy Lewandowski (2024) arXiv:2403.01910

Introduction

Collapse of a spherically symmetric cloud of pressureless dust

- Modified Einstein's equations
- Oppenheimer-Snyder model

EINSTEIN'S EQUATIONS

Classical theory

General spherically symmetric line element (ADM decomposition):

$$ds^2 = -Nd\tau^2 + \frac{(E^\varphi)^2}{E^x}(dx + N^x\, d\tau)^2 + E^x d\Omega^2$$

(PG coordinates)

$$\{K_x(y_1), E^x(y_2)\} = 2\gamma\delta(y_1 - y_2)$$

$G = c = 1$

$$\{K_\varphi(y_1), E^\varphi(y_2)\} = \gamma\delta(y_1 - y_2)$$

Dust Gauge ($dust\ field = \tau$) → $N = 1$

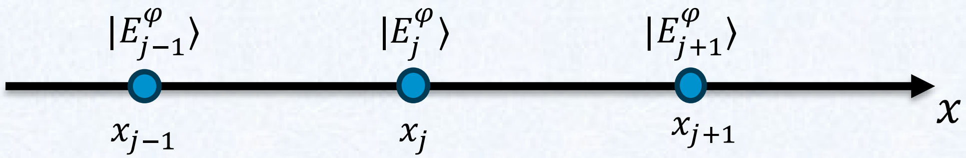
Areal Gauge → $E^x = x^2$

$$E^\varphi = \pm\frac{x}{\sqrt{1 + \varepsilon(\tau, x)}}$$

Polymerization

J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2020)
V. Husain, J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2022)

1-dimensional graph:



- Discretization: $x \longrightarrow x_j$

- Operators: $E^{\varphi} \Longrightarrow \hat{E}^{\varphi}_j$

$$K_{\varphi} \Longrightarrow \hat{U}_j = e^{i\,\bar{\mu}_j K_{\varphi}(x_j)} \qquad \bar{\mu}_j = \frac{\sqrt{\Delta}}{x_j}$$

- Polymerization: $\hat{U}_j = e^{i\,\bar{\mu}_j K_{\varphi}(x_j)}$ ⟶ $\frac{\hat{U}_j - \hat{U}_j^{\dagger}}{2\,i\,\bar{\mu}_j} = \frac{\sin(\bar{\mu}_j K_{\varphi}(x_j))}{\bar{\mu}_j}$

Semiclassical theory

J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2020)
V. Husain, J. G. Kelly, R. Santacruz, E. Wilson-Ewing. (2022)

$$ds^2 = -d\tau^2 + \frac{(E^\varphi)^2}{x^2}(dx + N^x\, d\tau)^2 + x^2 d\Omega^2$$

$$\beta := \frac{\sqrt{\Delta}}{x} K_\varphi$$

$$\dot{E}^\varphi = -\frac{x^2}{\gamma\sqrt{\Delta}} \partial_x \left(\frac{E^\varphi}{x}\right) \sin\beta \cos\beta$$

$$\dot{K}_\varphi = \frac{\gamma x}{2(E^\varphi)^2} - \frac{\gamma}{2x} - \frac{\partial_x (x^3 sin^2 \beta)}{2\gamma\Delta x}$$

Polymerized Einstein Field Equations (PEFE)

$$\dot{F} = \{F, H\}$$

$$\rho = -\frac{\mathcal{H}}{4\pi x E^\varphi} \longrightarrow \rho = \frac{1}{8\pi x E^\varphi}\left[\frac{E^\varphi}{\gamma^2 \Delta x}\partial_x (x^3 sin^2 \beta) + \frac{x}{E^\varphi} + \frac{E^\varphi}{x} - 2\partial_x\left(\frac{x^2}{E^\varphi}\right)\right]$$

$$N^x = -\frac{K_\varphi}{\gamma} \longrightarrow N^x = -\frac{x}{\gamma\sqrt{\Delta}} \sin\beta\cos\beta$$

Interior $(\rho \neq 0, \quad \partial_x \rho = 0)$

M. Bojowald, J. D. Reyes, R. Tibrewala (2009)
K. Giesel, H. Liu, E. Rullit, P. Singh, S. A. Weigl (2023)

$$E^{\varphi} = \pm \frac{x}{\sqrt{1+\varepsilon(\tau, x)}}$$

PEFE

$$\begin{cases} \dot{\varepsilon} = \varepsilon' \sqrt{\frac{8\pi}{3}\rho x^2 + \varepsilon} \sqrt{1 - \frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{\varepsilon}{x^2}} \quad \star \\ sin^2 \beta = \gamma^2 \Delta \left(\frac{8\pi}{3}\rho + \frac{\varepsilon}{x^2} \right) \end{cases}$$

$$\rho_c := \frac{3}{8\pi\gamma^2\Delta}$$

$x = \xi(T, R)$
$\tau = T$
$N^x = -\partial_T \xi$
$\varepsilon = E(R)$

$\longrightarrow$

$$ds^2 = -dT^2 + \frac{(\partial_R \xi)^2}{1+E(R)} dR^2 + \xi^2 d\Omega^2$$

(LTB coordinates)

$$\star \quad \left(\frac{\partial_T \xi}{\xi}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{E}{\xi^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{E}{\xi^2}\right)$$

Interior $(\rho \neq 0, \quad \partial_\chi \rho = 0)$

From $ds^2 = -dT^2 + \frac{(\partial_R \xi)^2}{1+E(R)} dR^2 + \xi^2 d\Omega^2$

$\xi = a(T)\chi_k(R)$
$E(R) = -k\chi_k^2$

The Friedmann dust ball: $ds^2 = -dT^2 + a^2 dR^2 + a^2 \chi_k^2 d\Omega^2$ with $\chi_k(R) = \frac{1}{\sqrt{k}} \sin\left(\sqrt{k}R\right)$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{k}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right)$$

- $\rho \propto {}^1/_{a^3}$ (pressureless dust)
- Classical Friedmann equation for $\rho_c \longrightarrow \infty$ $(\Delta \longrightarrow 0)$

Static Exterior $(\rho = 0)$

$$\dot{E}^{\varphi} = -\frac{x^2}{\gamma\sqrt{\Delta}}\partial_x\left(\frac{E^{\varphi}}{x}\right)\sin\beta\cos\beta = 0$$

$$E^{\varphi} = Ax = \pm\frac{x}{\sqrt{1+B}} \longrightarrow \dot{K}^{\varphi} = 0$$

$$(N^x)^2 = \frac{2M}{x} - \frac{\alpha}{x^2}\left(\frac{M}{x}+\frac{B}{2}\right)^2 + B \qquad \alpha \coloneqq 4\gamma^2\Delta$$

The metric is fully determined by these 2 expressions

$$ds^2 = -d\tau^2 + A^2(dx + N^x\, d\tau)^2 + x^2 d\Omega^2$$

In Schwarzschild coordinates: $ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r}+\frac{B}{2}\right)^2$$

Time dependent Exterior $(\rho = 0)$

$$\dot{E}^{\varphi} = -\frac{x^2}{\gamma\sqrt{\Delta}}\partial_x\left(\frac{E^{\varphi}}{x}\right)\sin\beta\cos\beta \neq 0 \qquad \longrightarrow \qquad E^{\varphi} = \pm\frac{x}{\sqrt{1+\varepsilon(\tau,x)}}$$

PEFE
$$\begin{cases} \dot{\varepsilon} = \varepsilon'\sqrt{\varepsilon + \frac{2M}{x}}\sqrt{1-\gamma^2\Delta\left(\frac{\varepsilon}{x^2}+\frac{2M}{x^3}\right)} \\ sin^2\beta = \gamma^2\Delta\left(\frac{\varepsilon}{x^2}+\frac{2M}{x^3}\right) \end{cases}$$

$\varepsilon = const$ or $\varepsilon = \varepsilon(\tau, x)$

If $\varepsilon \neq \varepsilon(\tau, x)$ then the only line element is given by $f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r}+\frac{B}{2}\right)^2$

BIRKHOFF'S THEOREM !!

Oppenheimer-Snyder Model

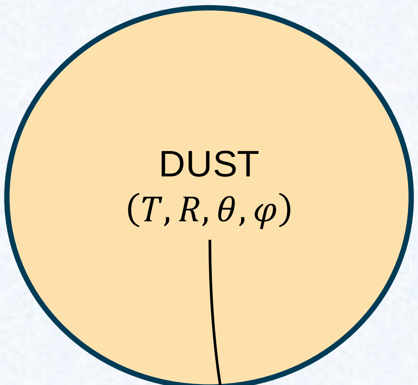
Oppenheimer-Snyder model

H. Ziaie, Y. Tavakoli (2020)
A. Parvizi, T. Pawlowski, Y. Tavakoli, J. Lewandowski (2022)
J. Lewandowski, Y. Ma, J. Yang, C. Zhang (2023)

VACUUM (t, r, θ, φ) →

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} - \frac{k\chi_{k,0}^2}{2}\right)^2$$



$$\rho = \frac{C}{a^3}$$

$$ds^2 = -dT^2 + a^2 dR^2 + a^2 \chi_k^2 d\Omega^2 \quad \text{with} \quad \chi_k(R) = \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}R\right)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{k}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right)$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right) + \left(\frac{8\pi}{3}\rho - \frac{k}{a^2}\right)\left(\frac{3}{2}\frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{k}{a^2}\right)$$

Critical mass and horizons

$$f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} - \frac{k\chi_{k,0}^2}{2}\right)^2$$

- If $M_- \leq M \leq M_+ \quad \Rightarrow \quad \nexists$ real solutions of $f(r) = 0$ (no horizons)

$$M_\pm^2 = \frac{\alpha}{216}\left[64 - 96\,k\chi_{k,0}^2 + 30\,k^2\chi_{k,0}^4 + k^3\chi_{k,0}^6 \pm \left(16 - 16\,k\chi_{k,0}^2 + k^2\chi_{k,0}^4\right)^{3/2}\right]$$

- If $M \geq M_+ \quad \cup \quad M \leq M_- \quad \Rightarrow \quad \exists$ 2 real solutions to $f(r) = 0$

$$r_- = \left(\frac{\alpha M}{2}\right)^{1/3} + \frac{1 - 2\,k\chi_{k,0}^2}{6M}\left(\frac{\alpha M}{2}\right)^{2/3} + \frac{\left(1 - k\chi_{k,0}^2\right)^2}{24\,M}\alpha + O\left(\alpha^{4/3}\right)$$

$$r_+ = 2M - \frac{\left(1 - k\chi_{k,0}^2\right)^2}{8\,M}\alpha + O\left(\alpha^{4/3}\right)$$

k=0

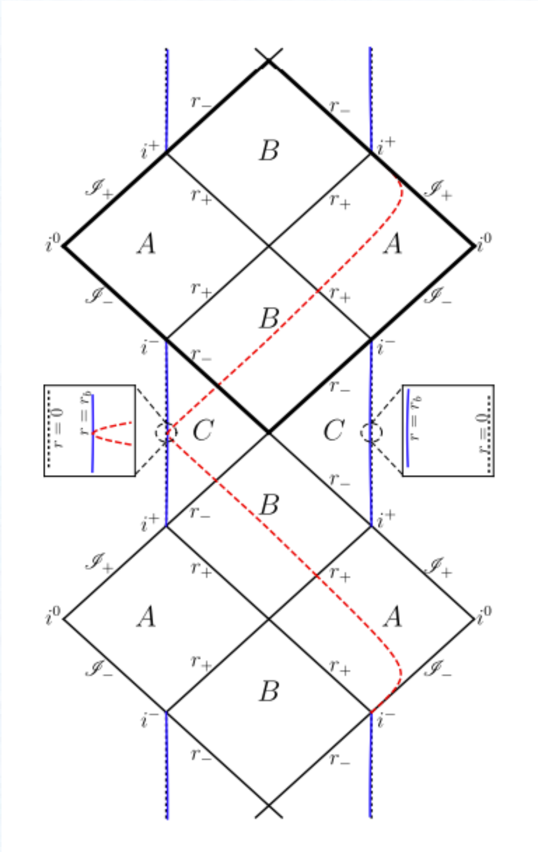
H. Ziaie, Y. Tavakoli (2020)
A. Parvizi, T. Pawlowski, Y. Tavakoli, J. Lewandowski (2022)
J. Lewandowski, Y. Ma, J. Yang, C. Zhang (2023)

- $f(r) = 1 - \frac{2M}{r} + \alpha \frac{M^2}{r^4}$
- $M_- = 0, \quad M_+ = \frac{4}{3\sqrt{3}}\sqrt{\alpha}$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho\right)\left(1 - \frac{\rho}{\rho_c}\right)$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right) + 4\pi\frac{\rho^2}{\rho_c}$$

Exact solution to the PEFE with $B = 0$

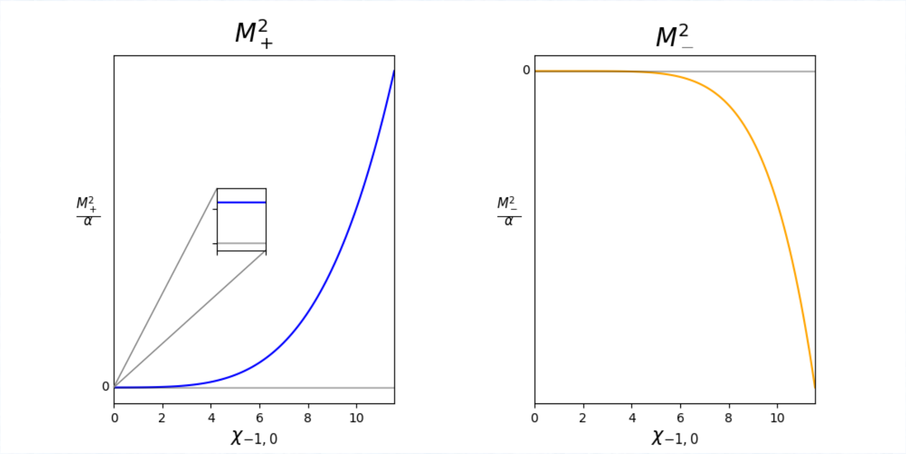


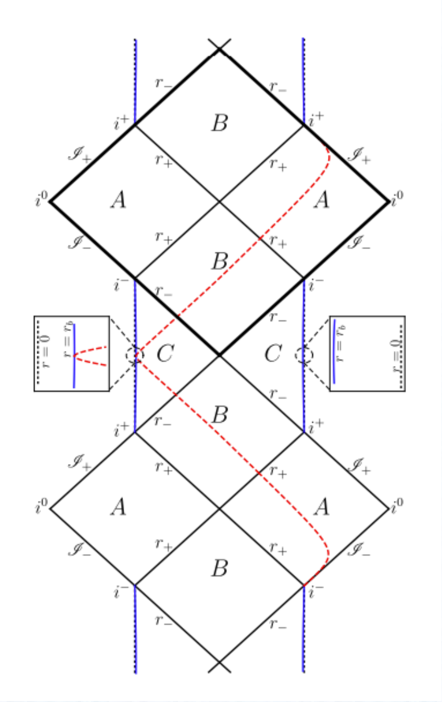
Credits:

J. Lewandowski,
Y. Ma,
J. Yang,
C. Zhang

k= -1

- $f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} + \frac{\chi^2_{-1,0}}{2}\right)^2$ Exact solution to the PEFE with $B = \chi^2_{-1,0}$





$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho + \frac{1}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

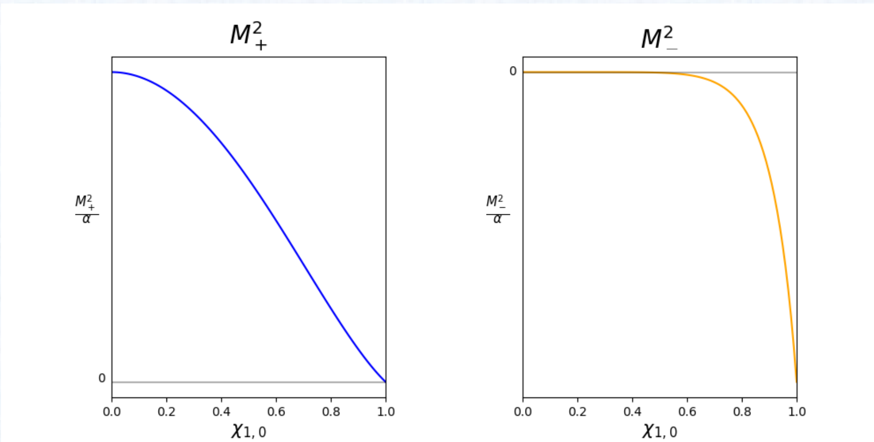
$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right) + \left(\frac{8\pi}{3}\rho + \frac{1}{a^2}\right)\left(\frac{3}{2}\frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

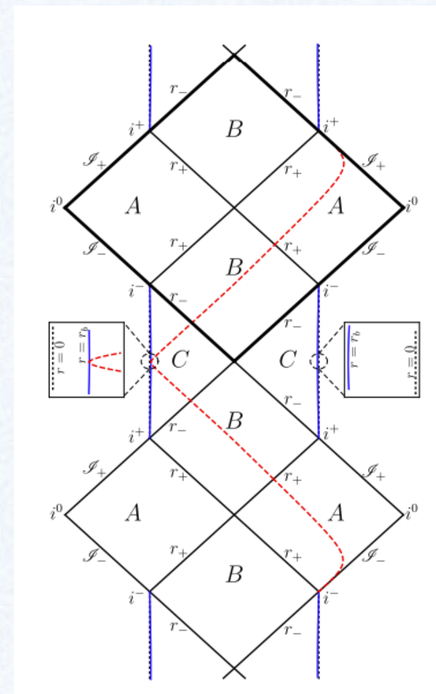
Credits:

J. Lewandowski, Y. Ma, J. Yang, C. Zhang

k=1

- $f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{r^2}\left(\frac{M}{r} - \frac{\chi_{1,0}^2}{2}\right)^2$ Exact solution to the PEFE with $B = -\chi_{1,0}^2 < 0$





Credits:

J. Lewandowski, Y. Ma, J. Yang, C. Zhang

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi}{3}\rho - \frac{1}{a^2}\right)\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3}\rho\left(1 - \frac{\rho}{\rho_c} + \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right) + \left(\frac{8\pi}{3}\rho - \frac{1}{a^2}\right)\left(\frac{3}{2}\frac{\rho}{\rho_c} - \frac{3}{8\pi\rho_c}\frac{1}{a^2}\right)$$

Conclusion

- Two independent methods lead to the very same metric.
- Static exterior solutions to the Einsten's equation are Schwarzschild-like but depend on two parameters (M and B).
- There may exist other non-static solutions.
  If this possibility is ruled out ⇒ Birkhoff's theorem.

Classical theory

Gravity + Dust:

$$S = \int d\tau \int dx \left[\frac{\dot{K}_x E^x + 2\dot{K}_\varphi E^\varphi}{2\gamma} + 4\pi\, \dot{\mathcal{T}} p_{\mathcal{T}} - N(\mathcal{H}^g + \mathcal{H}^d) - N^x(\mathcal{H}_x^g + \mathcal{H}_x^d)\right]$$

$$\mathcal{H}^g = -\frac{1}{2\gamma^2}\left[2K_x K_\varphi \sqrt{E^x} + \frac{E^\varphi}{\sqrt{E^x}}\left(K_\varphi^2 + \gamma^2\right) - \frac{\gamma^2}{4}\frac{(\partial_x E^x)^2}{E^\varphi \sqrt{E^x}} - \gamma^2 \sqrt{E^x}\, \partial_x\left(\frac{\partial_x E^x}{E^\varphi}\right)\right]$$

$$\mathcal{H}^d = 4\pi \sqrt{p_{\mathcal{T}}^2 + \frac{E^x}{(E^\varphi)^2} p_{\mathcal{T}}^2 (\partial_x \mathcal{T})^2}$$

$$\mathcal{H}_x^g = \frac{1}{2\gamma}\left(2E^\varphi \partial_x K_\varphi - K_x \partial_x E^x\right)$$

$$\mathcal{H}_x^d = -4\pi p_{\mathcal{T}} \partial_x \mathcal{T}$$

Classical theory

Gravity + Dust: $S = \int d\tau \int dx \left[\frac{\dot{K}_x E^x + 2\dot{K}_\varphi E^\varphi}{2\gamma} + 4\pi\, \dot{\mathcal{T}} p_{\mathcal{T}} - N(\mathcal{H}^g + \mathcal{H}^d) - N^x(\mathcal{H}_x^g + \mathcal{H}_x^d)\right]$

Dust Gauge $(\mathcal{T} = \tau)$ → $N = 1$

Areal Gauge $(E^x = x^2)$ → $N^x = -\frac{K_\varphi}{\gamma}$

$$S = \int d\tau \int dx \left[\frac{\dot{K}_\varphi E^\varphi}{\gamma} - \mathcal{H}\right] \quad \longrightarrow \quad \{K_\varphi(y_1), E^\varphi(y_2)\} = \gamma\delta(y_1 - y_2)$$

$$\mathcal{H} = -4\pi p_{\mathcal{T}} = -\frac{1}{2\gamma}\left[\frac{E^\varphi}{\gamma x}\,\partial_x\left(x K_\varphi^2\right) + \frac{\gamma E^\varphi}{x} + \frac{\gamma x}{E^\varphi} - 2\gamma\partial_x\left(\frac{x^2}{E^\varphi}\right)\right]$$

Dust density ρ

From the Dust Gauge: $\mathcal{H}^d = 4\pi p_\mathcal{T}$

By solving the Scalar Constraint: $\mathcal{H}^g = -4\pi p_\mathcal{T} = \mathcal{H}$

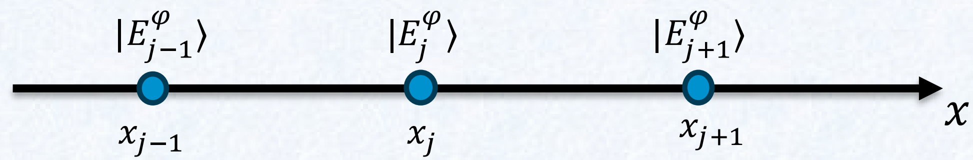
The density ρ is defined by $\mathcal{H}^d = \int d\Omega \sqrt{q}\rho$

$$\rho = \frac{p_\mathcal{T}}{xE^\varphi} = -\frac{\mathcal{H}}{4\pi x E^\varphi}$$

Quantum theory

Quantization: $\left(E^{\varphi}, K_{\varphi}, \frac{1}{E^{\varphi}}\right) \longrightarrow \left(\hat{E}_j^{\varphi}, \hat{U}_j, \widehat{\frac{1}{E_j^{\varphi}}}\right)$

1-dimensional graph:



- Triad operator: $\hat{E}_j^{\varphi}\,|E_j^{\varphi}\rangle = E_j^{\varphi}|E_j^{\varphi}\rangle$

- Holonomy: $\hat{U}_j = e^{i\,\bar{\mu}_j K_{\varphi}(x_j)}$ —Polymerization→ $\frac{\hat{U}_j - \hat{U}_j^{\dagger}}{2\,i\,\bar{\mu}_j} = \frac{\sin(\bar{\mu}_j K_{\varphi}(x_j))}{\bar{\mu}_j}$

  $\bar{\mu}_j = \frac{\sqrt{\Delta}}{x_j}$ $\qquad \hat{U}_j|E_j^{\varphi}\rangle = |E_j^{\varphi} + \bar{\mu}_j\rangle$

- Inverse triad:

$$\widehat{\frac{1}{E_j^{\varphi}}}|E_j^{\varphi}\rangle = \begin{cases} 0 & \text{If } \hat{E}_j^{\varphi}\,|E_j^{\varphi}\rangle = 0 \\ {}^{1}\!/_{E_j^{\varphi}}\,|E_j^{\varphi}\rangle & \text{If } \hat{E}_j^{\varphi}\,|E_j^{\varphi}\rangle = E_j^{\varphi}|E_j^{\varphi}\rangle \end{cases}$$