Lattice for Curved Manifolds: Ising Model and The Affine Conjecture

ZAKOPANE, June 22, 2024

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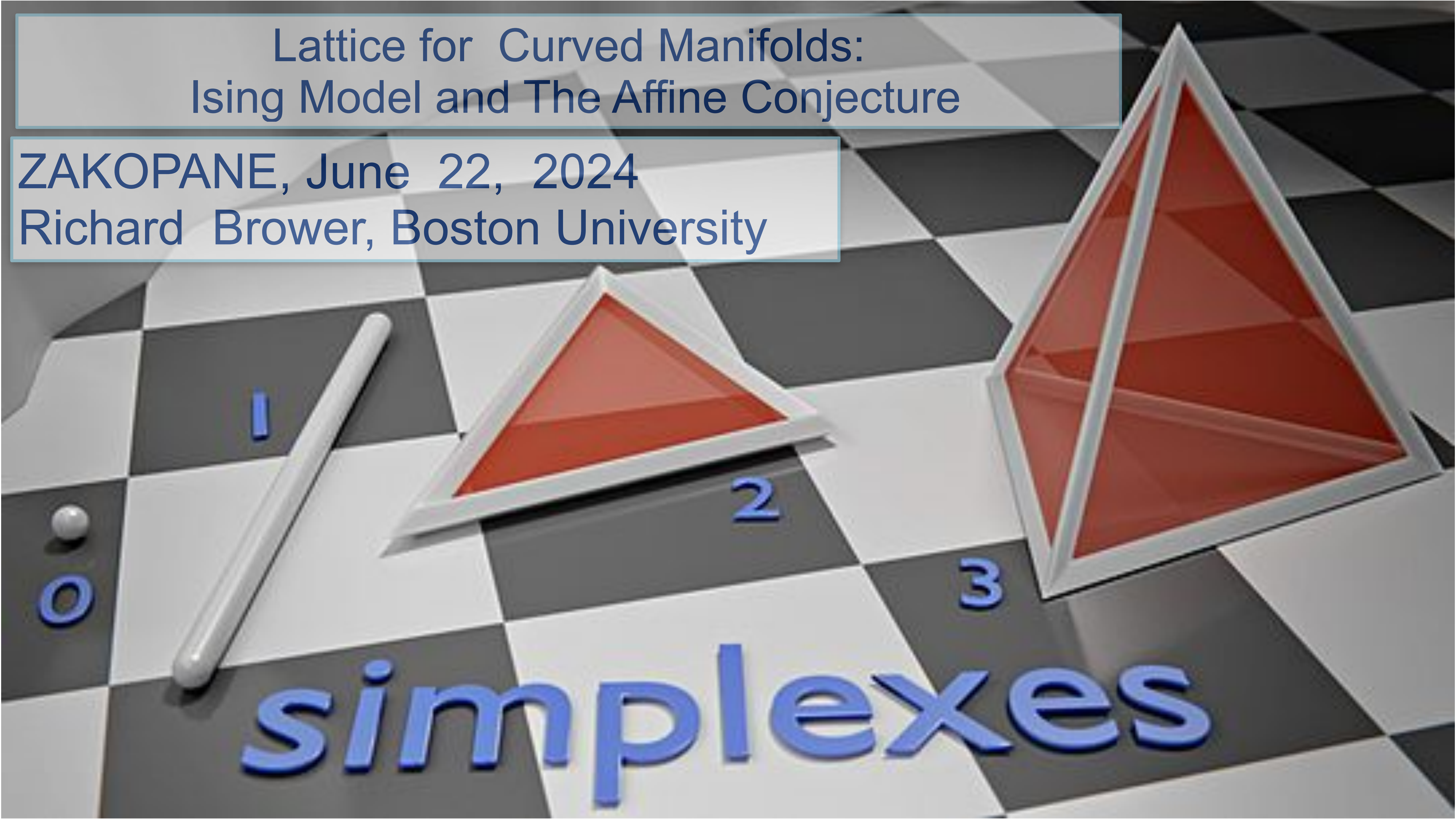


LATTICE FIELD THEORY ON CURVED MANIFOLDS

1. MOTIVATION & OVERVIEW
2. GEOMETRY: REGGE GR meets QUANTUM LATTICE FIELDS
3. ONE (exact) SOLUTION on S2
4. FUTURE TEST AND DEVELOPMEMS
5. "PROVE" THE AFFINE CONJECTURE

Acknowledgements

I would like to thank my collaborators and co-authors

- George T. Fleming, Fermi Laboratory
- Anna-Marie Gluk, Heidelberg University
- Evan Owen, Boston University
- Nobuyuki Matsumoto ,Boston University
- Jin-Yun Lin, Carnegie Mellon University
- Chung-I Tan, Brown University

REFERENCES

1985 CARDY "Universal Amplitude in Finite Size Scaling"

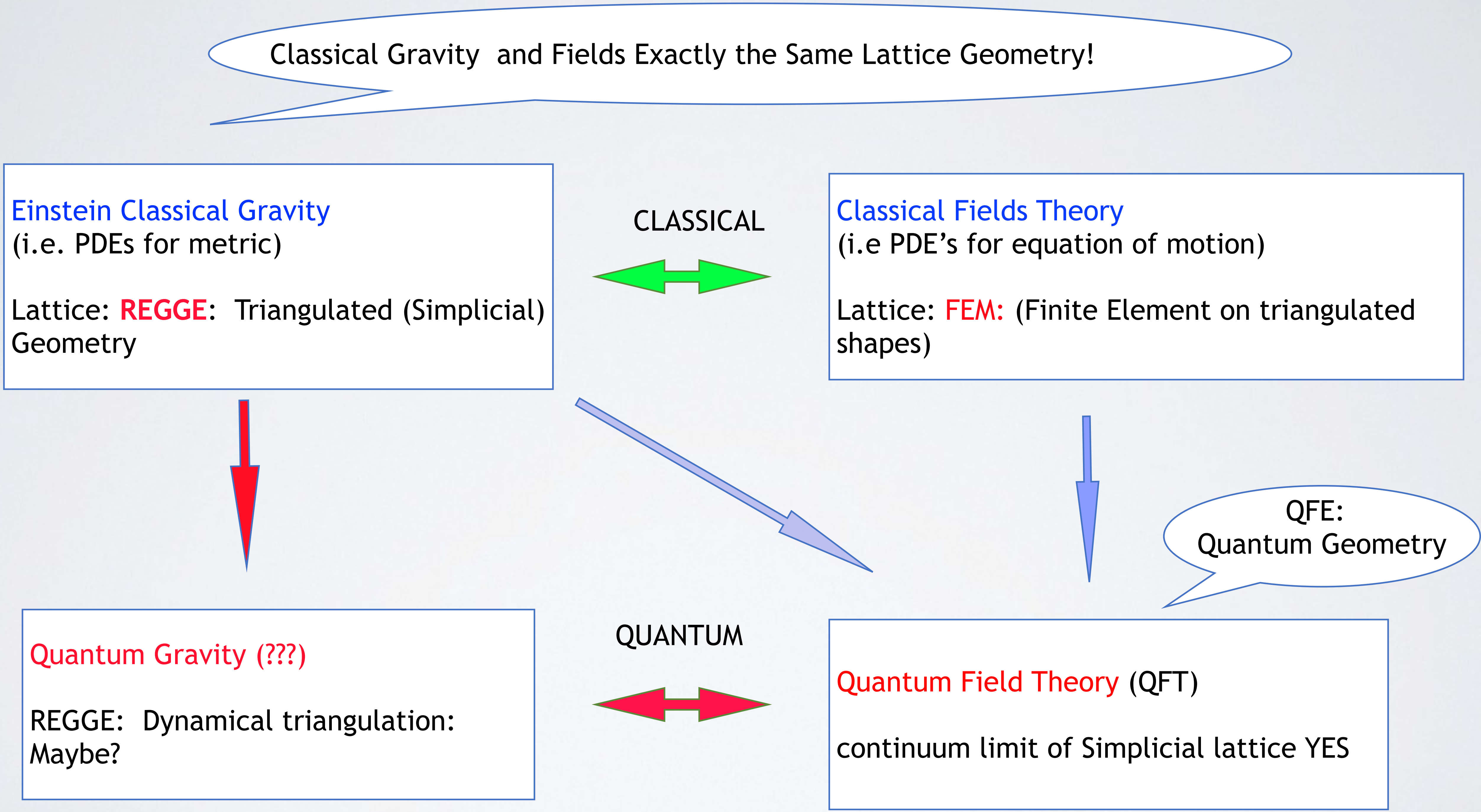
1961 REGGE "General Relativity without Coordinates"

1984 FEINBERG, FRIEBERG, T D LEE and H. C REN " Lattice Gravity Near the Continuum"

2024++ Ising Solution* and "The Affine Conjecture"?

*Ising model on the affine plane Richard C. Brower and Evan K. Owen Phys. Rev. D **108**, 014511 – Published 20 July 2023

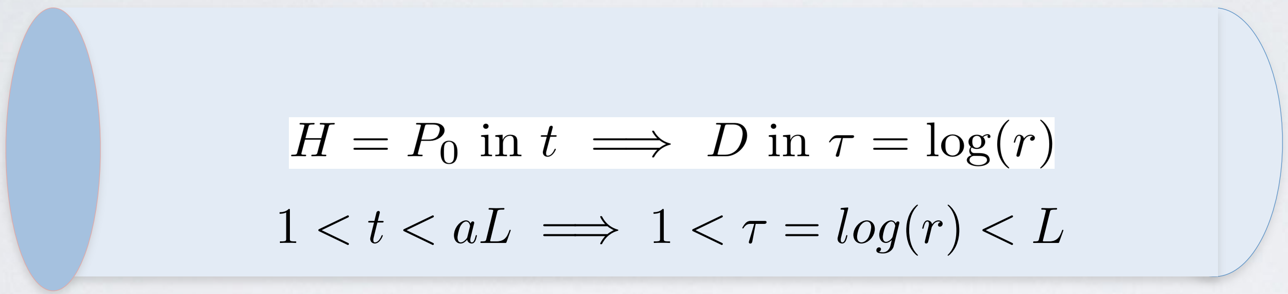
*Ising Model on S2 R.C.B and Evan Owen (thesis) -- paper posted next week.


Classical Gravity and Fields Exactly the Same Lattice Geometry!
Einstein Classical Gravity
(i.e. PDEs for metric)
Lattice: REGGE: Triangulated (Simplicial) Geometry
CLASSICAL
Classical Fields Theory
(i.e PDE's for equation of motion)
Lattice: FEM: (Finite Element on triangulated shapes)
QFE:
Quantum Geometry
Quantum Gravity (???)
REGGE: Dynamical triangulation: Maybe?
QUANTUM
Quantum Field Theory (QFT)
continuum limit of Simplicial lattice YES

MOTIVATION* RADIAL QUANTIZATION OF CONFORMAL FIELD THEORY

$$\mathbb{R} \times \mathbb{T}^3 \quad \text{vs} \quad \mathbb{R} \times \mathbb{S}^3$$

Boundary of Global AdS5



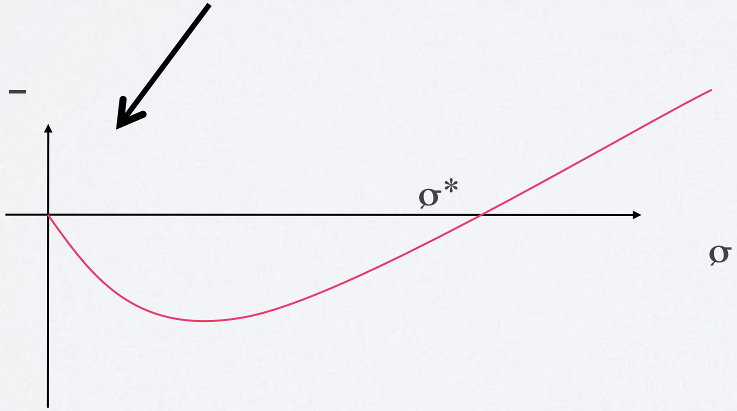
$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

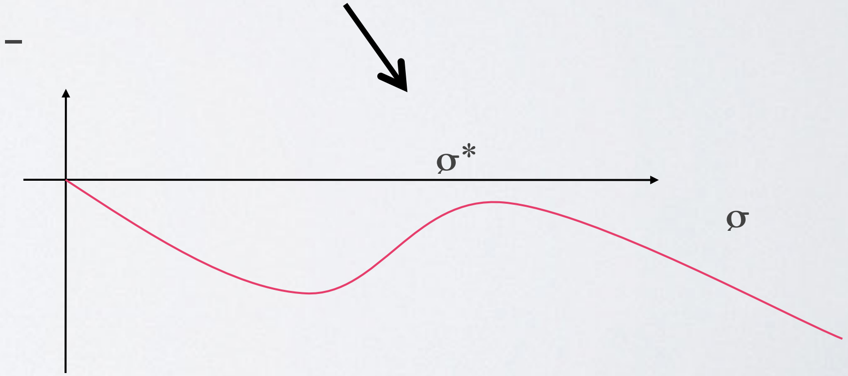
Isometries are the conformal group O(5,1)

On lattice mass scales exponentially

$$b_0 < 0 \text{ for } N_f < 11N_c/2 = 16.5$$



$$b_1 > 0 \text{ for } N_f > 153/14$$



* Near IR conformal
BSM gauge theories for Higgs,
Dark Matter, Cosmology?

Problem

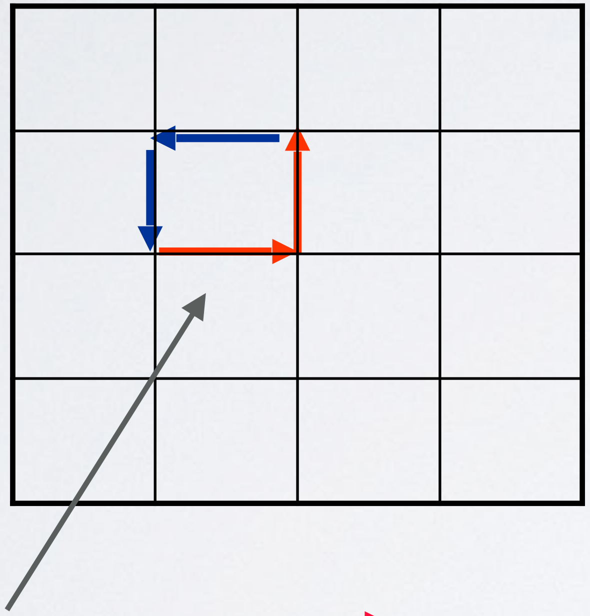
- Define LATTICE FIELD THEORIES on CURVED MANIFOLDS
- Is it possible (e.g like Euclidean lattice QCD in flat space).
- Exact and Polynomial complexity : $\text{error} \sim O(a^n)$

- Conformal Field Theories are more easily studied on Sphere, Cylinders (Radial Quantization) and Hyperbolic Spaces (Gauge/Gravity Duality)

$\mathbb{S}^d$ $\mathbb{R} \times S^{d-1}$ $\mathbb{A}d\mathbb{S}^{d+1}$

*FIRST 50 YEARS: WILSON'S LATTICE QCD**

$$Z_{wilson} = \int_{Haar} dU \; \mathbf{e}^{\frac{6}{g^2} \sum_{\Box} Tr[U_{\Box} + U_{\Box}^{\dagger}]}$$

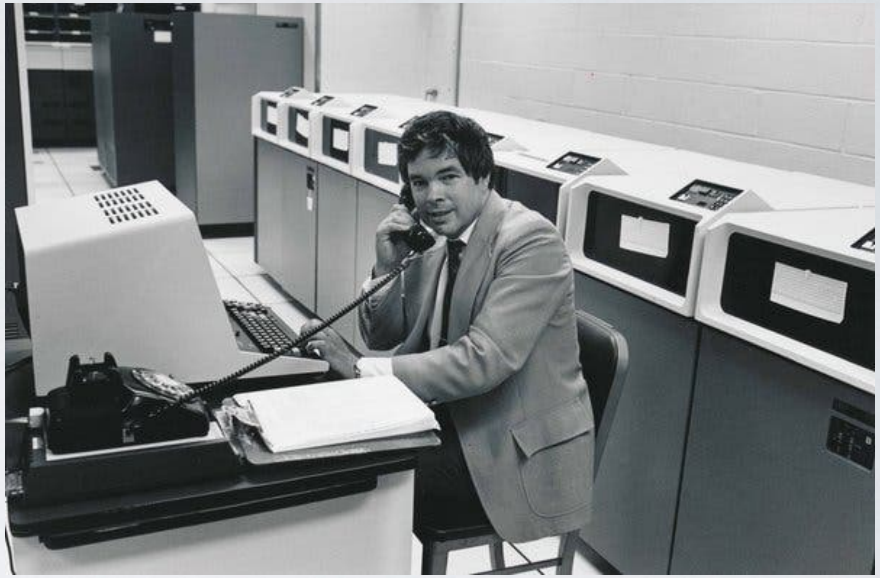


$$U^{ij}(x, x+\mu) = e^{iagA_\mu^{ij}(x)}$$

$$i, j = 1, 2, 3$$

SU(3) Gauge Transport on each link.
Exact per site gauge invariance

$$U_{\Box_{\mu\nu}}(x) = [U(x, x+\mu)U(x+\mu, x+\mu+\nu)][U(x, x+\nu)U(x+\nu, x+\nu+\mu)]^{\dagger}$$
$$\simeq 1 + a^2 iF_{\mu\nu} - (a^4/2)F_{\mu\nu}^2 + \cdots$$



* K.G. Wilson, Phys. Rev. D 10 (1974), 2445.

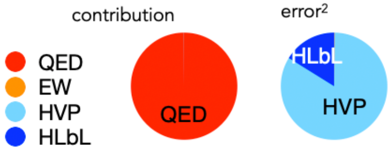
WILSON LATTICE QCD IS EXACT & POLYNOMIAL COMPLEX*

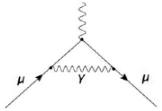
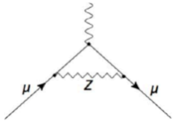
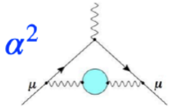
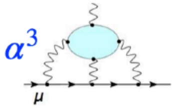
- The Euclidean Wilson Lattice is (believed) on hypercubic lattice to be exact solution as UV cut-off 1/a and Volume (a L^4) / M_proton to infinity.
- Monte Carlo methods are polynomial: Errors = O(a^n) with n = O(10)
- e.g. Precision Results
  - g-2 QCD contribution
  - alpha_strong(M_HIGGS),
  - all quark mass > 0; theta angle problem?

* THE QCD ABACUS: A New Formulation for Lattice Gauge Theories
Maybe the Quantum Link Qubit Hamiltonian is Exact and Polynomial as well?
R.C.B., S. Chandrasakeran and U-J Wiese 1997.

Standard Model Contribution: Calculating the Anomaly

$$a_\mu = a_\mu(QED) + a_\mu(EW) + a_\mu(hadronic)$$

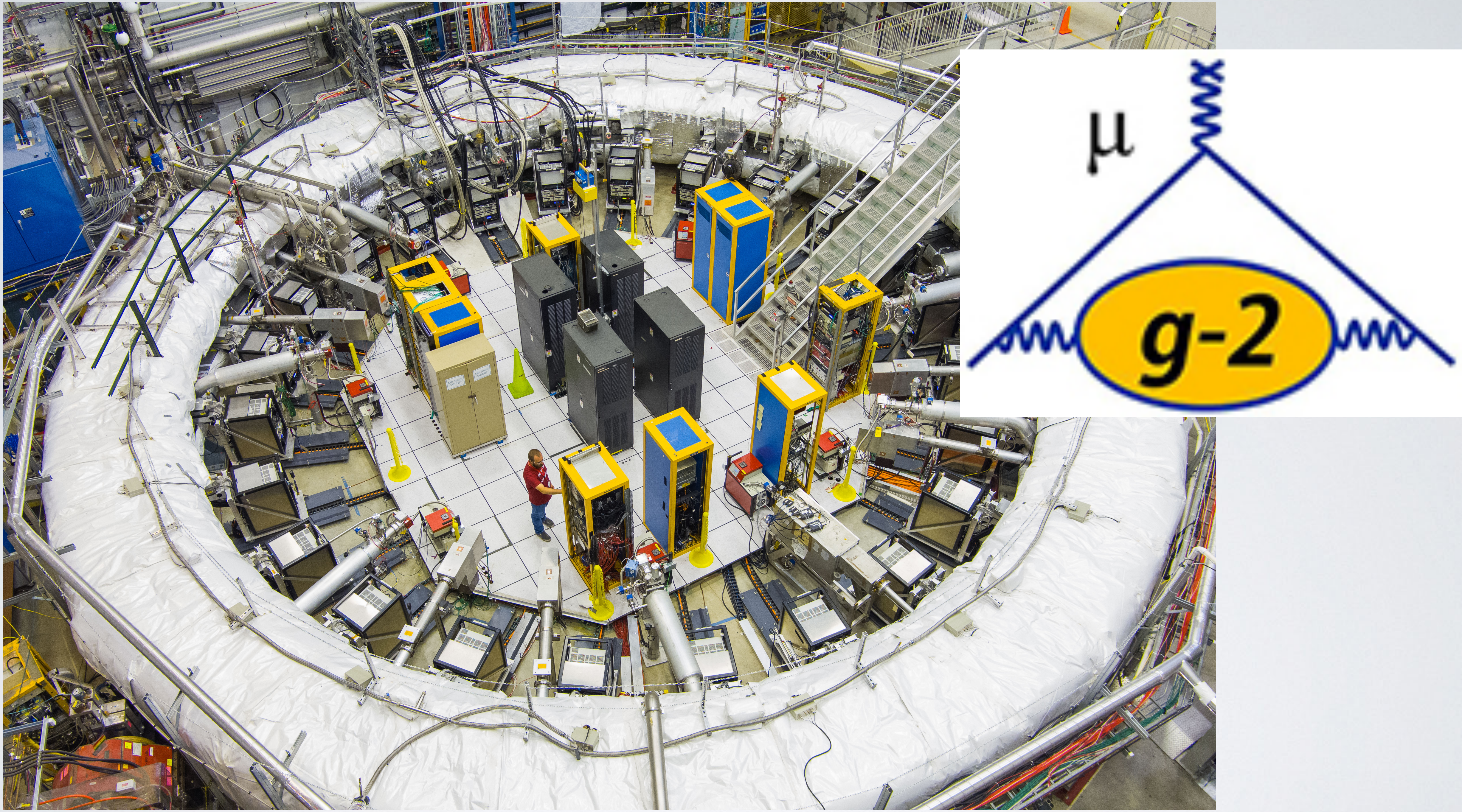


QED		+…	(5 loops)	$116\,584\,718.9\,(1) \times 10^{-11}$	0.001 ppm
EW		+…	x	$153.6\,(1.0) \times 10^{-11}$	0.01 ppm
HVP	α^2 	+…	(NNLO)	$6845\,(40) \times 10^{-11}$ [0.6%]	0.34 ppm
HLbL	α^3 	+…	(NLO)	$92\,(18) \times 10^{-11}$ [20%]	0.15 ppm

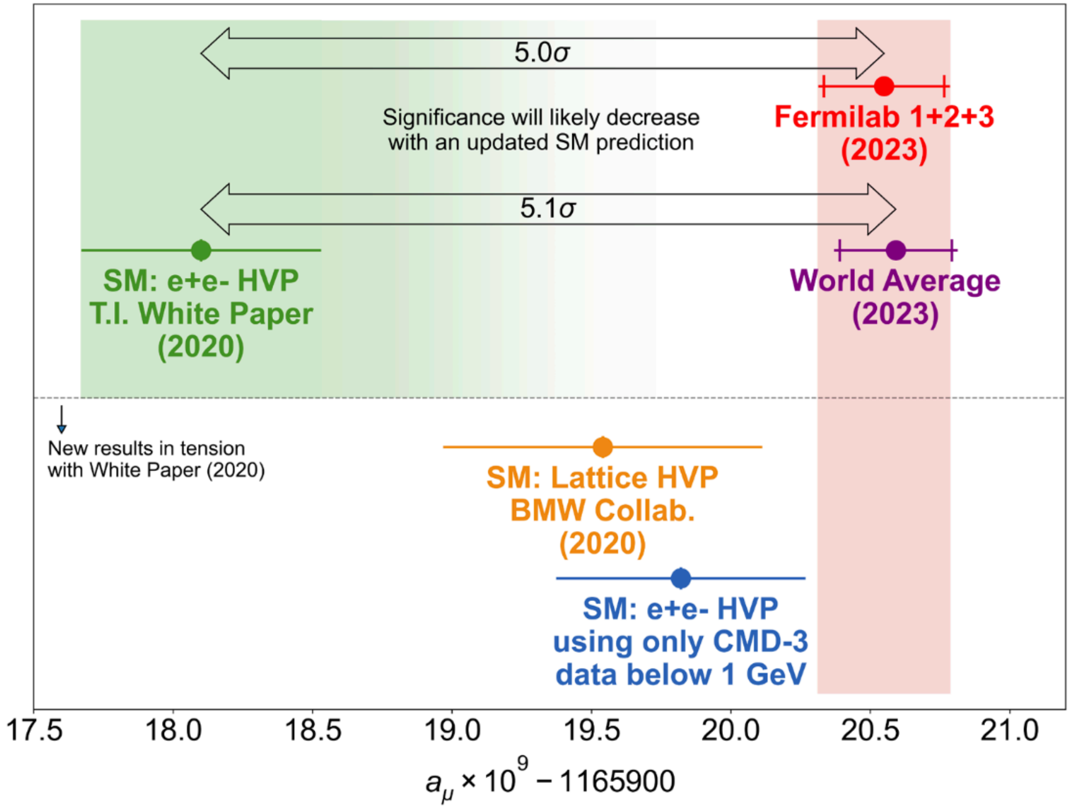
QED and EW: Well-known

HVP and HLbL: Non-perturbative (Data-driven & lattice QCD)

- QED and EW contributions are very well-known with small uncertainties
- Hadronic contribution error dominates the uncertainty budget
- HVP needs to be on the 0.5% precision to keep up with the experiment uncertainties
- HLBL precision demand is less thank HVP, only 10% would be good enough
- Refining the SM calculations means refining the HVP calculation
- **Muon g-2 Theory Initiative** was formed to determine SM value of a_μ. Produce a single consensus theoretical value which is comparable to the experimental value.


μ
g-2

A new challenge for theory

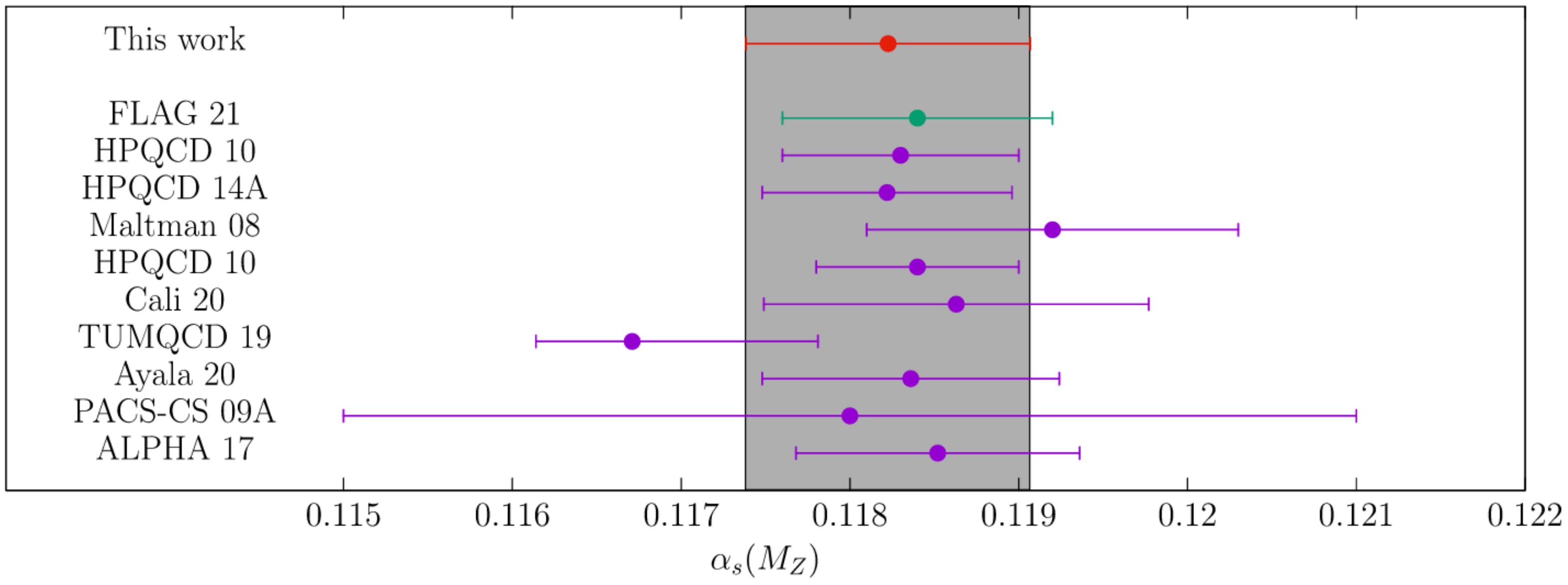


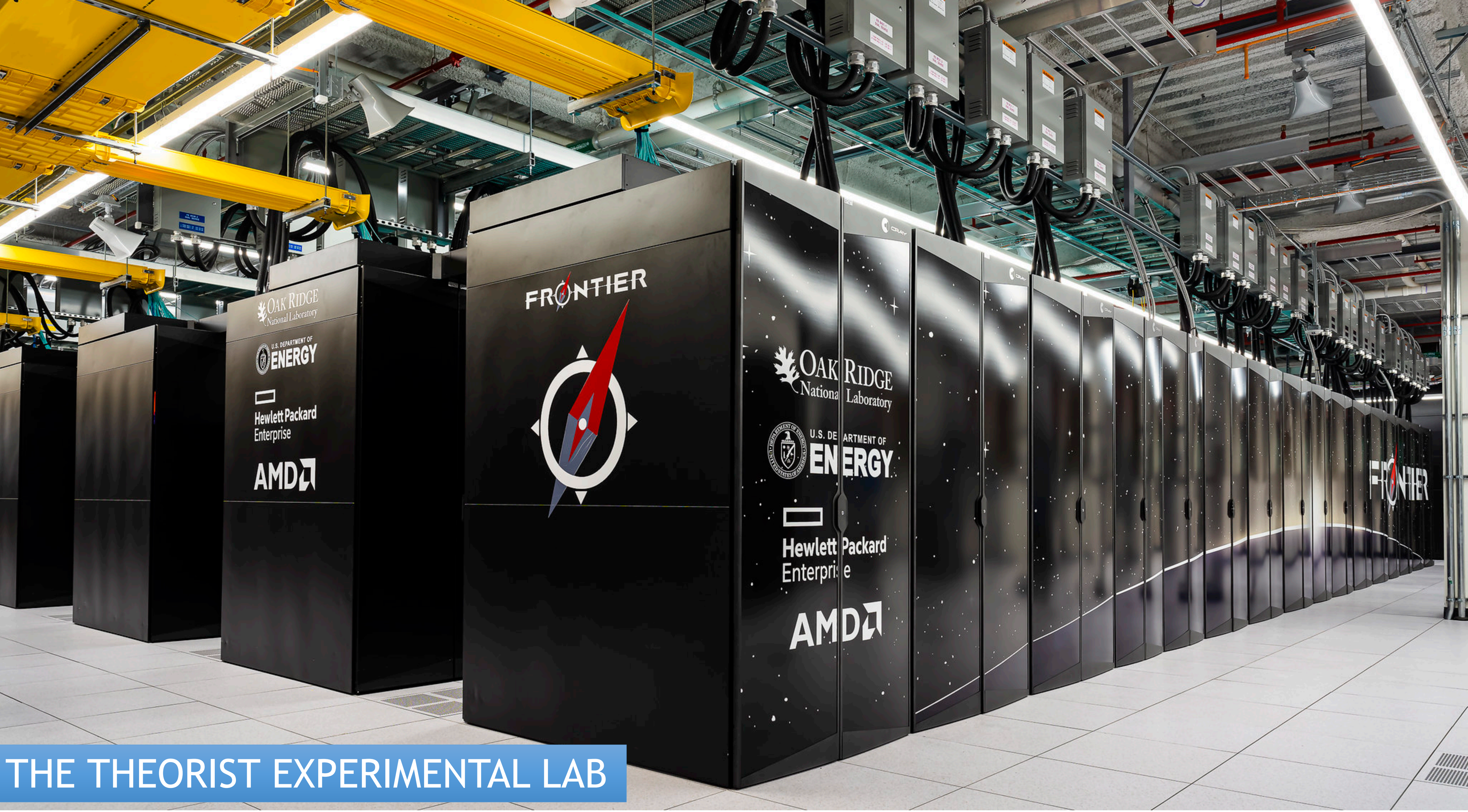
[Mott 08/2023]

Decoupling for α_s

- Decoupling allows to relate $n_f = 3$ QCD to $n_f = 0$ QCD
- Step scaling methods in $n_f = 0$ QCD allow high-precision results
- Result dominated by statistical errors

[Dalla Brida 22]





THE THEORIST EXPERIMENTAL LAB

Ok, BE REALISTIC TO GET GOING!

The art of doing mathematics consists finding that special case which contains all the germs of generality.

David Hilbert Mathematician, Physicist, Philosopher

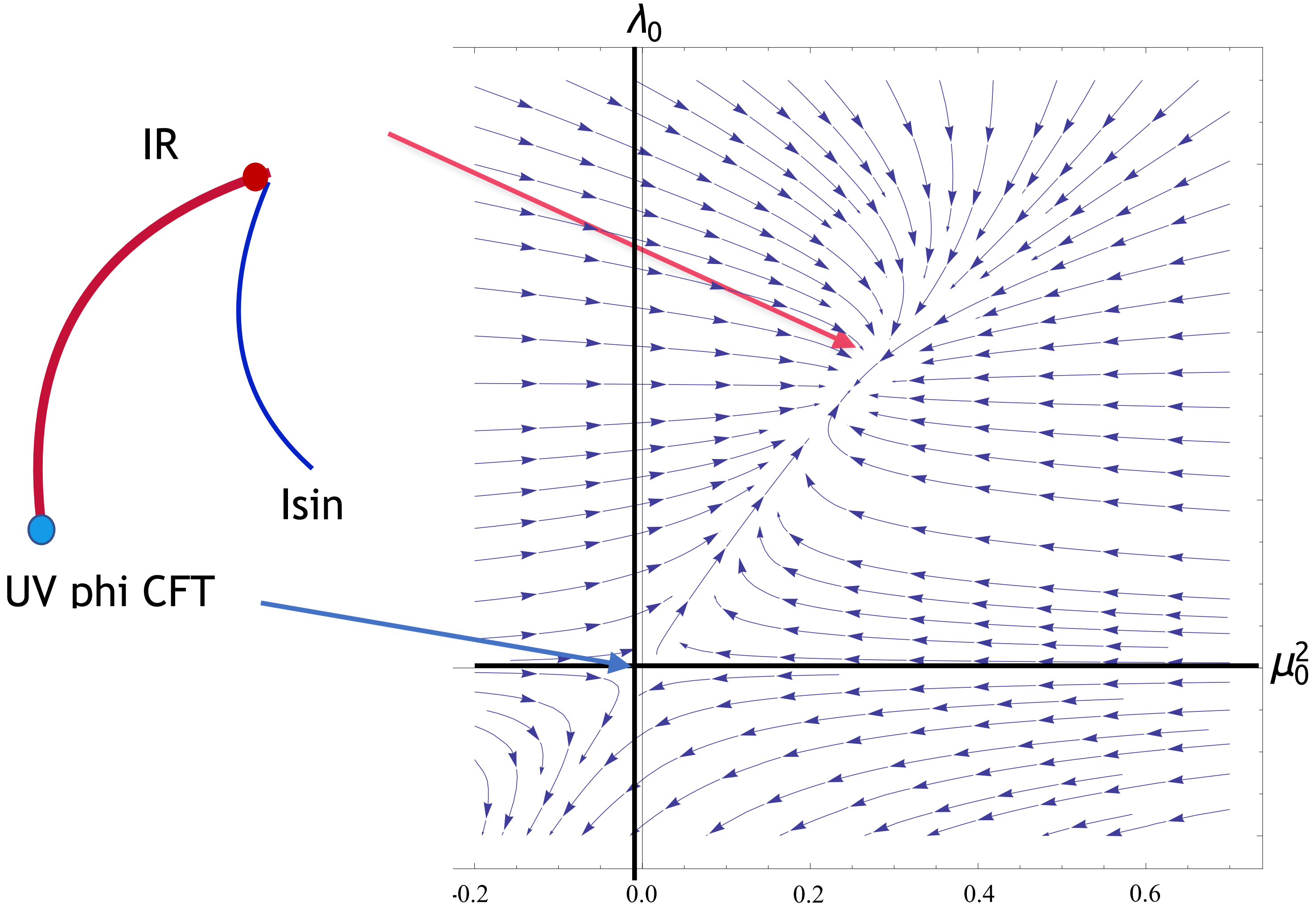
Author of Geometry and the Imagination

Scalar Phi4/Ising Universality

$$\lambda_0 \to \infty$$

$$S[\phi_i] = -\frac{1}{2}\sum_{\langle i,j\rangle}(\phi_i - \phi_j)^2 - \frac{\lambda_0}{2}(\phi_i^2 - \mu_0/\lambda)^2$$

$$S_{Ising} = -K\sum_{\langle i,j\rangle} s_i s_j = \frac{K}{2}\sum_{\langle i,j\rangle}(s_i - s_j)^2$$



$$S[\phi_i] = \frac{K}{2}\sum_{\langle i,j\rangle}(\phi_i - \phi_j)^2 + \frac{\lambda_0}{2}(\phi_i^2 - 1)^2$$

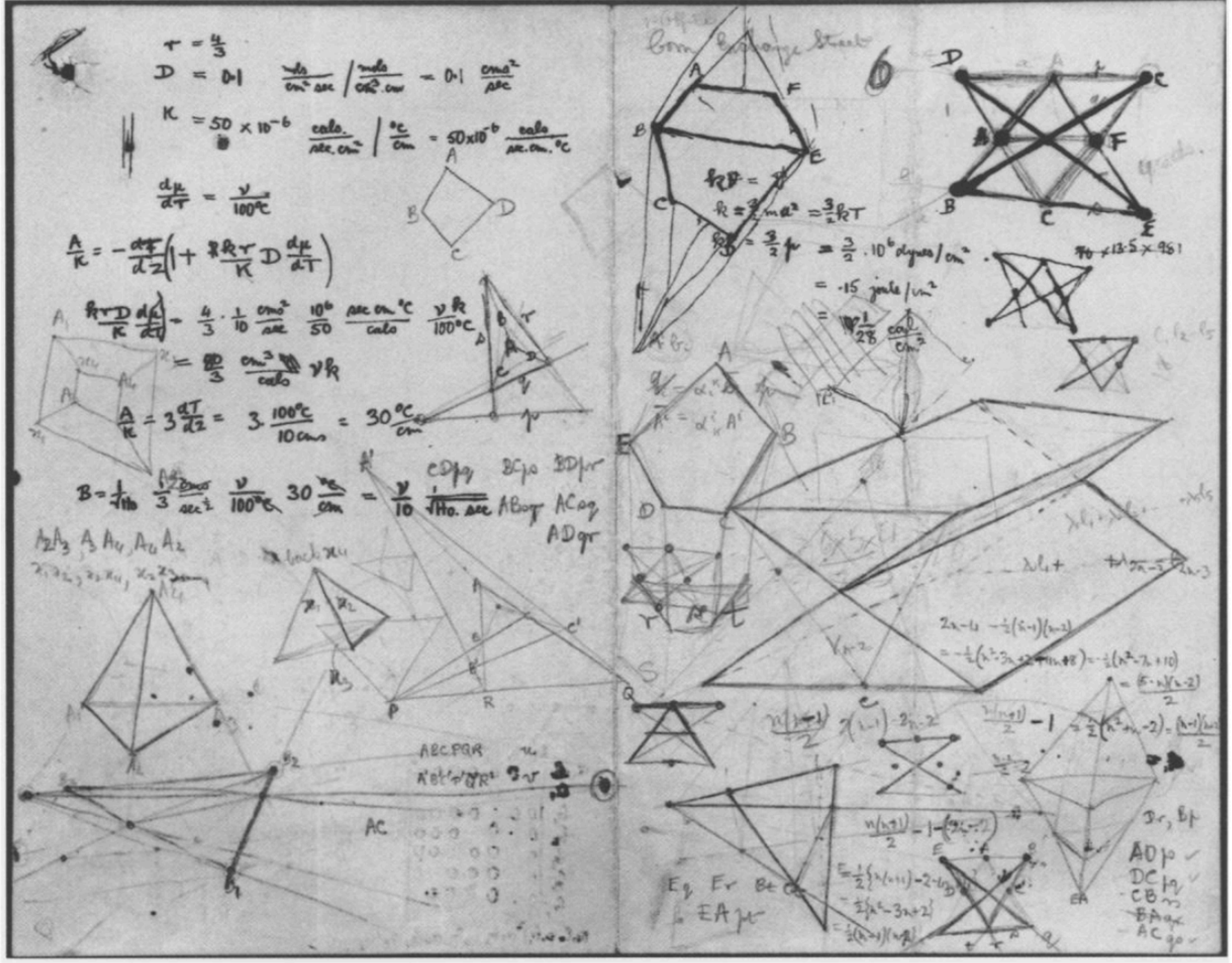


FIGURE 1. Paul Dirac, Geometrical Sketches, in the Paul A. M. Dirac Papers, Florida State University, Tallahassee, Florida; hereafter PDP. By permission of the Florida State University Libraries.

PART 1 : FIRST ATTEMPT

First step: Construct the Classical Simplicial Action

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x \sqrt{g(x)}\,[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)]$$

$g_{\mu\nu}(x)$

Regge Calc Geometry

Quantum field $\phi(x)$

Finite Element Method

Classical Simplicial Action

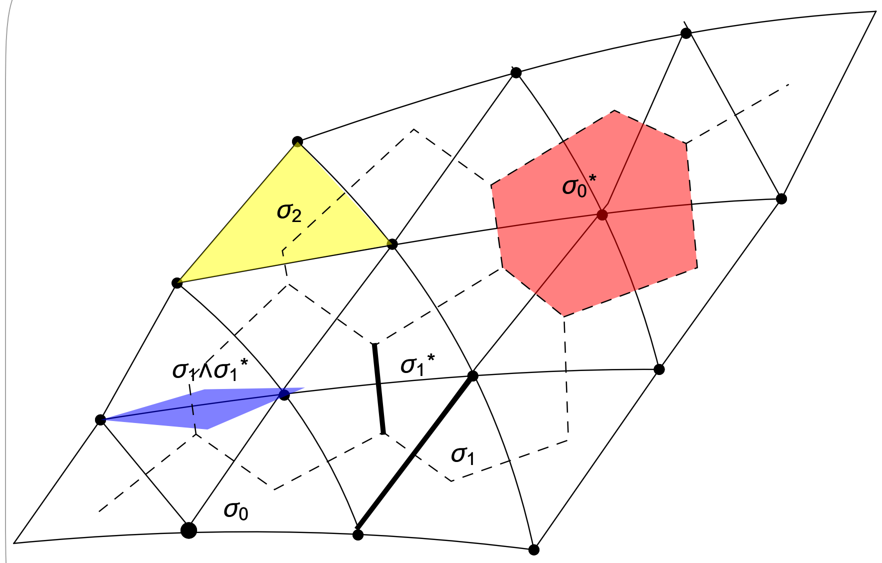
$$S_{FEM} = \frac{1}{2}[\sum_{y\in\langle\, x,y\rangle} K_{xy}(\phi_x - \phi_y)^2 + \sqrt{g_x}[\xi\ Ric\ \phi_x^2 + m_0^2\phi_x^2 + \lambda_0\phi_{t,x}^4]]$$

Start with Classical Simplicial Lattice

Gravitation Metric Manifold

REGGE: Piecewise linear metric

$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$

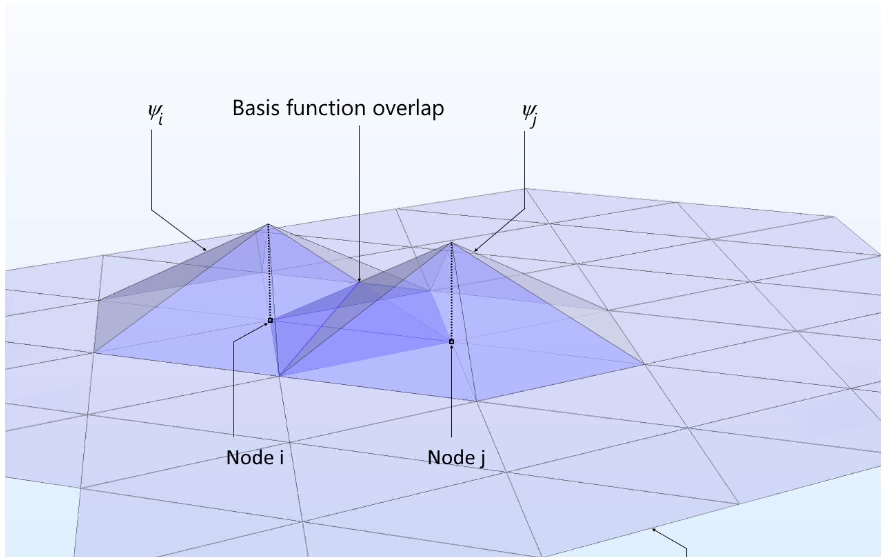


Simplicial Complex/Delaunay Dual Complex + Regge flat metric on each Simplex

Classical Fields: PDEs

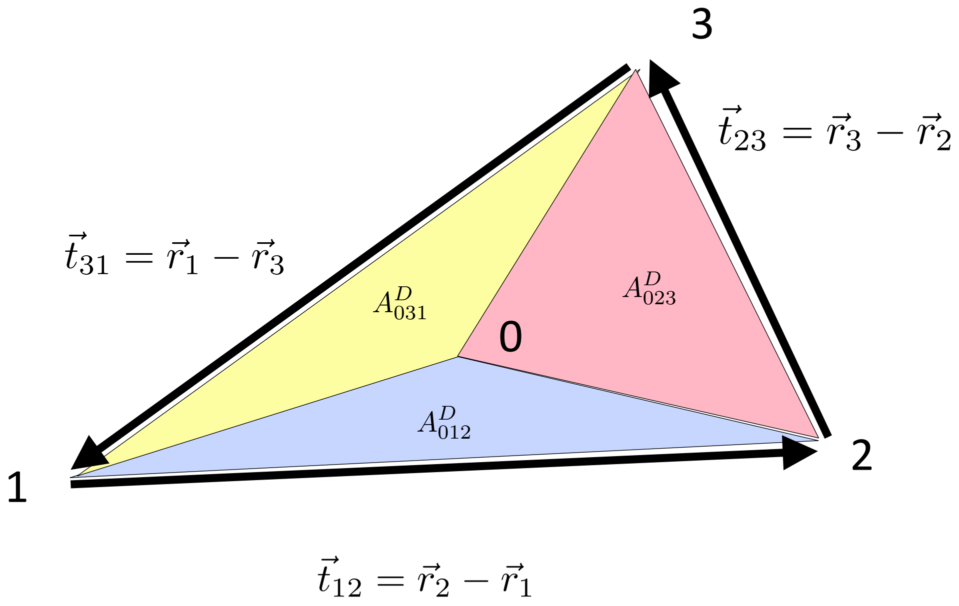
FEM: Piecewise linear fields

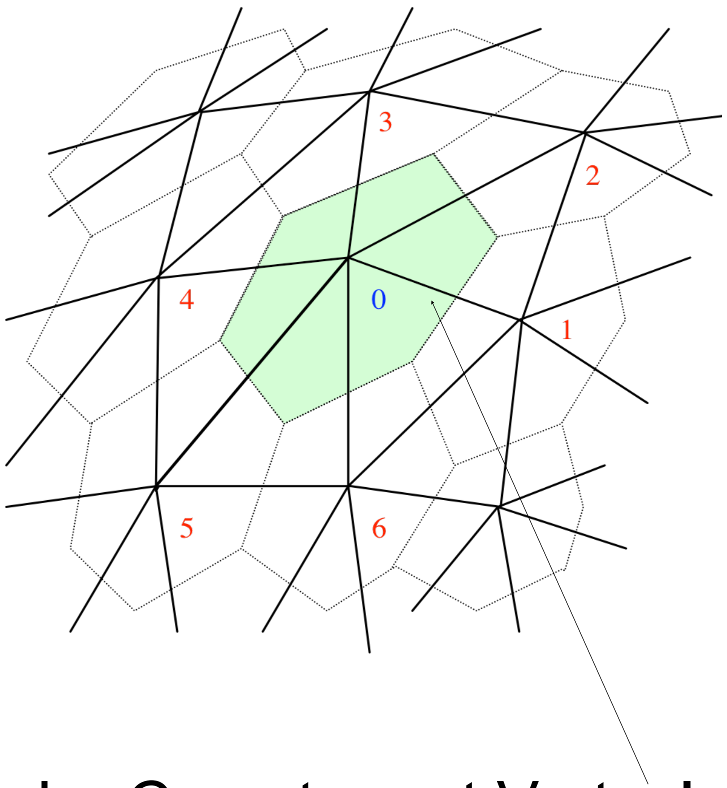
$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$



Actually fancier methods: **Discrete Exterior Calculus** (scalar), Spin connection (Fermion), Wilson links (gauge) , etc.

The Affine Triangle





Singular Curvature at Vertex!

The l's fix metric and the local co-ordinates (diffeomorphism) and the angles the intrinsic curvature.

SUMMARY OF CLASSICAL FEM SIMPLICIAL LATTICE FIELDS

J = 0

$$S_{scalar} = \frac{1}{2}\sum_{\langle i,j\rangle} \frac{V_{ij}}{l_{ij}^2}(\phi_i - \phi_j)^2 \;,$$

$$l_{ij}^2 = |\sigma_1(ij)|^2$$

J = 1/2

$$S_{Wilson} = \frac{1}{2}\sum_{\langle i,j\rangle} \frac{V_{ij}}{l_{ij}}(\bar{\psi}_i \hat{e}_a^{j(i)}\gamma^a\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\hat{e}_a^{i(j)}\gamma^a\psi_i)$$

J = 1

$$S_{gauge} = \frac{1}{2g^2N_c}\sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}]$$

FFdual

$$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl}\epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0)F_{\rho\sigma}(0)]$$

But Dirac needs Spin Connection (Kahler Dirac doesn't)

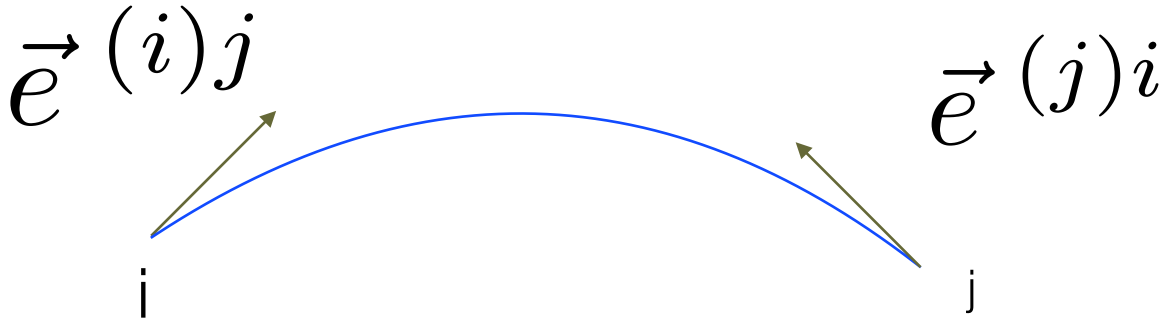
$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$$

Verbein & Spin connection*

$$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab} \quad , \quad \sigma_{ab} = i[\gamma_a, \gamma_a]/2$$

$$S_{naive} = \frac{1}{2}\sum_{\langle i,j\rangle}\frac{V_{ij}}{l_{ij}}[\bar{\psi}_i\vec{e}^{\,(i)j}\cdot\vec{\gamma}\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\vec{e}^{\,(i)j}\cdot\vec{\gamma}\psi_i] + \frac{1}{2}mV_i\bar{\psi}_i\psi_i$$



Simplicial Tetrad Hypothesis

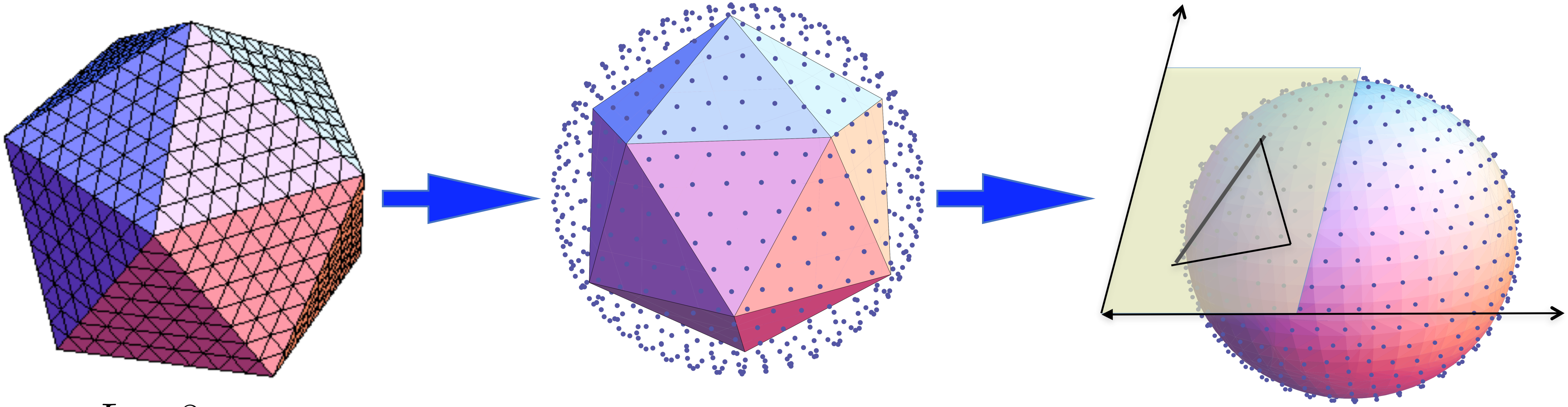
$$e_a^{(i)j}\gamma^a\Omega_{ij} + \Omega_{ij}e_a^{(j)i}\gamma^a = 0$$

Gauge Invariance under Spin(D) transformations

$$\psi_i \to \Lambda_i\psi \quad , \quad \bar{\psi}_j \to \bar{\psi}_j\Lambda_j^\dagger \quad , \quad \mathbf{e}^{(i)j} \to \Lambda_i\mathbf{e}^{(i)j}\Lambda_i^\dagger \quad , \quad \Omega_{ij} \to \Lambda_i\Omega_{ij}\Lambda_j^\dagger$$

Note generalization to Domain Wall straight forward. Add an extra flat direction. Limit of extra dimension is overlap Fermion.

First Attempt (with good results) on refined octahedron



$L = 8$

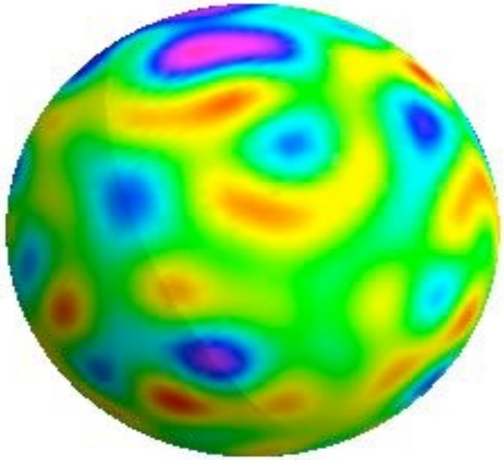
l = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

$N - F + E = 2$

$$F = N_{\triangle} = 20L^2 \text{ and dof: } 2N = 4 + 20L^2$$

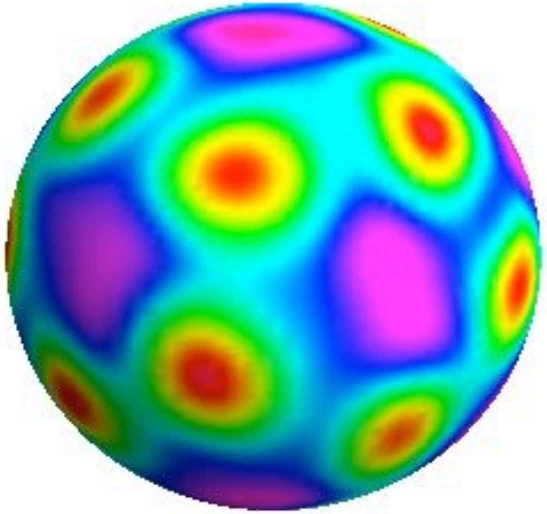
Now add $\lambda\phi^4$ term: What happens to FEM?

$\phi^2(x)$

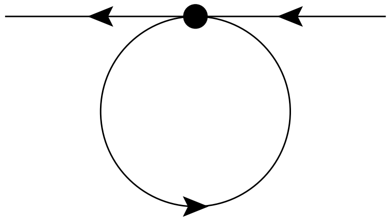


one configuration

$\langle \phi^2(x) \rangle$

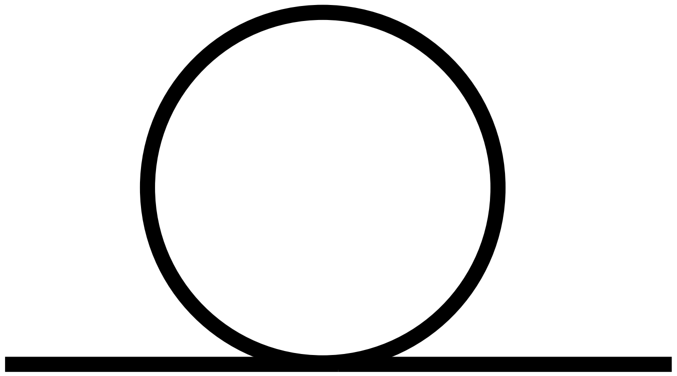


Average of config.



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$

Perturbative CT on the Sphere



$$\Delta m_i^2 = 6\lambda \left[K^{-1}\right]_{ii} \simeq \frac{\sqrt{3}}{8\pi}\lambda \log(1/m_0^2 a_i^2) = \frac{\sqrt{3}}{8\pi}\lambda \log(N_s) + \frac{\sqrt{3}}{8\pi}\lambda \log(a^2/a_i^2)$$

$$\delta\mu_i^2 = -6\lambda(\left[K^{-1}\right]_{ii} - \frac{1}{N_s}\sum_{j=1}^{N_s}\left[K^{-1}\right]_{jj})$$

NUMERICAL TEST against Exact c=1/2 Ising CFT

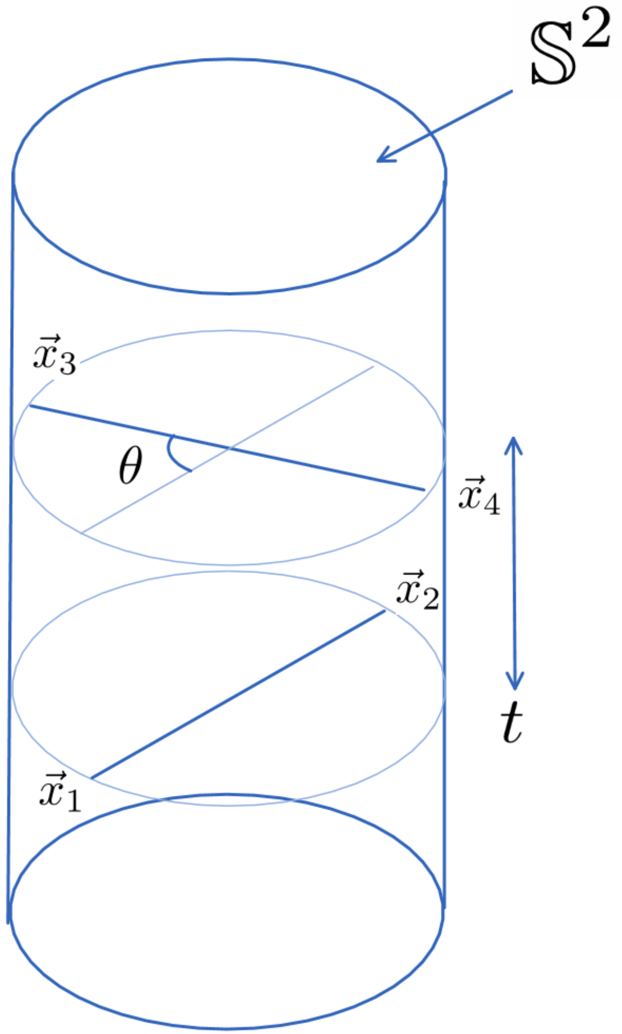
μ^2	s	$r_{\min} \le r \le r_{\max}$	norm	Δ_ϵ	λ_ϵ^2	c
1.82241	9	$0.25 \le r \le 0.75$	0.2900	1.075	0.2536	0.4668
1.82241	9	$0.30 \le r \le 0.70$	0.2901	1.075	0.2533	0.4704
1.82241	9	$0.35 \le r \le 0.65$	0.2902	1.077	0.2533	0.4738
1.82241	9	$0.40 \le r \le 0.60$	0.2902	1.016	0.2427	0.4747
1.82241	18	$0.25 \le r \le 0.75$	0.2051	1.068	0.2563	0.4866
1.82241	18	$0.30 \le r \le 0.70$	0.2051	1.056	0.2544	0.4878
1.82241	18	$0.35 \le r \le 0.65$	0.2051	1.050	0.2535	0.4904
1.82241	18	$0.40 \le r \le 0.60$	0.2051	1.046	0.2526	0.4884
1.82241	36	$0.25 \le r \le 0.75$	0.1457	1.031	0.2528	0.4926
1.82241	36	$0.30 \le r \le 0.70$	0.1458	1.026	0.2519	0.4932
1.82241	36	$0.35 \le r \le 0.65$	0.1458	1.018	0.2508	0.4931
1.82241	36	$0.40 \le r \le 0.60$	0.1458	1.007	0.2486	0.4933

Lattice Sizes: N = 32+ 10 s^2 sites

Antipodal 4-point function on $\mathbb{R} \times \mathbb{S}^2 \ni (t, \vec{x})$

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle \sigma(x_1)\sigma(x_2)\rangle\langle \sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} G_{\mathcal{O}}(\Delta_{\mathcal{O}}; x_1, x_2, x_3, x_4)$$

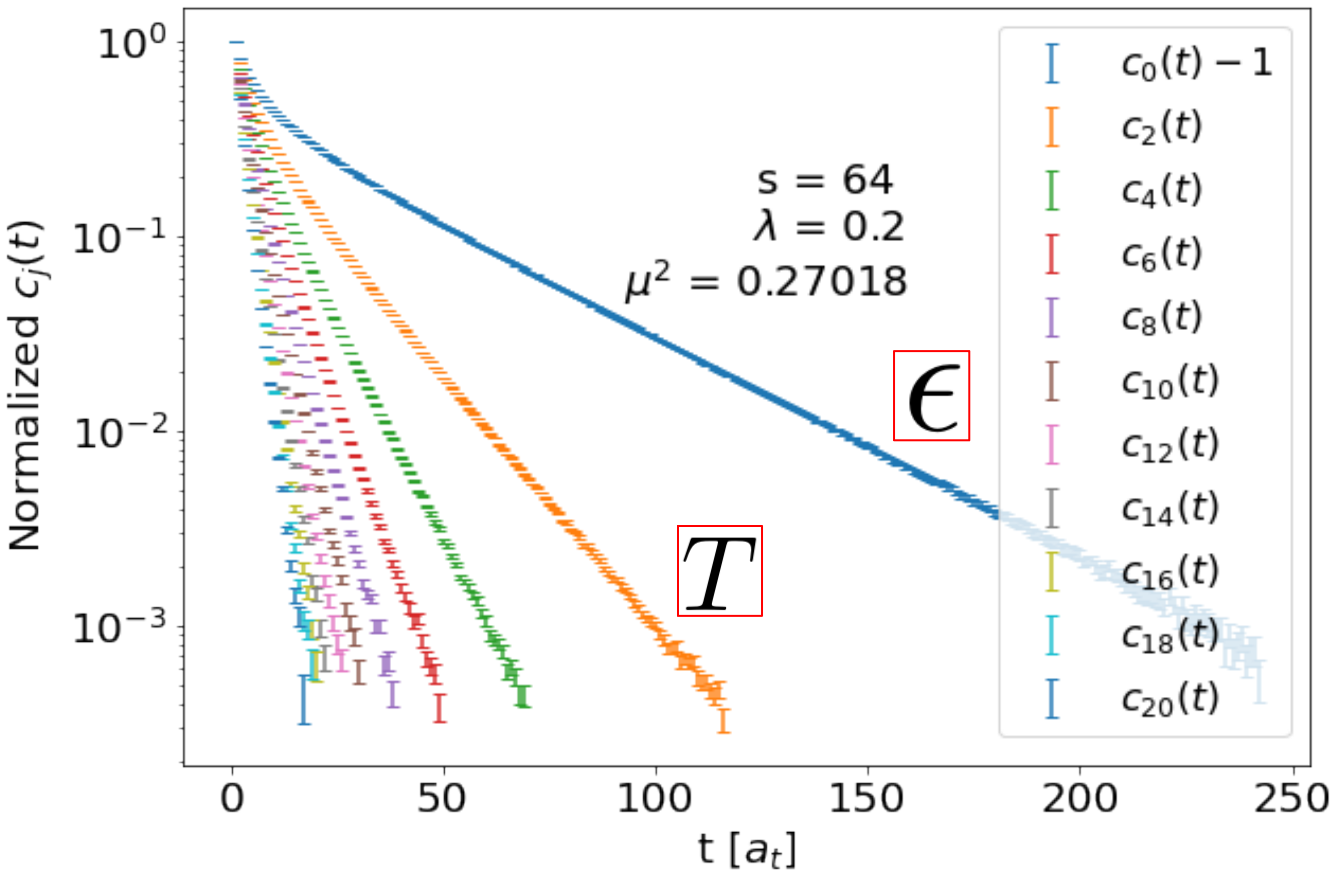
$$G_{\mathcal{O},l} = \sum_{n=0,2,4,\ldots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$



Numerical results

$$j \in \{\max(0, l-n), ..., l+n-2, l+n\}$$

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle \sigma(x_1)\sigma(x_2)\rangle\langle \sigma(x_3)\sigma(x_4)\rangle} = \sum_{\text{even } j} c_j(\Delta t) P_j(\cos(\theta)) = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \sum_{n=0,2,4,\dots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)c_R g t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$



Simultaneous fits of $c_0(t)$ and $c_2(t)$ using primaries $\epsilon, \quad T, \quad \epsilon', \quad T'$ up to n=20

PART II
GEOMETRY: AFFINE &SIMPLITIAL

The Problem of Classic vs Quantum Geometry

Classical Field Geometry

$$S = S_{EH} + S_M = \int dx \sqrt{g} [\frac{1}{2\kappa} R + \mathcal{L}_M]$$

[put in scalar field

$$\frac{\delta}{\delta g^{\mu\nu}(x)} \implies R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = \kappa T_{\mu\nu}(x)$$

Quantum Field Geometry

$$\frac{\delta \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle}{\delta g^{\mu\nu}(x)} = \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) T_{\mu\nu}(x)\rangle$$

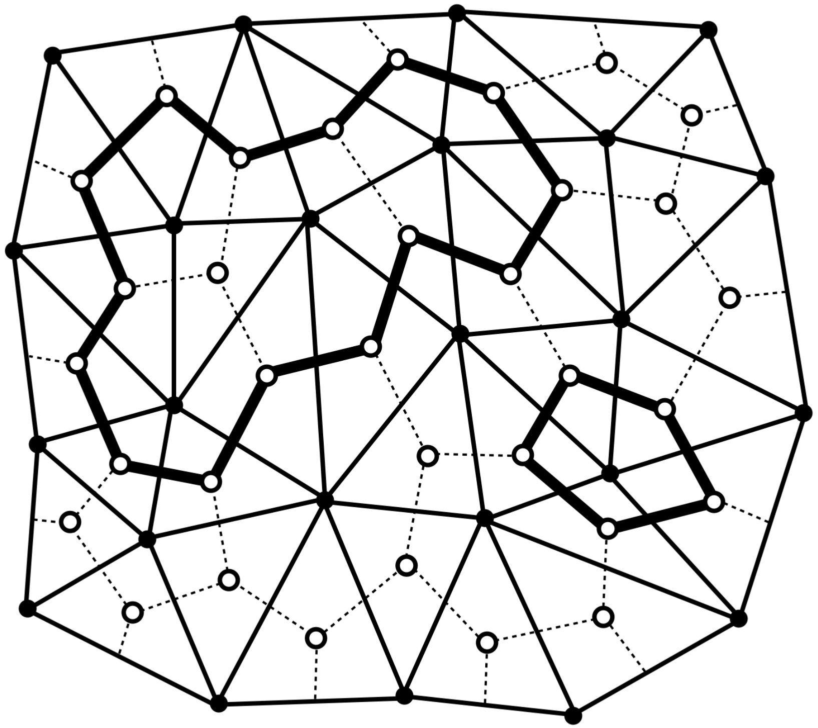
TWO BIG QUESTION?

GEOMETRY RECOVERED IN THE CONTINUUM:CLASSICAL GR VS QUANTUM MATTER

- REGGE CLASSICAL GR SIMPLICIAL FIXES THE PIECE WISE GEOMETRY:
  - BUT HOW DOES IT RECOVER THE DIFFERENTIAL MANIFOLD?
- THE LATTICE QUANTUM FIELD THEORY FIXES LATTICE ACTION COUPLINGS?
  - BUT HOW DOES QUNTUM MATTER MATCH GEOMETRY OF THE MANIFOLD?

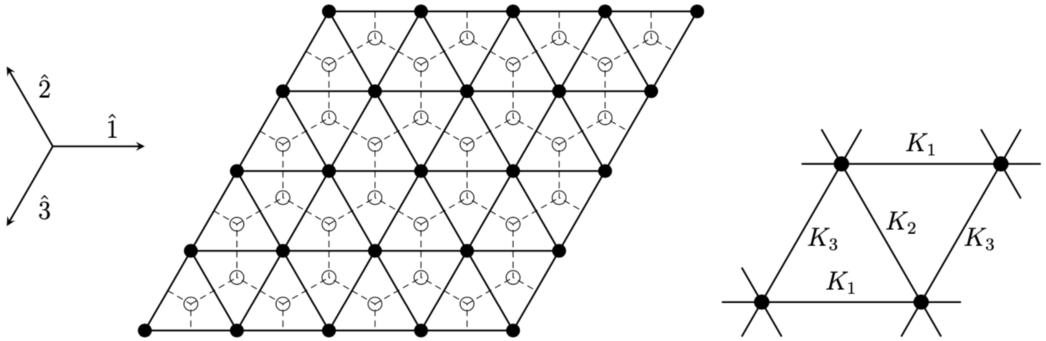
REGGE => DISCRETE GEOMETRY QUANTUM LATTICE FIELD TH ==> COUPLINGS

$$\{G, \ell_{ij}\}$$



Lattice Field Theory has NO dimensional parameter. Just topological topological graph

$$Z^{\triangle} = \sum_{s_n=\pm 1} e^{K_1 s_n s_{n+\hat{1}} + K_2 s_n s_{n+\hat{2}} + K_3 s_n s_{n+\hat{3}}},$$



REGGE:
“General Relativity without Coordinates” 1960

- The Simplicial Approximation Theory (L.E.J. Brouwer 1927?)

In mathematics, the **simplicial approximation theorem** is a foundational result for algebraic topology, guaranteeing that continuous mappings can be (by a slight deformation) approximated by ones that are piecewise of the simplest kind. It applies to mappings between spaces that are built up from simplices—that is, finite simplicial complexes. The general continuous mapping between such spaces can be represented approximately by the type of mapping that is (*affine*-) linear on each simplex into another simplex, at the cost (i) of sufficient barycentric subdivision of the simplices of the domain, and (ii) replacement of the actual mapping by a homotopic one.

Einstein: $\{\mathcal{M}, g_{\mu\nu}\}$

Regge: $\{G, \ell_{ij}\}$

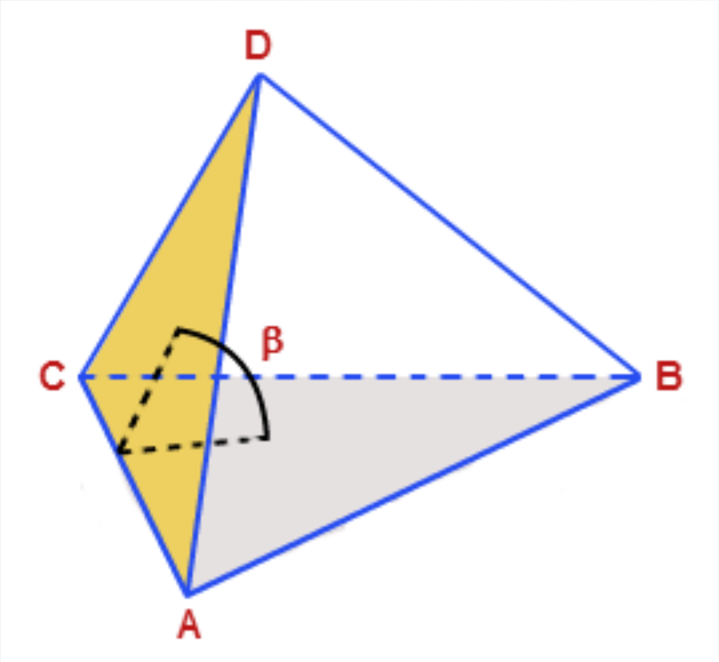
$$S_{EH} = \int d^d x \sqrt{g(x)} R(x)$$

$$S_{Regge}[\ell_{ij}] = 2 \sum_{h \in G} A_h^* \epsilon_h$$

Schlafli Identity in 2D and 3D

$$\sum_f V_f \hat{n}_f = \sum_f V_f h_f \vec{\nabla}\xi_f = 3V_T \sum_f \vec{\nabla}\xi_f = 0$$

$$\implies 0 = -\sum_f V_f \hat{n}_f \cdot \hat{n}_{f'} = \sum_f V_f \cos(\theta_{ff'})$$



$$V_D = \frac{D-1}{D}\frac{V_f V_{f'}}{V_h}\sin(\theta_{ff'})$$

Affine => The Metric & the d Simplex

$$X = A\xi + b \quad \Longrightarrow \quad dx^\mu = A^\mu_i d\xi^i$$

$$ds^2 = d\vec{X}\cdot d\vec{X} = (A^T A)_{ij} d\xi^i d\xi^j = \sum_\mu e^\mu_i e^\mu_j d\xi^i d\xi^j = \vec{e}_i \cdot \vec{e}_j d\xi^i d\xi^j$$

- The affine map is d(d+1)/2 Poincare+ d(d+1)/2 shearing.
- All simplexes are affine equivalent.
- d = 2 -> 3 edges, d = 3 -> 6 edges d = 4 -> 10 edges
- The Affine and Conformal Extension of Poincare group share scaling operator

In a simplex $\vec{X} = \vec{x}_i \xi_i + \xi_0 \vec{x}_0$ with $i = 1, .., d$ and $\xi_0 = 1 - \sum_i \xi_i$

PART III
ISING ON SPHERE

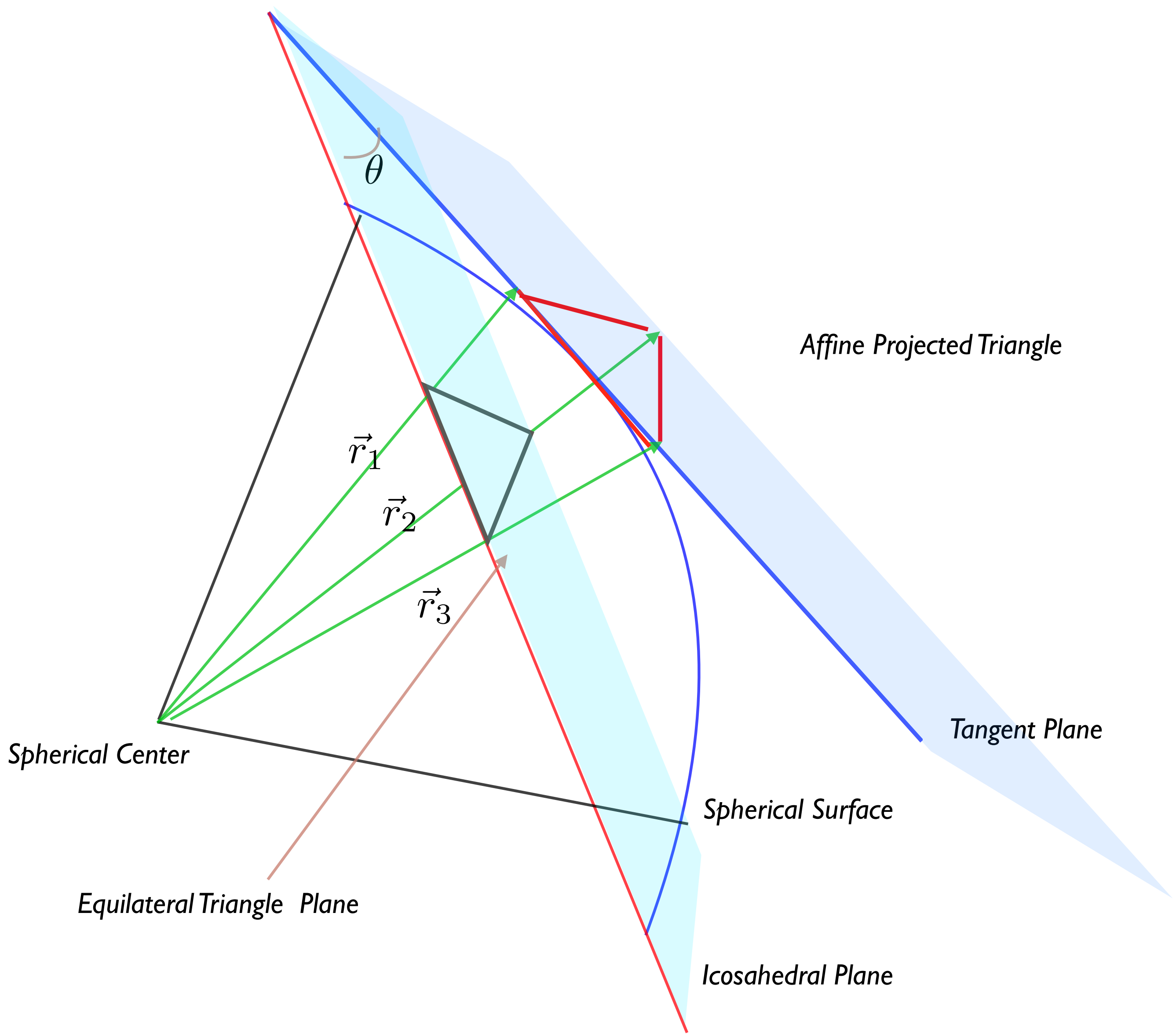
Ising Model on the Affine Plane

$$\mathbb{R}^2$$

EXTENSION OF POINCARE INTRANSFORMATION;

AFFINE VS CONMFORMAL EXTENSION:

To O(a^2) the tangent plane is an Affine lattice on each tangent plane.



EXACT Example of Emergent Geometry

- Free (FEM) scalar CFT.

$$S_{\text{free}} = \frac{1}{2}\sum_n [K_1(\phi_n - \phi_{n+\hat{1}})^2 + K_2(\phi_n - \phi_{n+\hat{2}})^2 + K_3(\phi_n - \phi_{n+\hat{3}})^2]$$

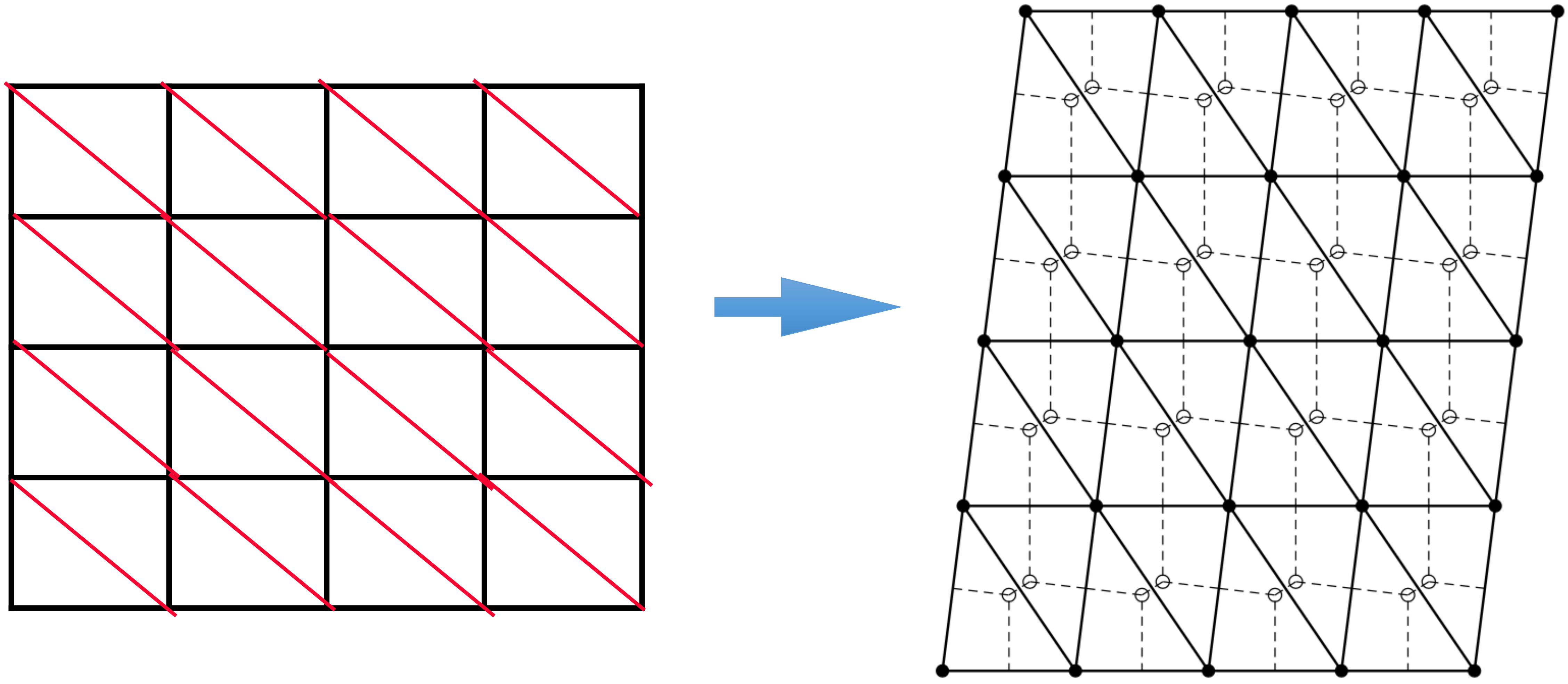
$$2K_1 = \ell_1^*/\ell_1 \quad , \quad 2K_2 = \ell_2^*/\ell_2 \quad , \quad 2K_3 = \ell_3^*/\ell_3 \ .$$

- Critical Ising at

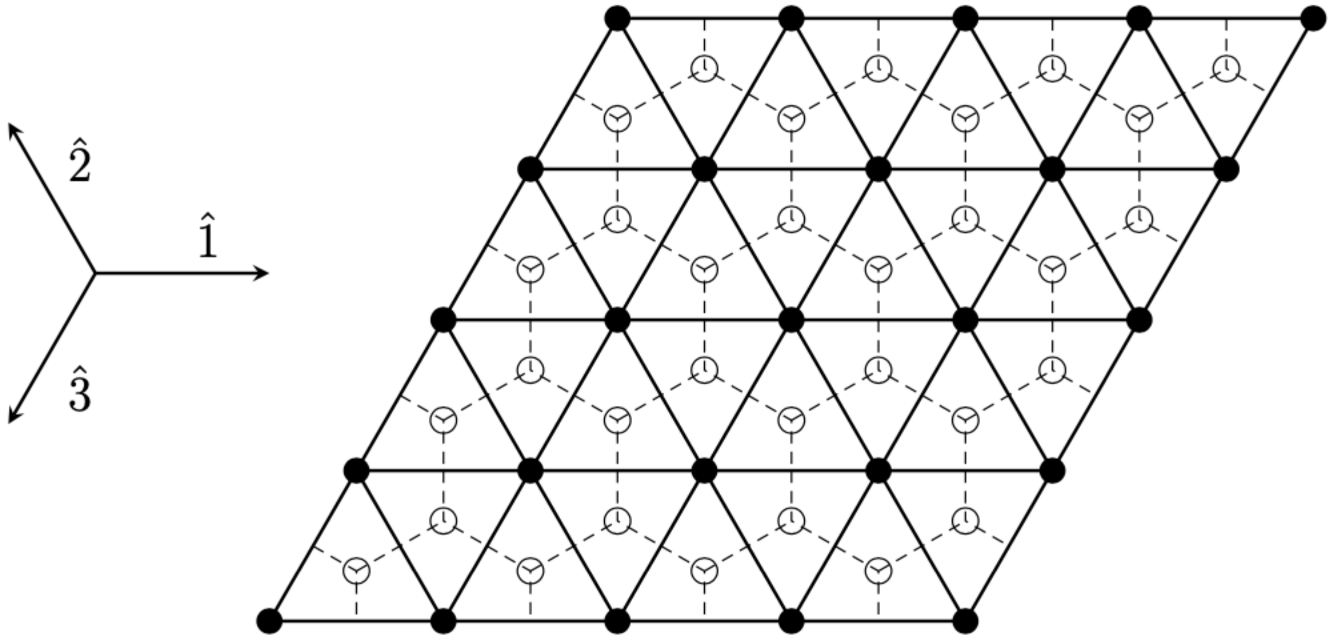
$$\sinh(2K_1) = \ell_1^*/\ell_1 \quad , \quad \sinh(2K_2) = \ell_2^*/\ell_2 \quad , \quad \sinh(2K_3) = \ell_3^*/\ell_3$$

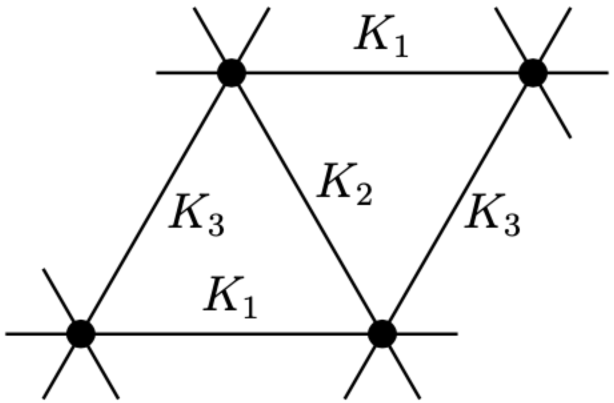
$$p_1 p_2 + p_2 p_3 + p_3 p_1 = 1 \quad \text{with} \quad p_i = \exp(-2K_i)$$

Affine: Square to triangle Circle to Ellipse



Ising Model on the Affine Plane





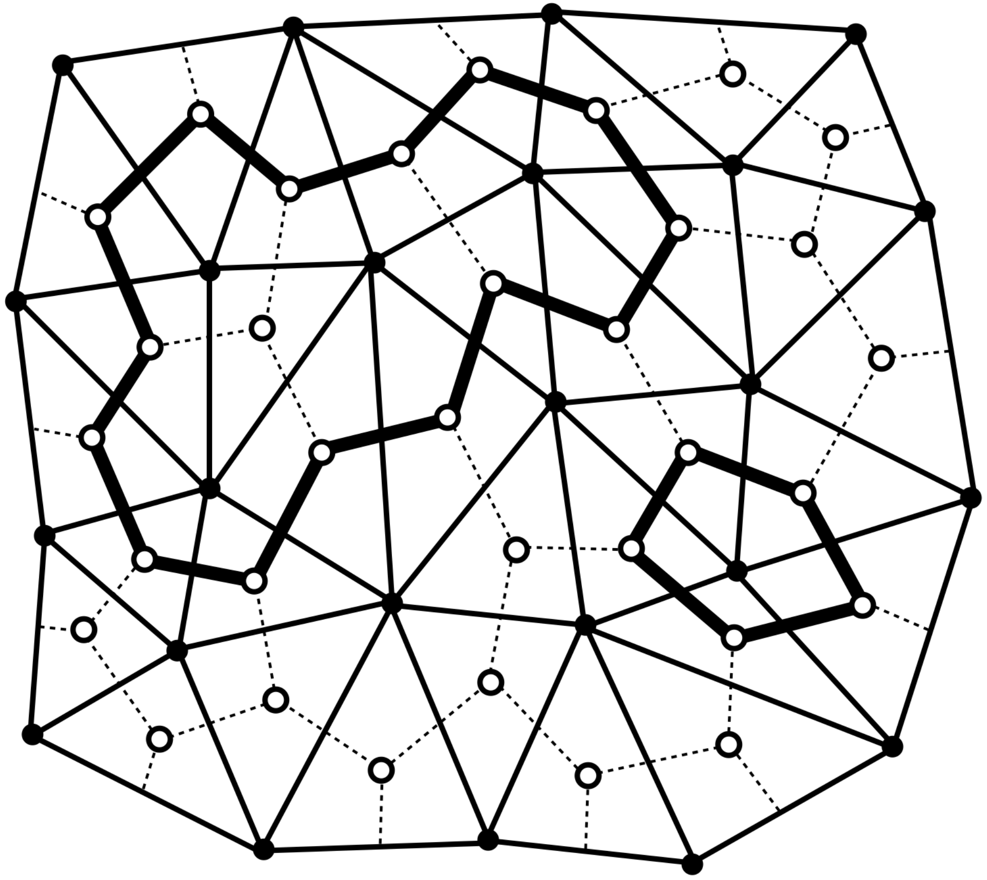
$$Z^{\triangle} = \sum_{s_n=\pm 1} e^{\,K_1 s_n s_{n+\hat{1}} + K_2 s_n s_{n+\hat{2}} + K_3 s_n s_{n+\hat{3}}}\ ,$$

Affine Parameters:
2d Affine transformation takes circle to ellipse:

$$\langle \phi(x,y)\phi(0)\rangle = \frac{1}{(x^2+y^2)^{\Delta_\phi}} \leftrightarrow \frac{1}{(ax^2+bxy+cy^2)^{\Delta_\phi}}$$

- d = 2 Poincare 1 rotation 2 translation
- New Affine plus 1 major/minor + 1 orientation + 1 scaling
- General Poincare d(d+1)/2 plus d(d+1)/2 the number of edge in d-simplex - local metric

3 Equivalent Loop Expansion for Partition Functions!



$$S_{\triangle} = -\sum_{\langle ij \rangle} K_{ij} s_i s_j$$

$$S_{\text{dual}} = -\sum_{\langle ij \rangle} L_{ij} s_i s_j$$

$$S_{\psi} = \frac{1}{2}\sum_{i} \bar{\psi}_i \psi_i - \sum_{\langle ij \rangle} \kappa_{ij} \bar{\psi}_i P_{ij} \psi_j \ ,$$

Kramers Wannier High T/Low T Loop expansion Wilson-Majorana Lattice Fermions

$$\sinh 2K_{ij} \sinh 2L_{ij} = 1$$

$$P_{ij} = \frac{1}{2}(1 + \hat{e}_{ij} \cdot \vec{\sigma})$$

Step II: Map Hexagonal Loop Expansion is easy to map to free Ising to Free Wilson-Majorana Fermion*

$$Z_N^\psi = \prod_n \iint d\psi_n^1 d\psi_n^2 \; e^{-S[\bar{\psi},\psi]} = \prod_n \int d^2\psi_n e^{-\frac{1}{2}\sum_n \bar{\psi}_n \psi_n} \prod_{n,i} [1 + \kappa_i \bar{\psi}_n P(\hat{e}_i)\psi_{n+\hat{i}}]$$

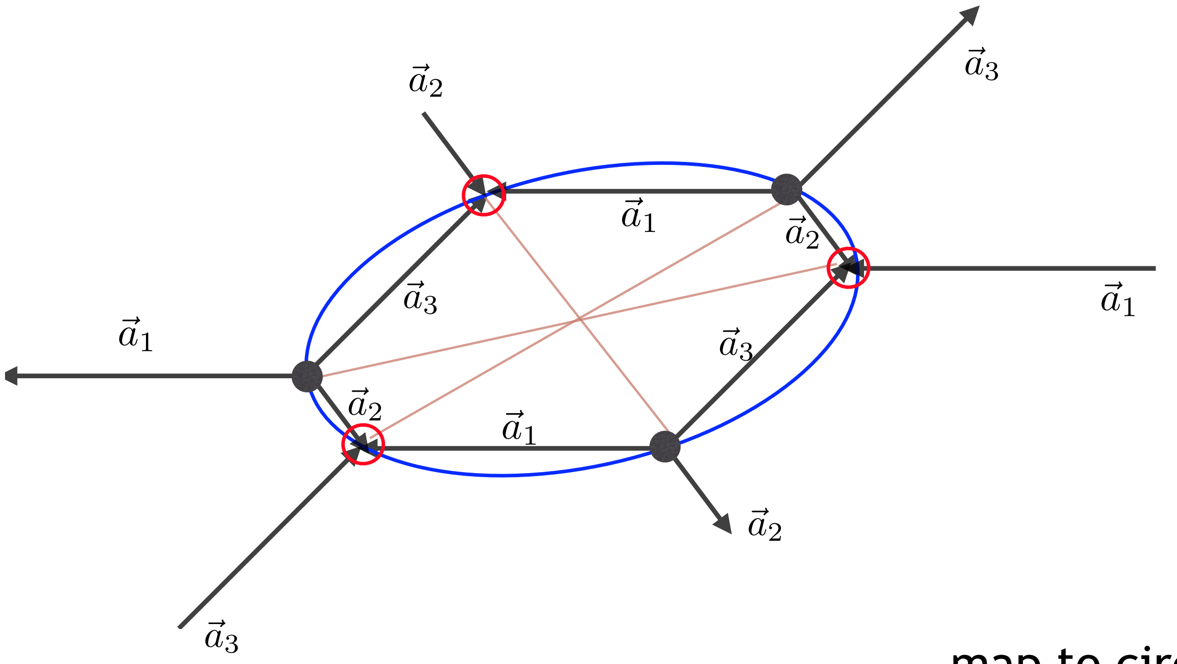
$$S[\psi] = \frac{1}{2}\sum_n \bar{\psi}_n \psi_n - \frac{1}{2}\sum_{n,i} \kappa_i \bar{\psi}_n (1 + \hat{e}_i \cdot \vec{\sigma})\psi_{n+\hat{i}} \; .$$

$$\tanh(L_1) = \frac{\kappa_1 \cos(\theta_{12}/2)\cos(\theta_{13}/2)}{\cos(\theta_{23}/2)}$$

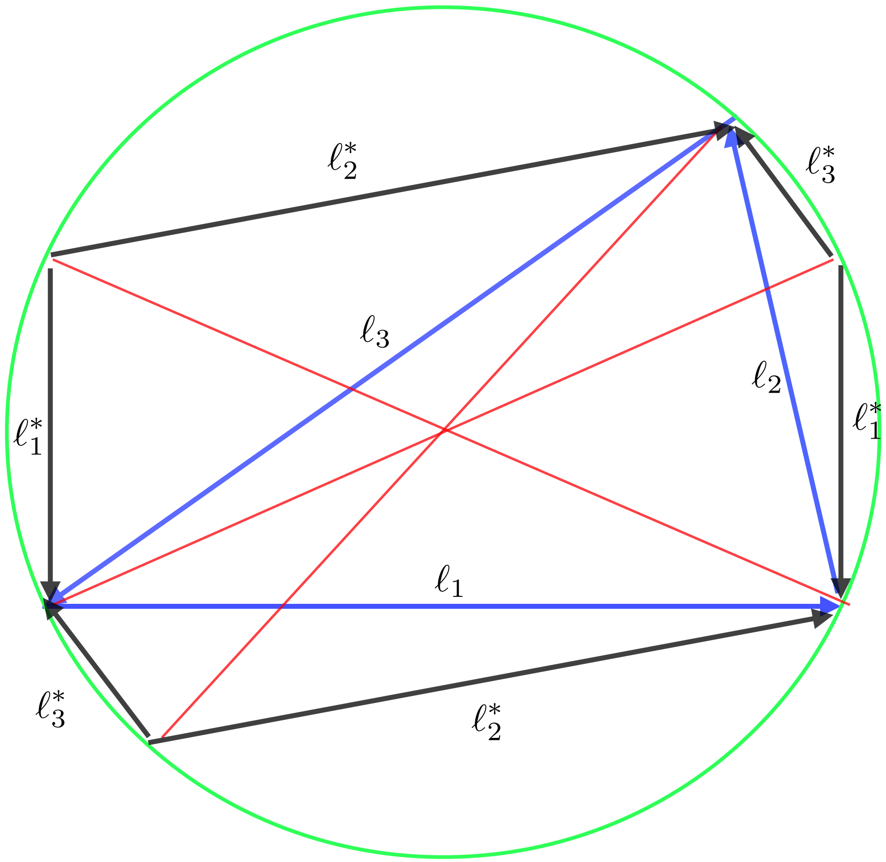
***Generalizing very nice paper by Ulli Wolff.**

Ising model as Wilson-Majorana Fermions. Nucl. Phys. B, 955:115061, 2020.

Elliptical Hexagon to a Circular Hexagon



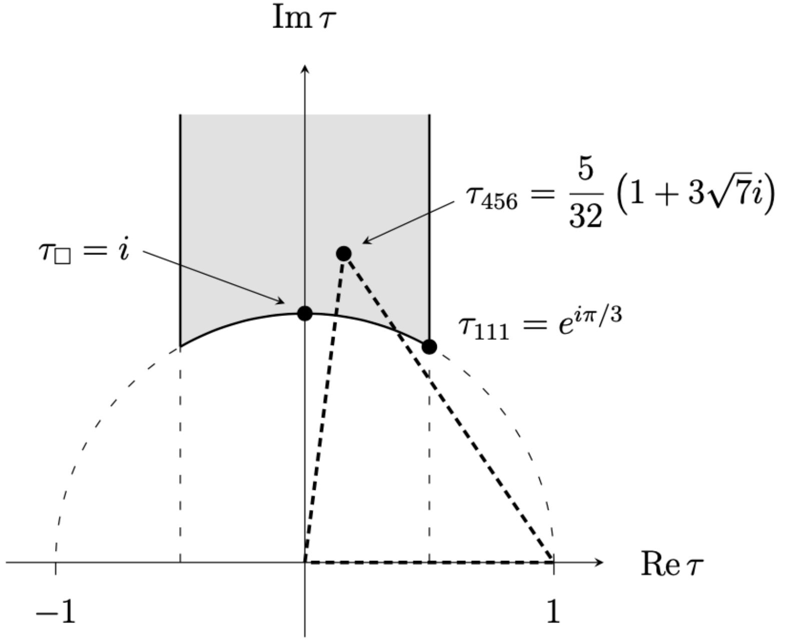
map to circle ==>

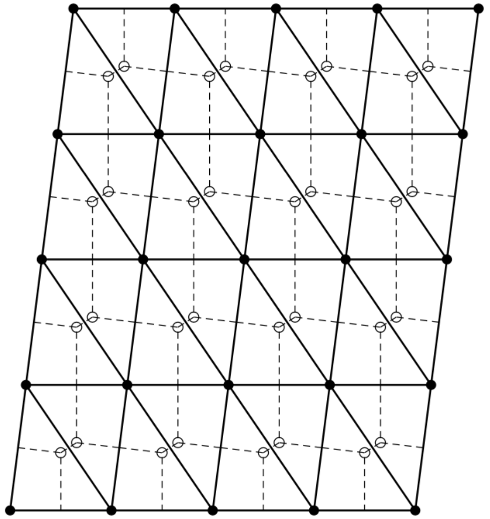


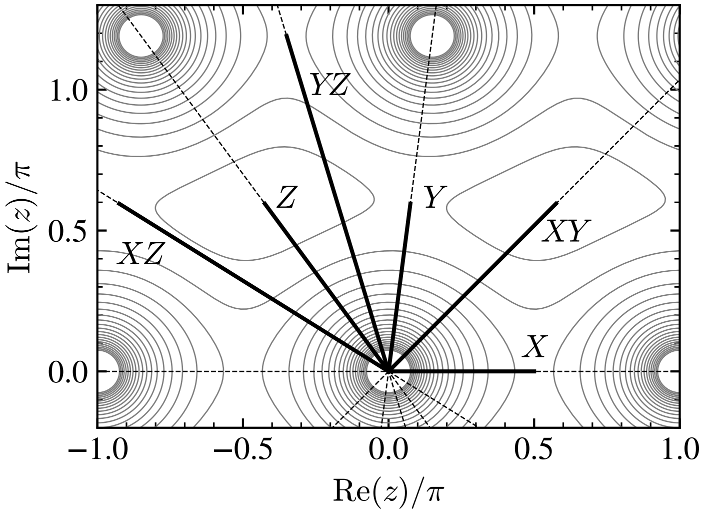
Basic algebra of Projective Geometry going back to Pascal in 1640!

- Blaise Pascal. Essay pour les conique. (facsimile) Nieders¨achsiche Landesbibliothek, Gottfried Wilhelm Leibniz Bibliothek, 1640.

Calculation Modular dependent on the torus

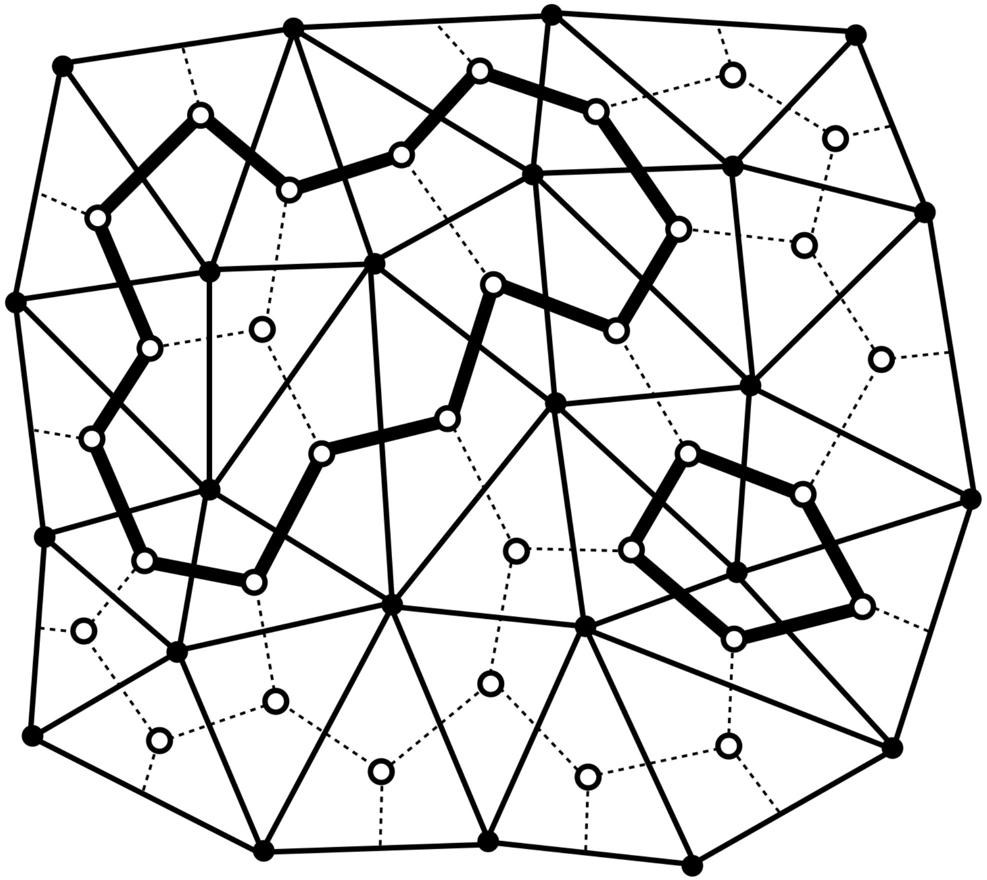






$$\langle \sigma(0)\sigma(z)\rangle = \left|\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)}\right|^{1/4} \frac{\sum_{\nu=1}^{4} |\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4} |\vartheta_\nu(0|\tau)|}$$

3 Equivalent Loop Expansion for Partition Functions!



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$$S_{\text{dual}} = -\sum_{\langle ij \rangle} L_{ij} s_i s_j$$

$$S_{\psi} = \frac{1}{2} \sum_i \bar{\psi}_i \psi_i - \sum_{\langle ij \rangle} \kappa_{ij} \bar{\psi}_i P_{ij} \psi_j \ ,$$

Kramers Wannier High T/Low T Loop expansion Wilson-Majorana Lattice Fermions

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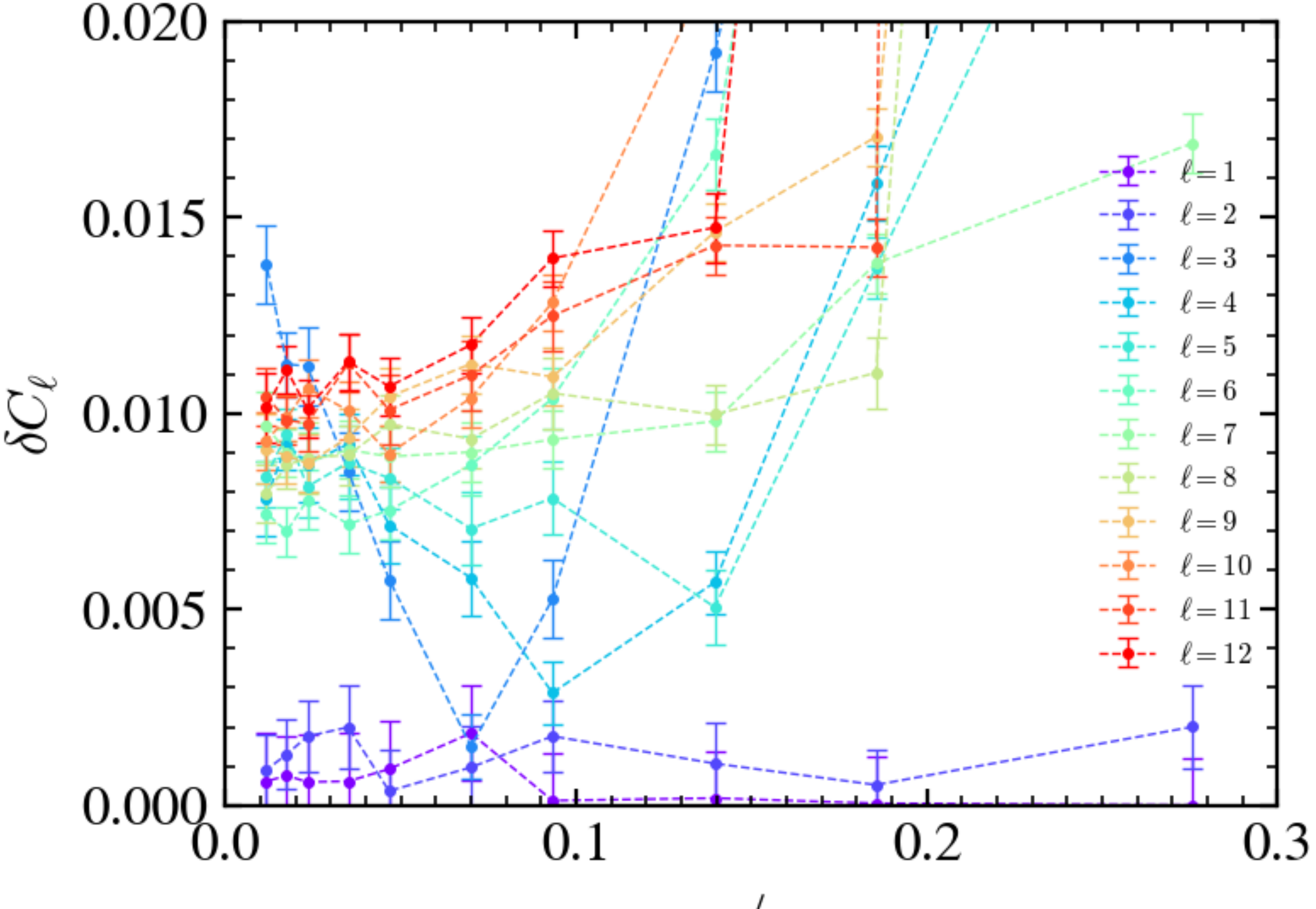

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$\ell=11$
$\ell=12$

COMMENT ON GEOMETRIC SMOOTHING

- The Sphere obeys N - E + F = 2
- N = 2 + 10 * L*L to deficit delta to smooth the scalar curvature
- But there are 2 N = 4 + 20*L*L D.O.F on the sphere
- So smoothing F = 20*L*L. areas is **one to one**

$$\frac{\partial A_\triangle(1,2,3)}{\partial \ell_{ij}^2} = \frac{\ell^*_{ij}}{\ell_{ij}}$$

$$\text{Same as FEM Beltrame Laplace operator}$$

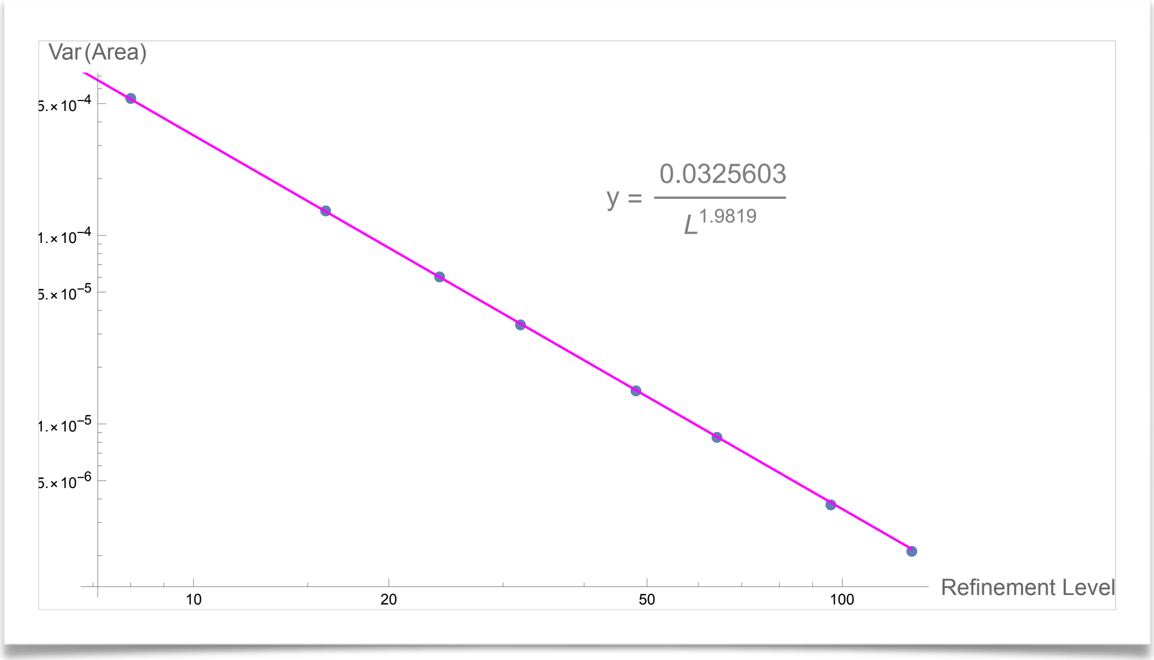
$$d * d\phi = \frac{\ell^*_{ij}}{\ell_{ij}}(\phi_i - \phi_j)^2$$

Area Optimization to smooth scalar curvature

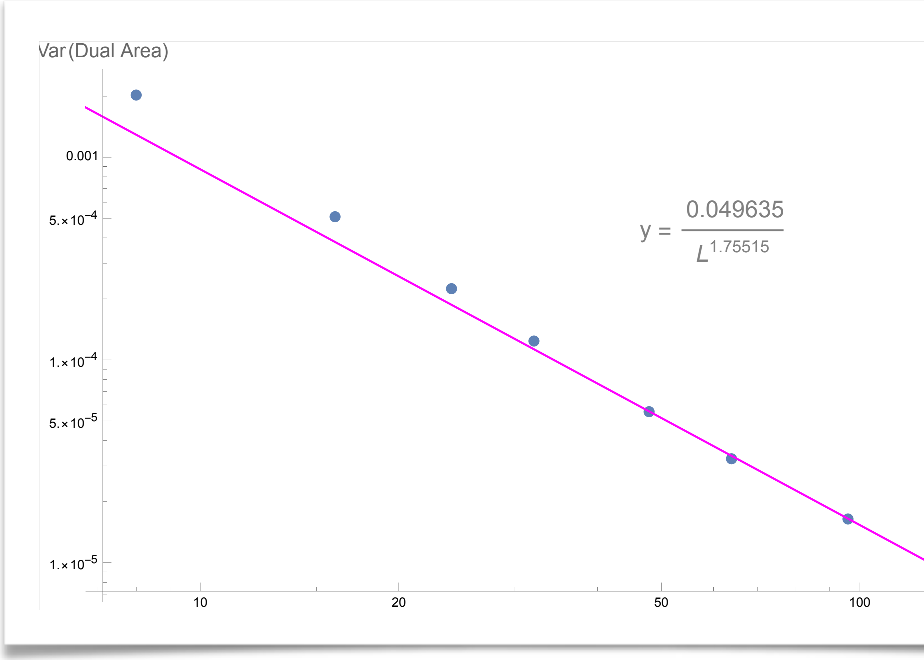
$$S(\ell_{ij}) = N^{-1} \sum_{\triangle} A^2_{\triangle}(l_{ij})$$

dof: $\quad 2N = 4 + 20L^3$

Area Variance



Dual Area Variance

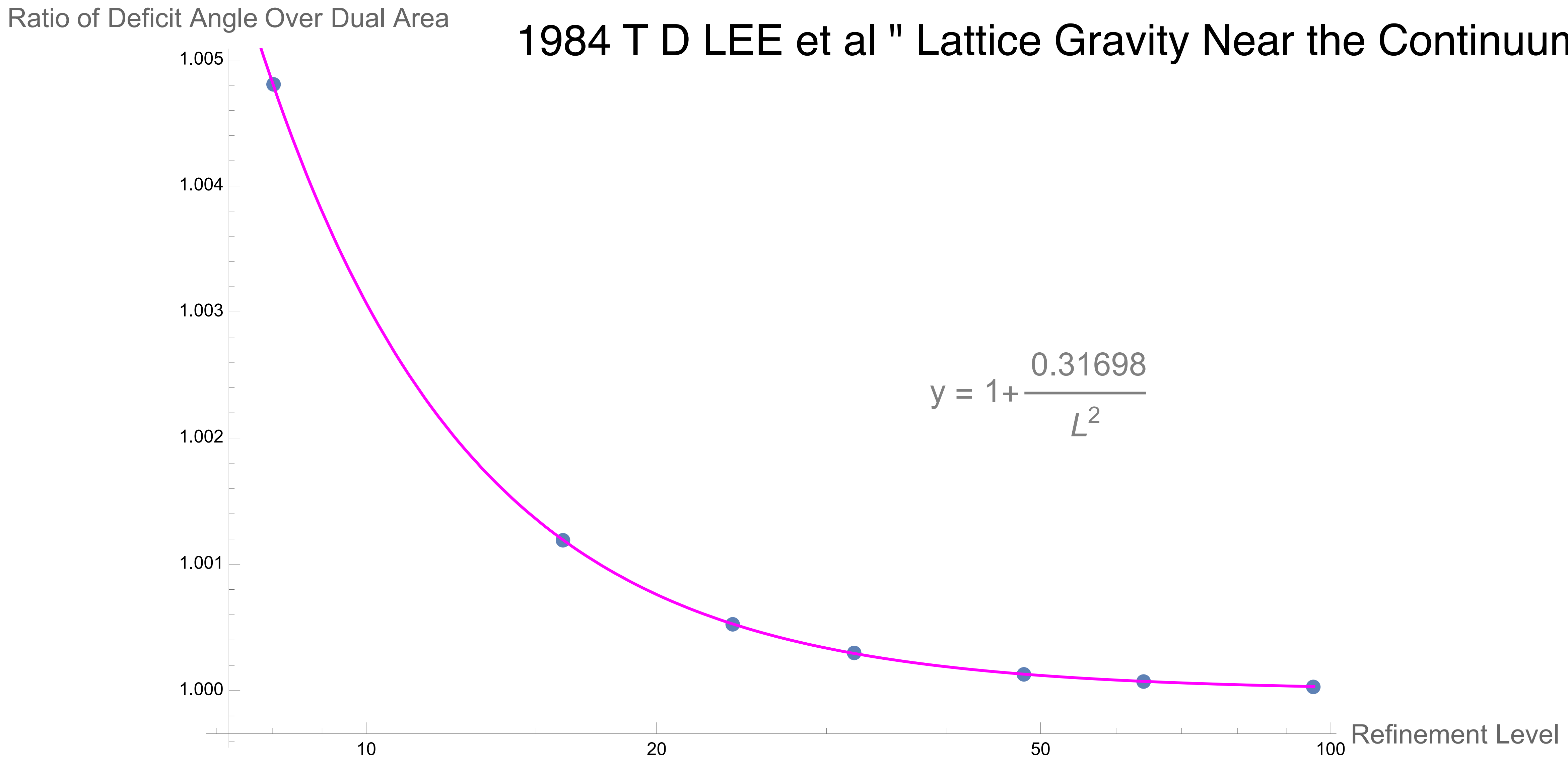


$$4A(a,b,c)^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$
$$= a^2 b^2 c^2 / R^2_{\triangle}$$

$$a^2 = \ell^2_{12} = |\vec{r}_1 - \vec{r}_2|^2 = 2 - 2\vec{r}_1 \cdot \vec{r}_2$$

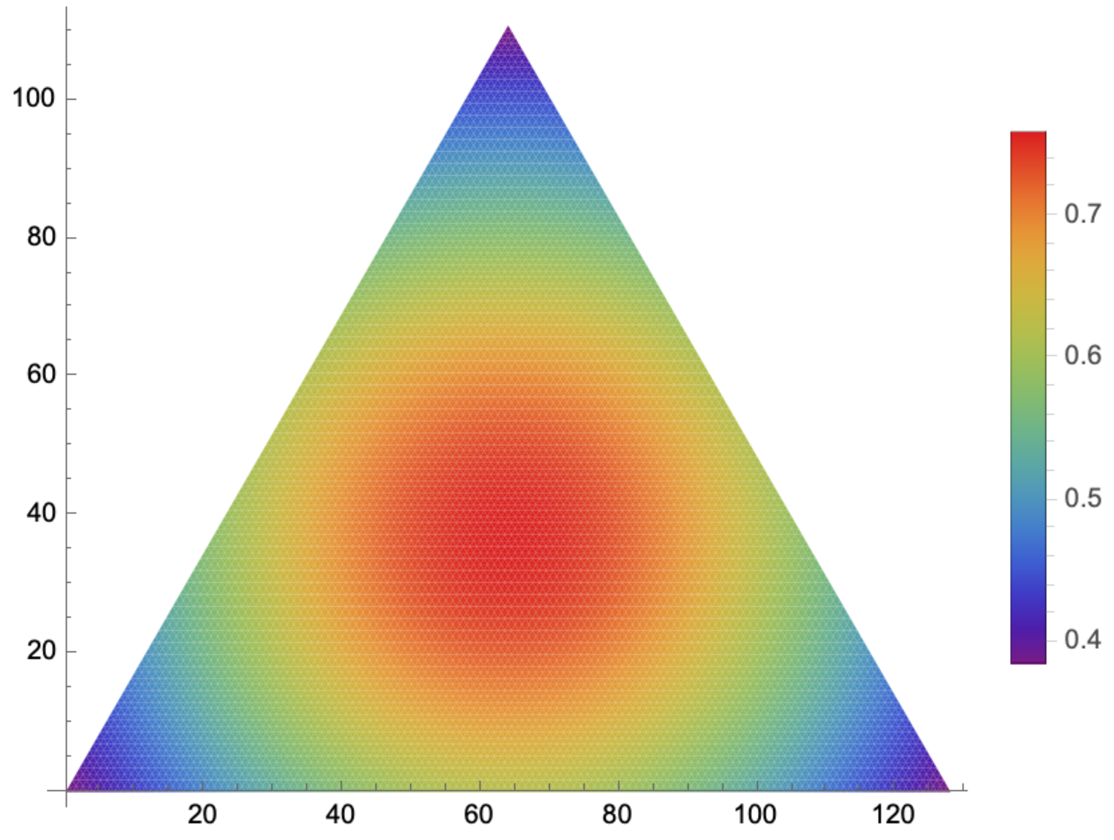
Smooth Scalar Curvature Theorem

1984 T D LEE et al " Lattice Gravity Near the Continuum"

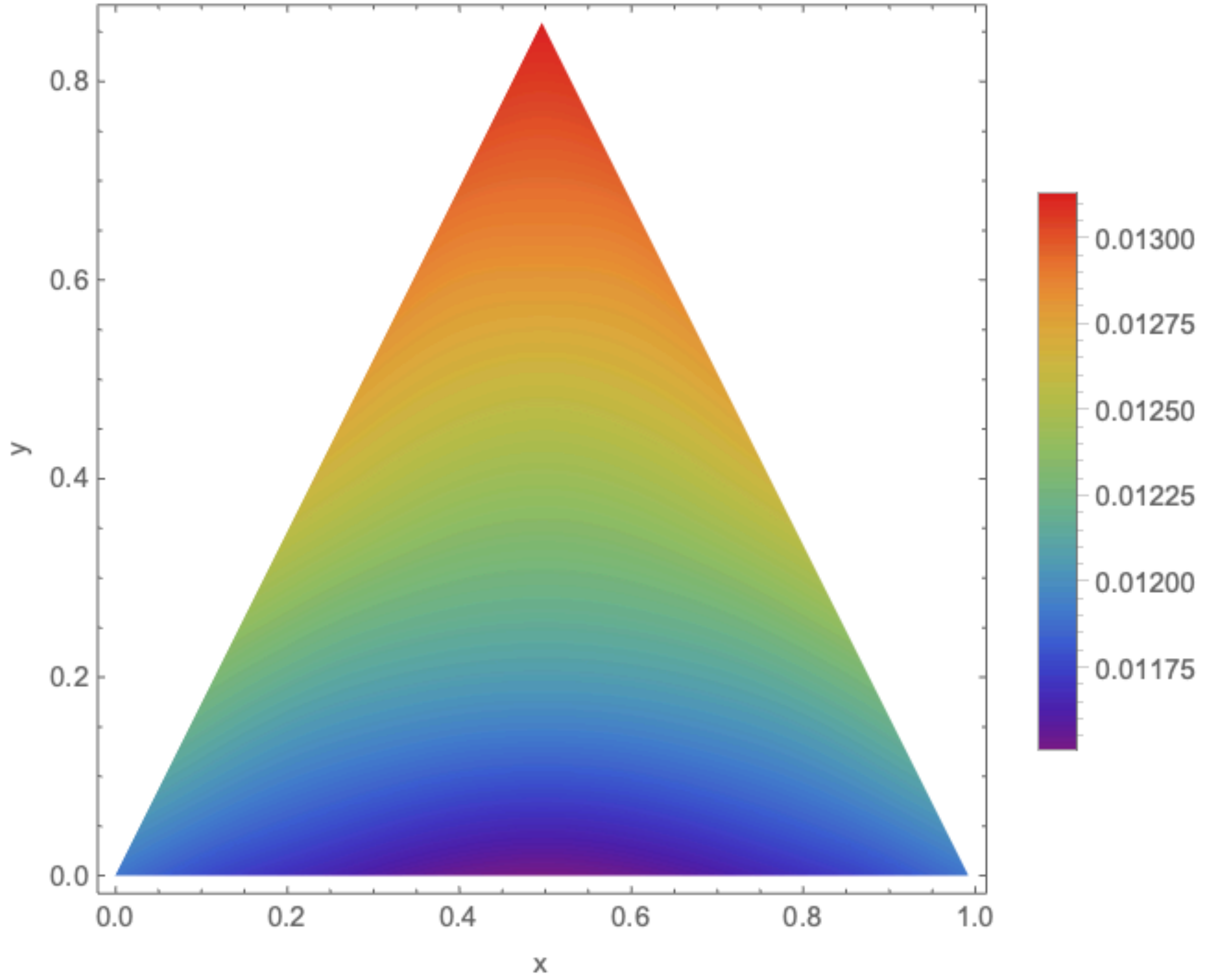


Smooth Link Weight K1 (K2 and K3 are rotated) before and after scalar smoothing

Projected From Icosahedron



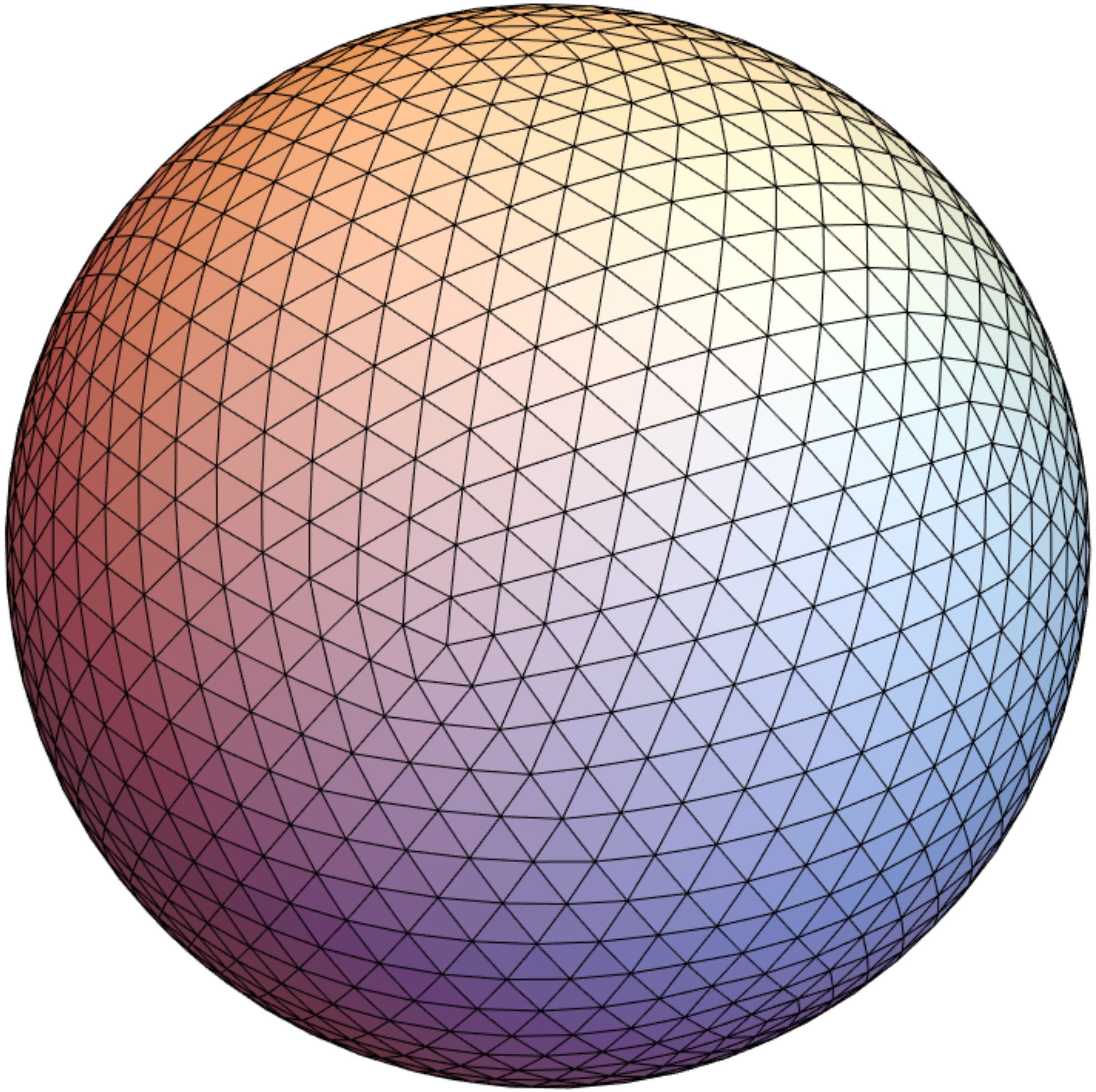
Smoothed Scalar Curvaturer

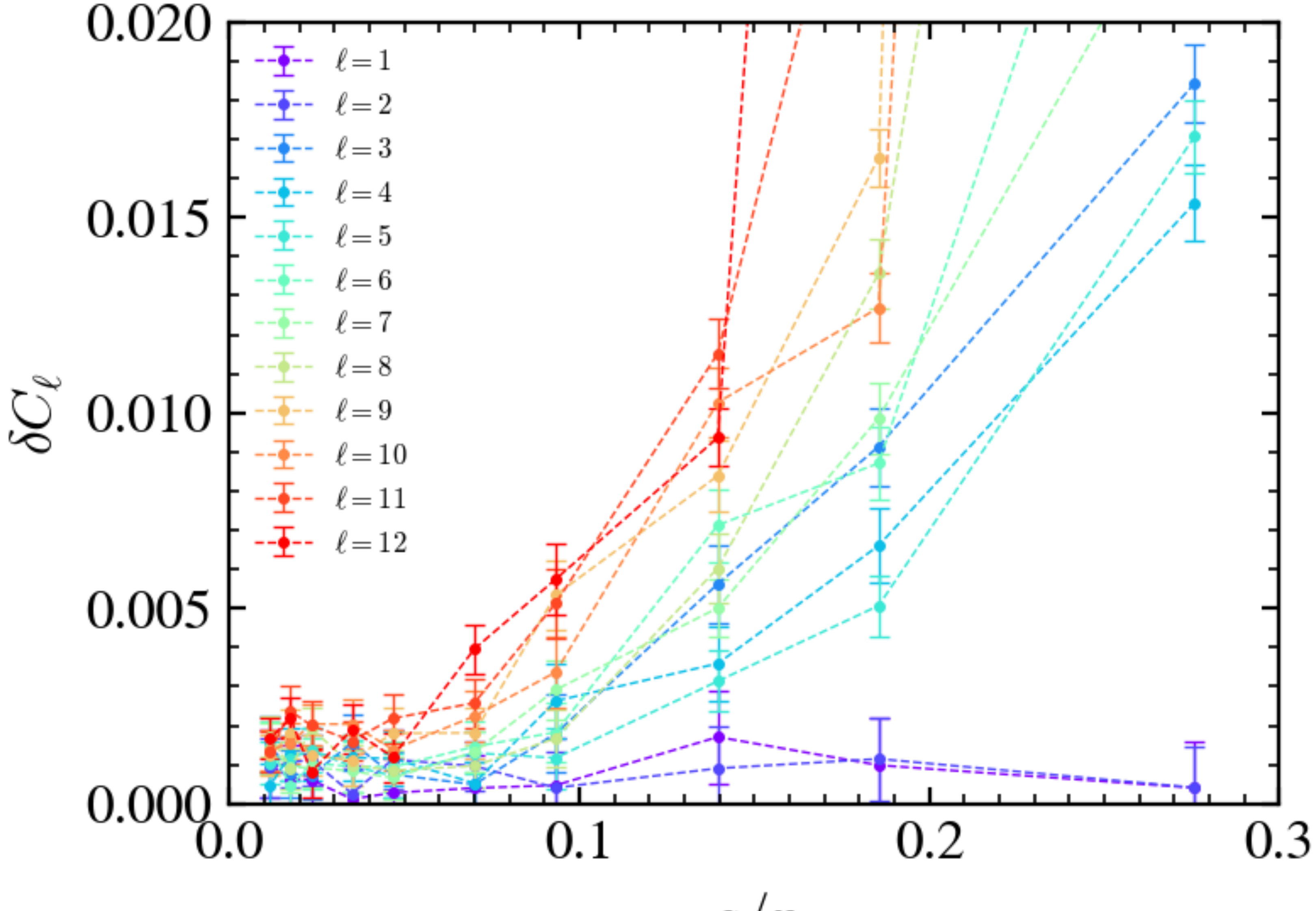


Back to Putting critical 2d Ising on the sphere

- Now set all the circumradii equal to converge to a differential affine tangent planes.

- Give we believe the Exact Ising CFT in the continuum limit.





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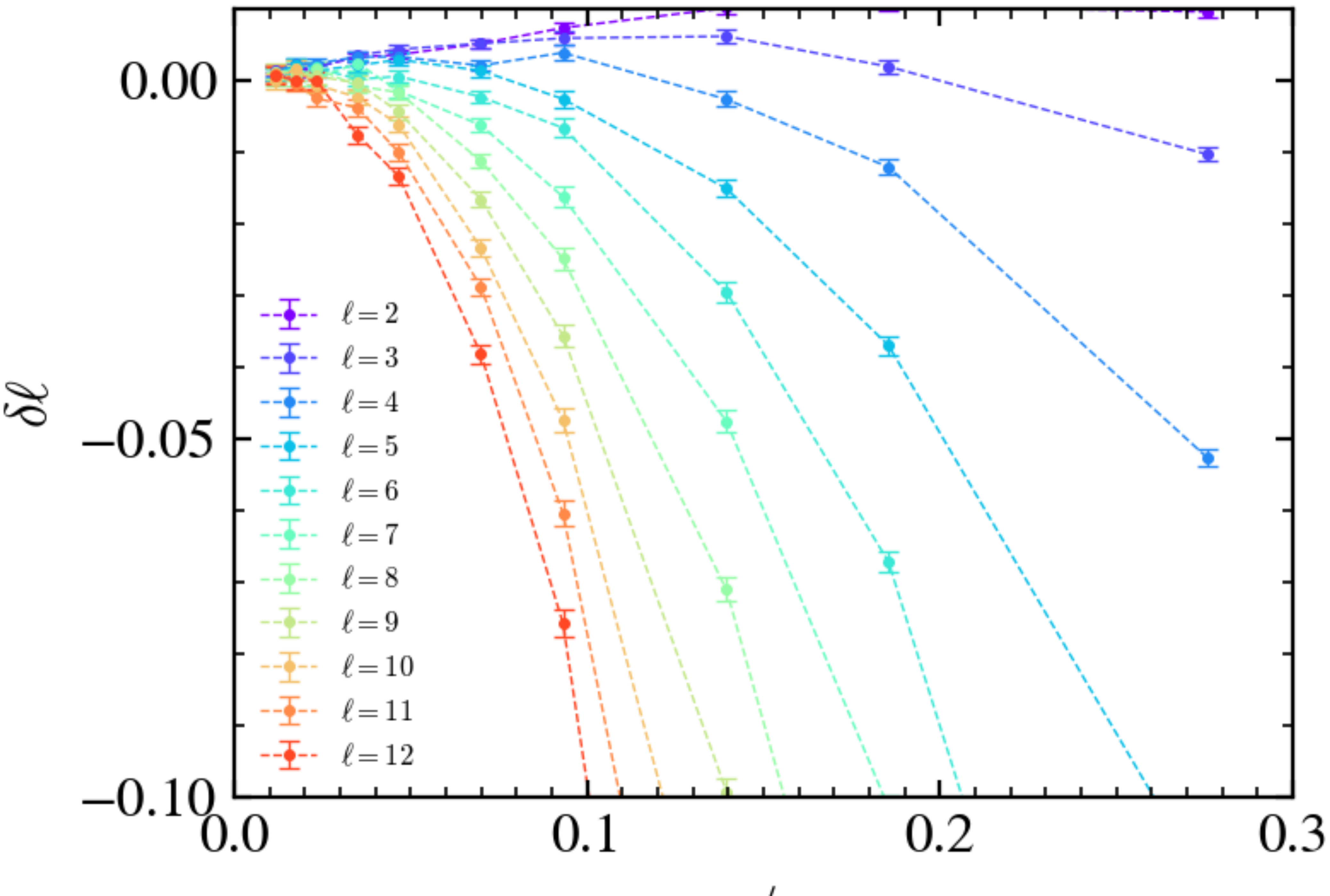

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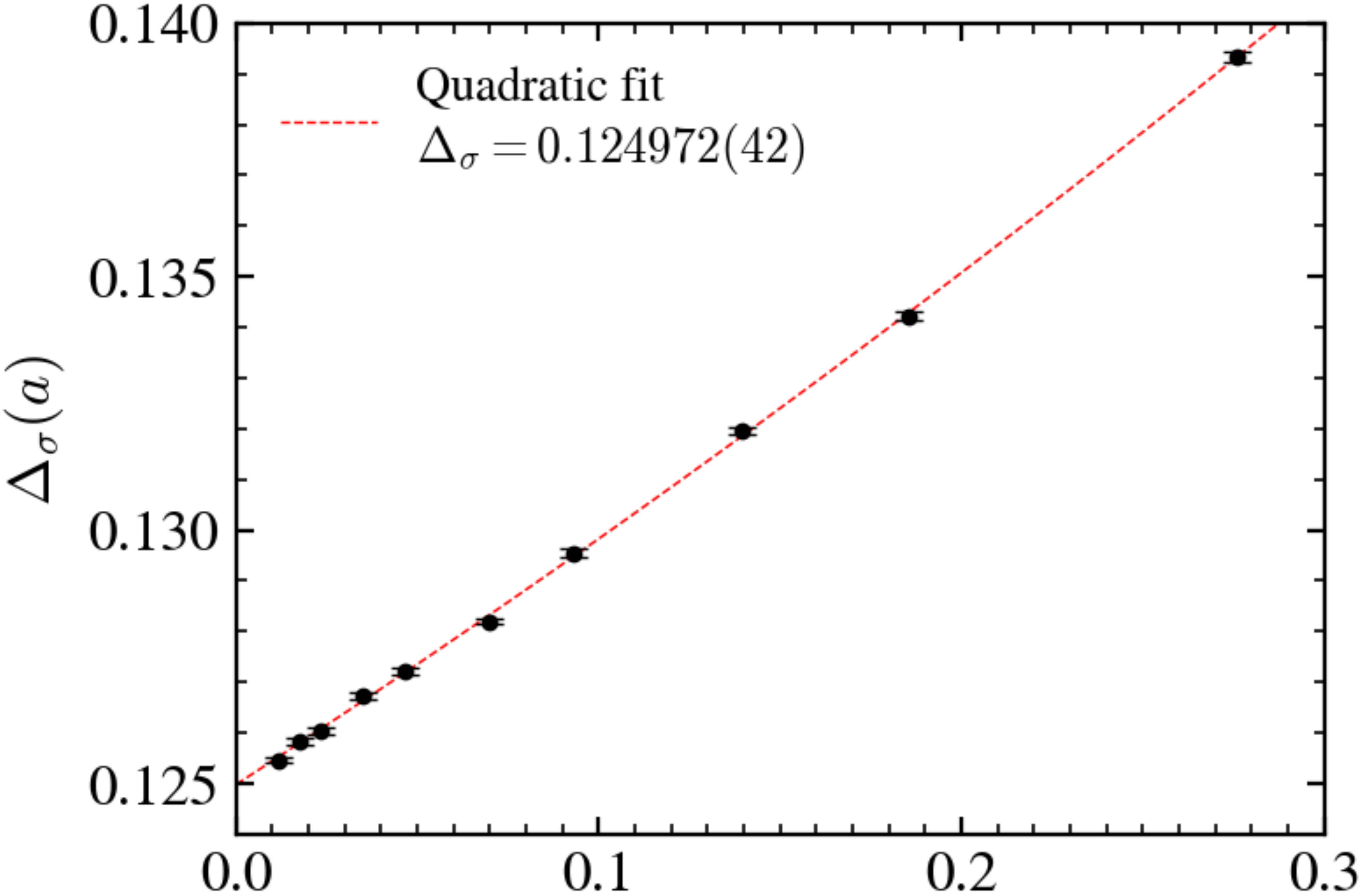


Quadratic fit
$\Delta_\sigma = 0.124972(42)$
$\Delta_\sigma(a)$
0.140
0.135
0.130
0.125
0.0
0.1
0.2
0.3

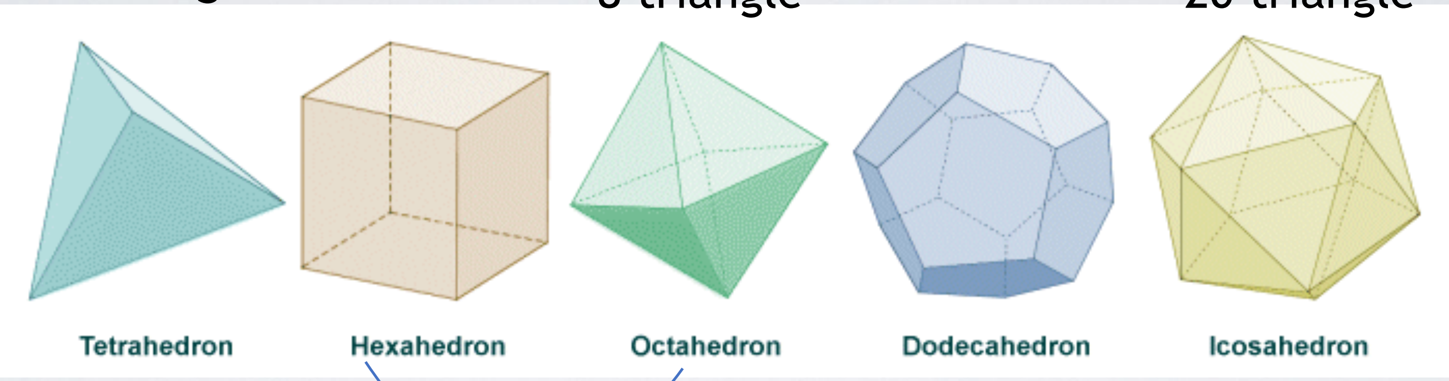
PART IV
GENERALIZATION 3D & 4D & CODES

2D & 3D SIMPLCIAL PLATONIC SOLIDS

4 triangle

8 triangle

20 triangle



self dual

dual

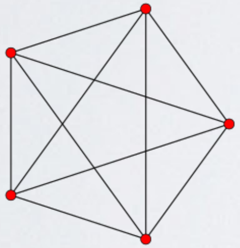
5 tetra

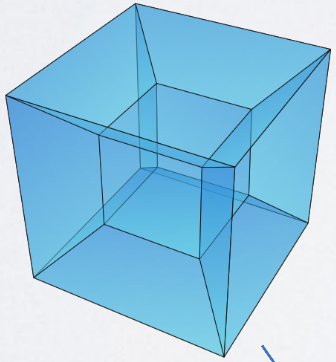
8 cubes

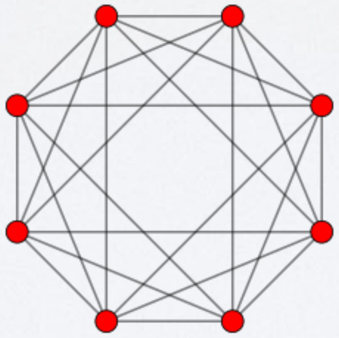
16 tetra

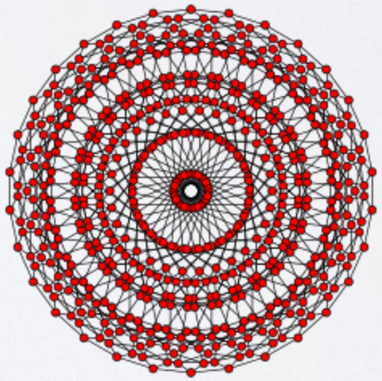
120 dedaca

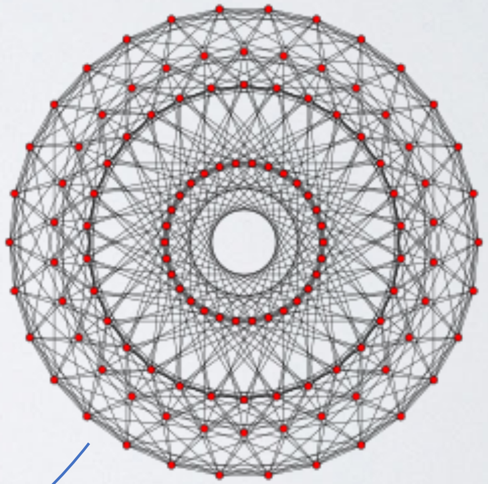
600 tetra











self dual

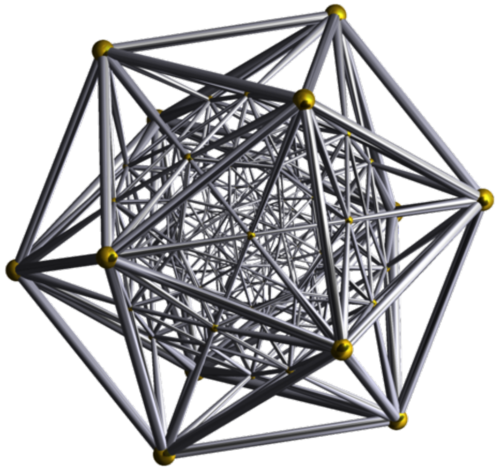
6th self dual with 24 octahedrons

Euler N - E + F - V = 0

https://en.wikipedia.org/wiki/Regular_4-polytope#

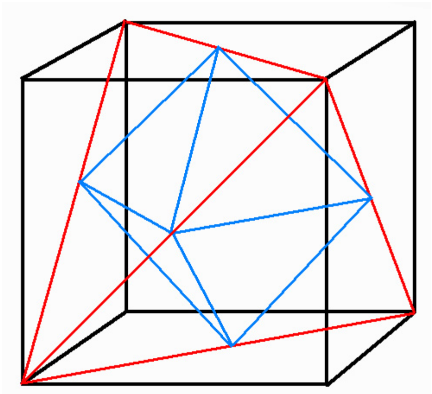
3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \implies \mathbb{R} \times \mathbb{S}^3$$



Aristotle’ s 2% Error!

$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$



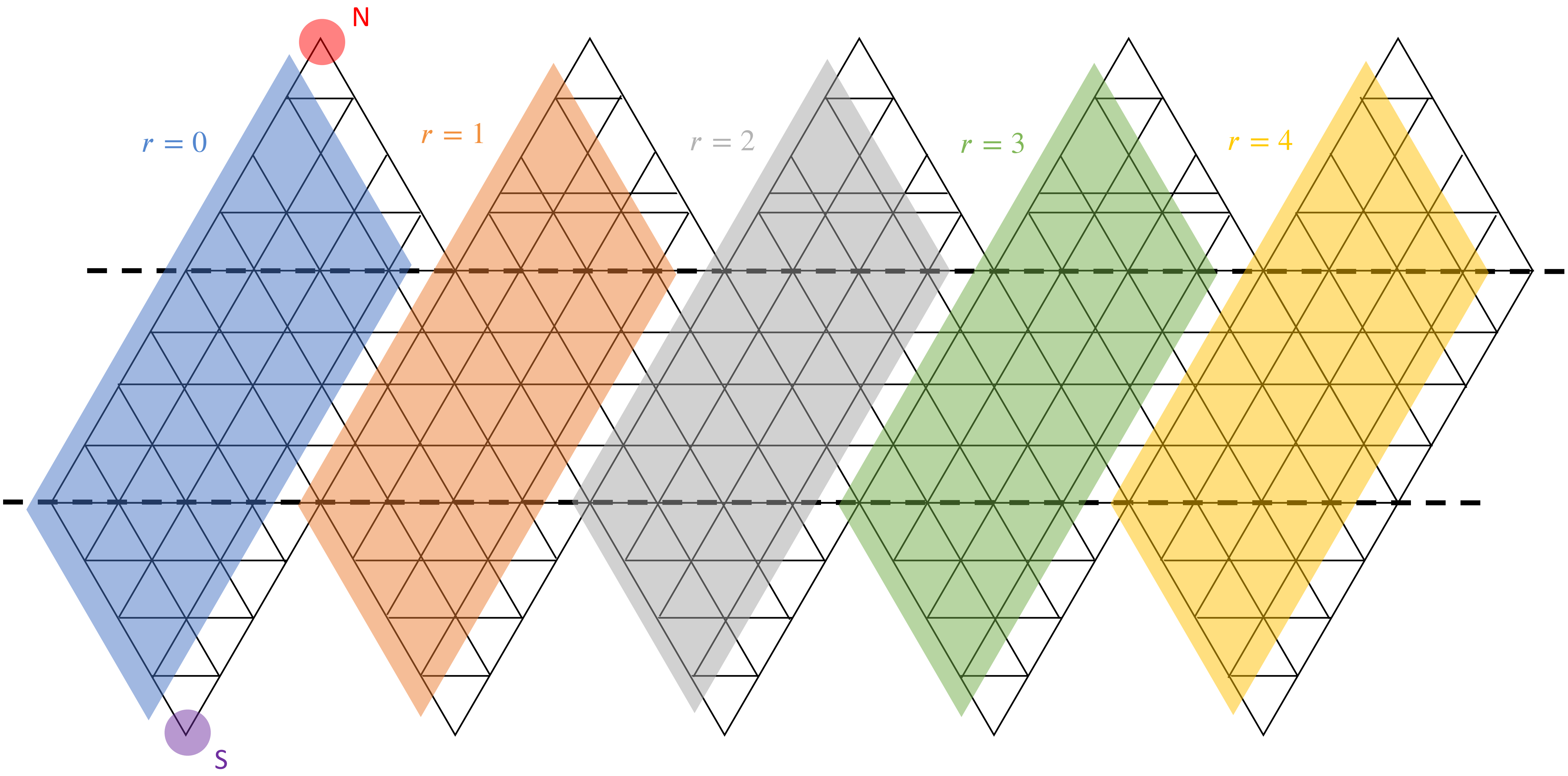
Fast Code Domains of
Regular 3D Grids on Refinement

600 cell: “Square of the icosahedron” –Symmetries 1440= 120 * 120 the 120 copies of icosahedron

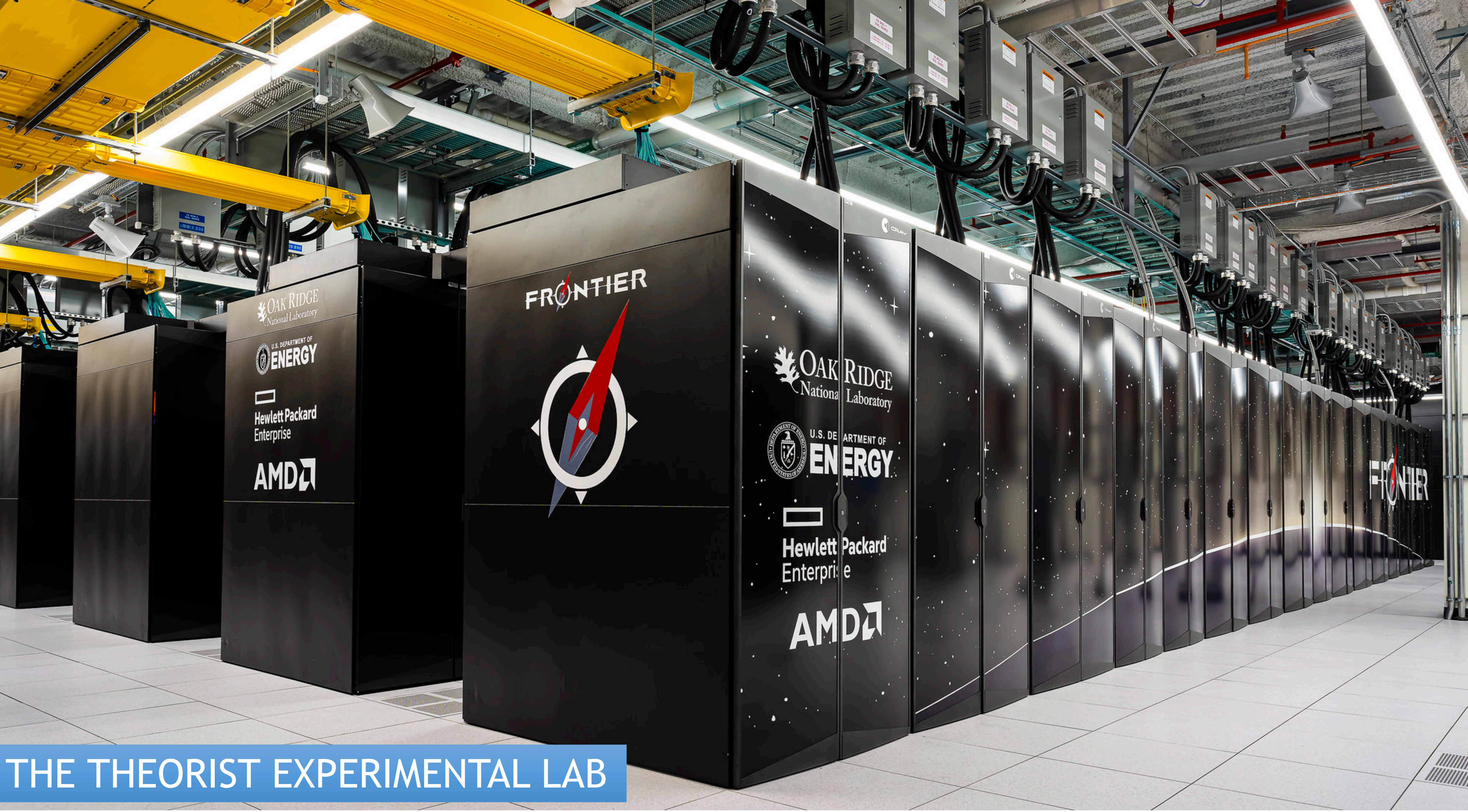
$$O(4) \sim SU(2) \times SU(2)$$

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.

Aiming to implement MPI, we divide the lattice into 5 patches + two pole points:



Uncolored points have an identical point somewhere else.



THE THEORIST EXPERIMENTAL LAB

WHAT'S NEXT?

- MORE PRECISION TESTS OF AFFINE CONJECTURE
- Non-integrable systemsPhi 4 2d, 3d Ising, Large J/Q etc.
- Exascale & Quantum Qubit Algorithm.
- HELP WANTED -- Thanks!

Review of Narrative

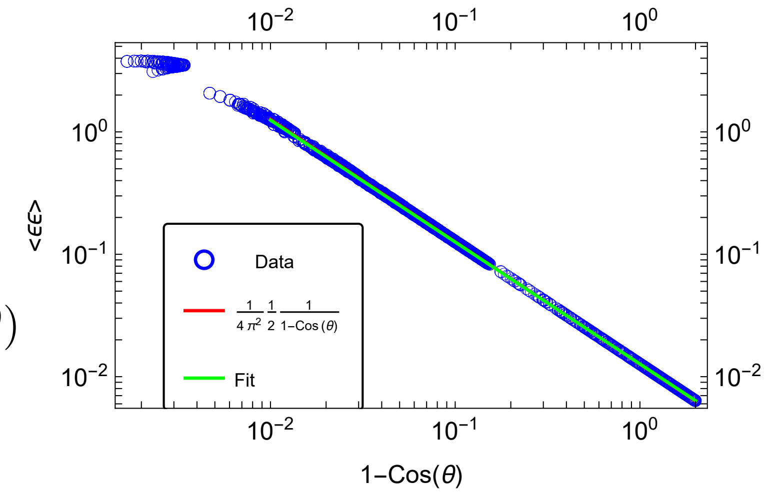
1. MOTIVATION:
    1. EXACT LATTICE THEORHY ON CURVED MANIFOLDS
    2. CFT on Sphere/Cylinder dual to AdS Space
    3. BSM Exascale project!
2. HILBERT'S ADVICE
    1. ph 4 Results on S2 and Rx S2 but ...
3. SIMPLICAL GEOMETRY
    1. Regge's. and FEM Manifold
    2. Affine Simplicial Structure Barycentric Invariants
    3. Classical EH acton and FEM Action vs Quantum
4. QUANTUM GEOMETRY
    1. Affine Action for Ising on R2
    2. Affine Action for Ising on S2
    3. Is Ising on S2 exact
5. NEXT STEPS
    1. What about phi ^4 theory on S2
    2. What about SUSY et al
    3. What about R x S2, S3 , R x S3
    4. The Affine Map problem --- Machine Learning?
    5. What about Lattice Codes for Exascale

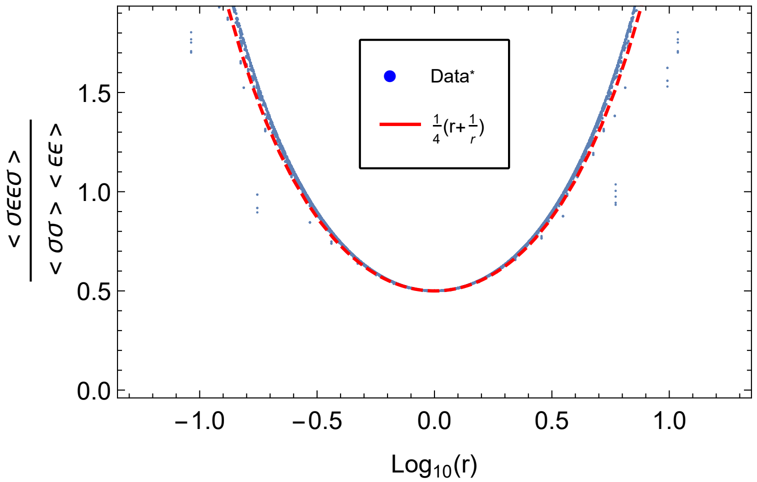
BACK UP SLIDES

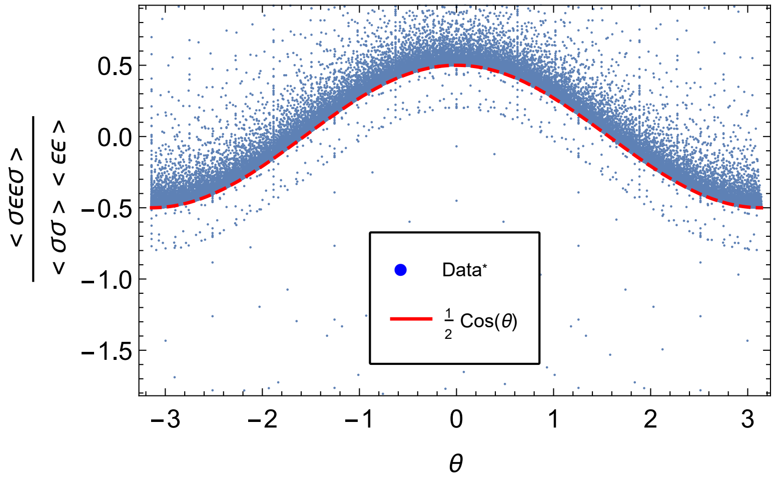
FREE MAJORANA FERMIONS ON S2

$$\langle \psi(z_1)\bar{\psi}(z_1)\bar{\psi}(z_1)\psi(z_2)\rangle = \left[\frac{1}{\partial}\right]_{z_1,z_2}\left[\frac{1}{\bar{\partial}}\right]_{z_1,z_2} = \frac{1}{4\pi^2}\frac{1}{|z_1-z_2|^2}$$

$$\frac{\langle \sigma(0)\epsilon(z_2)\epsilon(z_3)\sigma(\infty)\rangle}{\langle \epsilon(z_2)\epsilon(z_3)\rangle} = \frac{1}{4}|\sqrt{z_1/z_2}+\sqrt{z_2/z_1}|^2 = \frac{1}{4}(r+1/r+2\cos\theta)$$



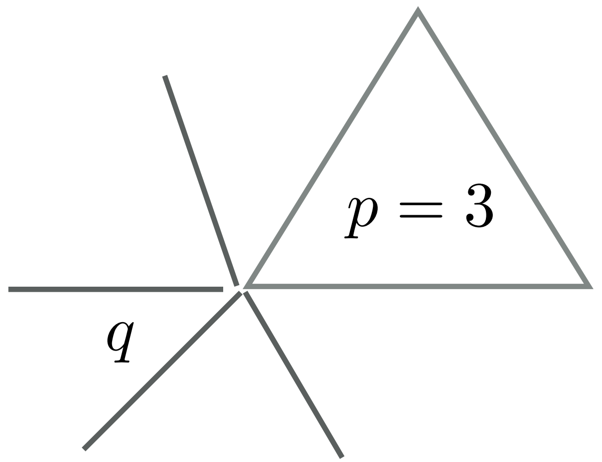




EQUILATERAL TRIANGULATION

Triangle case p = 3

Preserves Discrete
Subgroup of Isometries

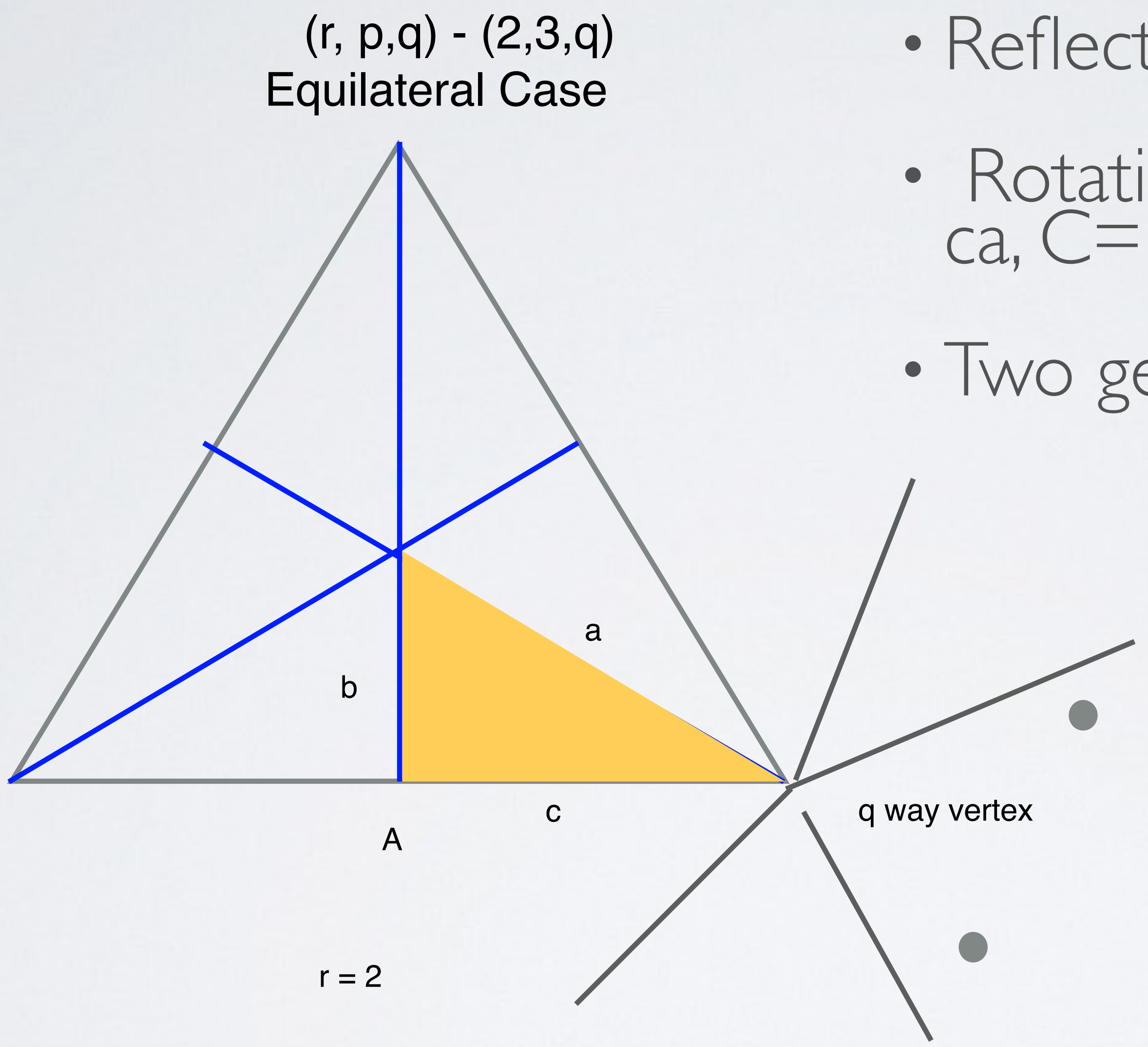


$$\frac{1}{p}+\frac{1}{q}>1/2 \qquad \text{de Sitter} \quad \mathbb{S}^2 \qquad \text{vertex} \quad q=3,4,5$$

$$\frac{1}{p}+\frac{1}{q}=1/2 \qquad \text{flat} \quad \mathbb{T}^2 \qquad \text{vertex} \quad q=6$$

$$\frac{1}{p}+\frac{1}{q}<1/2 \qquad \text{Hyperbolic} \quad \mathbb{A}dS^2 \qquad \text{vertex} \quad q=7,8,9,\cdots$$

Triangle Group Tiling

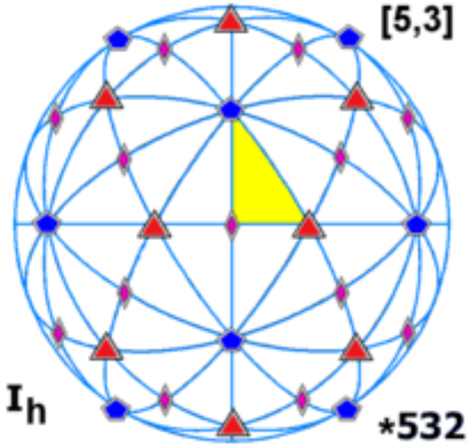


- Reflection: a, b, c
- Rotations: A = bc, B = ca, C= AB =ba
- Two generators

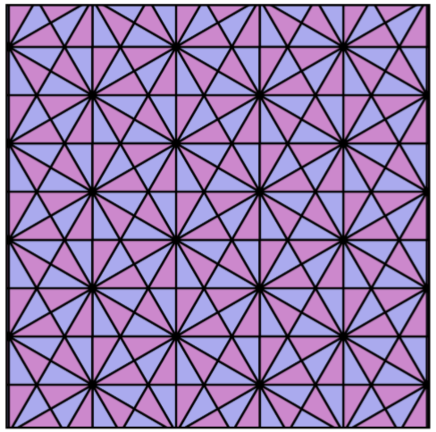
$$\Delta(p,q,r) = \{a,b,c | a^2 = b^2 = c^2 = A^p = B^q = C^r\}$$

DISCRETE ISOMETRIES & THE TRIANGLE GROUP

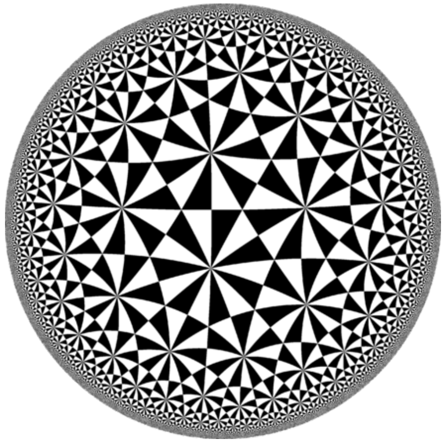
$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} \begin{cases} > \pi & \text{Postive curvature} \\ = \pi & \text{Zero curvature} \\ < \pi & \text{Negative Curvature} \end{cases}$$



$(2,3,5)$
120 element
Icosahedral in O(3)



$(2,3,6)$
Triangle Lattice
on Euclidean $\mathbb{R}^2$

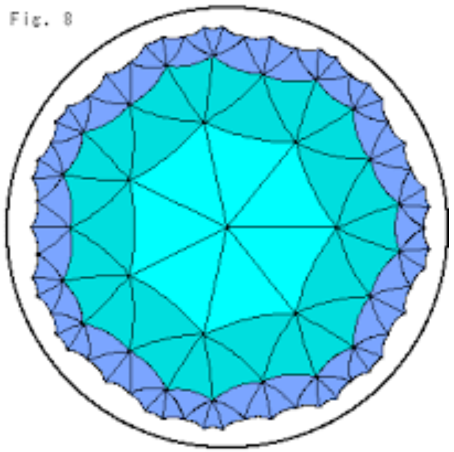


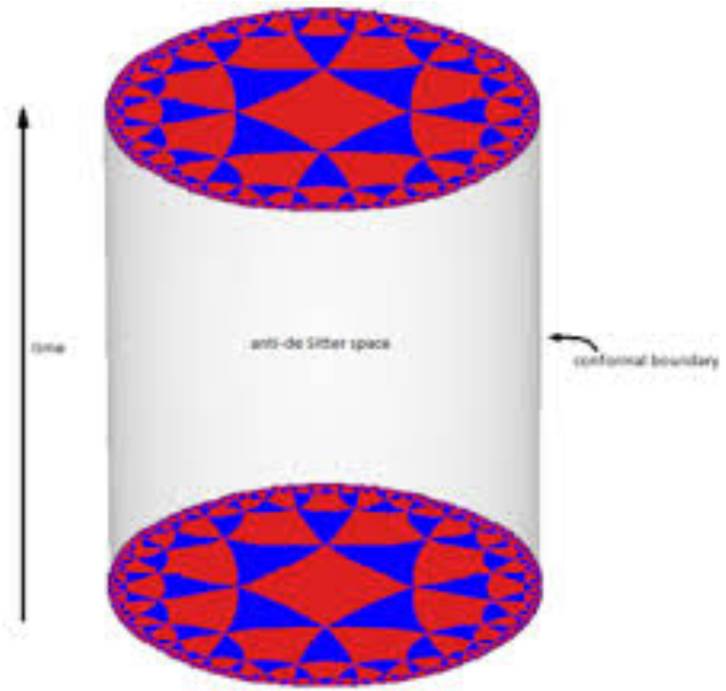
$(2,3,7)$
Subgroup of Modular
Group on $\mathbb{H}^2$

https://en.wikipedia.org/wiki/(2,3,7)_triangle_group

Hyperbolic (e.g. Poincare Disk) and Global AdS

$$\mathbb{H}^d \quad \rightarrow \quad \mathbb{A}dS^d$$



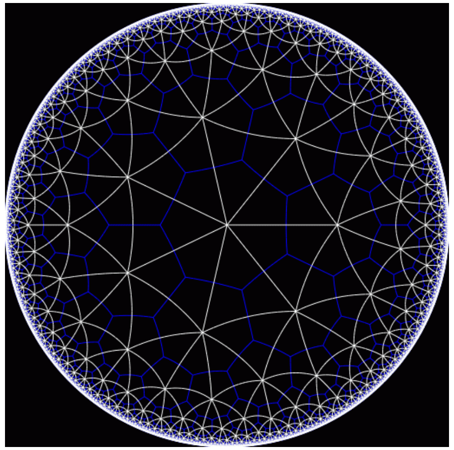


1/p + 1/q < 1/2

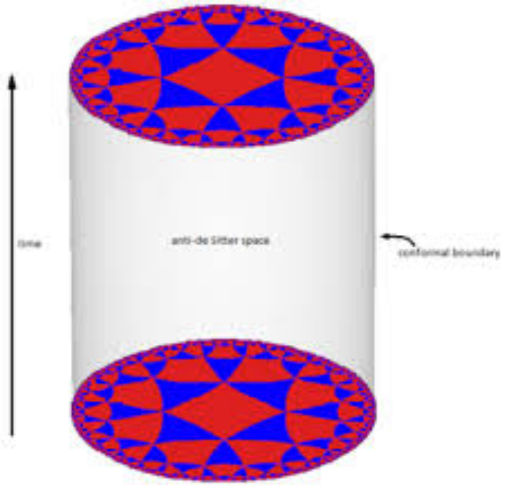
Triangle Group Tessellation: Preserve Finite subgroup of the Modular Group

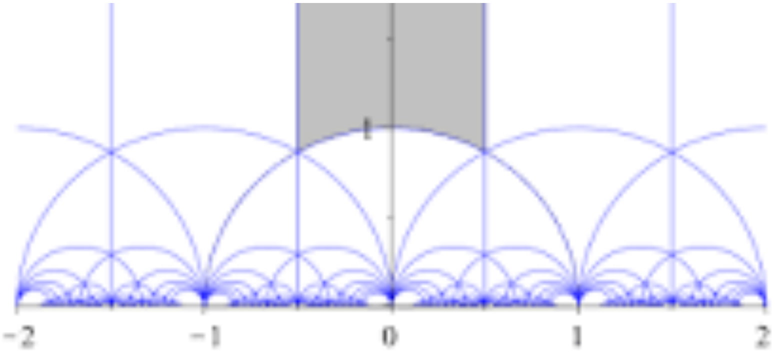
Hyperbolic (e.g. Poincare Disk) and Global AdS

q = 7



1/2 + 1/3 + 1/q < 1





Triangle Group Tessellation: Preserve Finite subgroup of the Modular Group

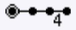
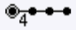
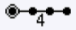
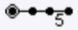
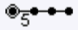
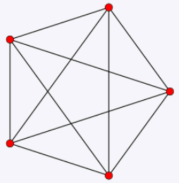
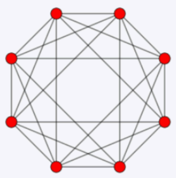
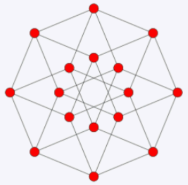
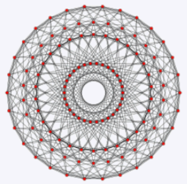
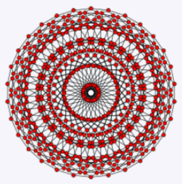
$$z \to \frac{az+b}{cz+d} \quad ad - bc = 1$$

$$a, b, c, d \in \mathbb{Z} \; mod \; q$$

Are these Tessellation "Tensor Networks" ?
YES: See Daniel Harlow's Slide from Wednesday

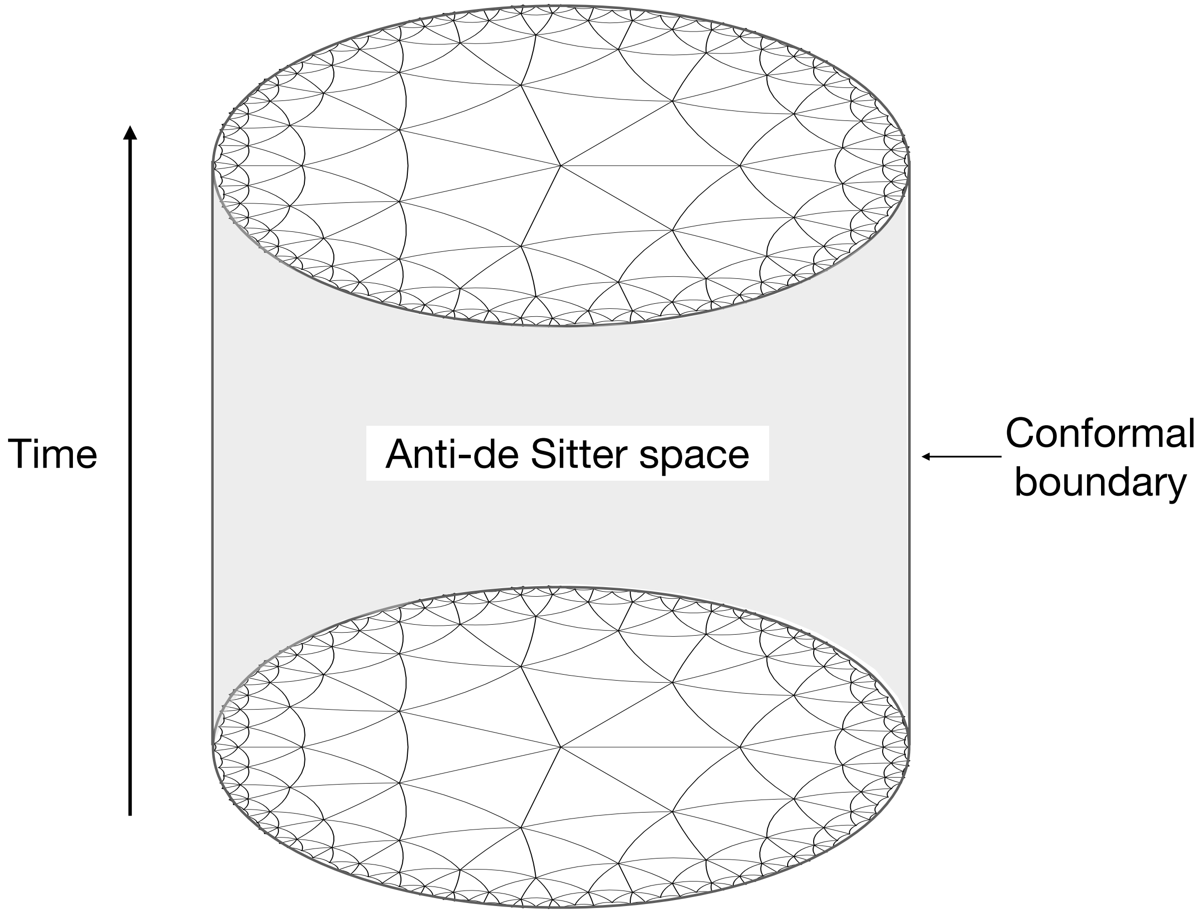
Can we do QC lattice Field Theories in AdS?
Classical YES /QC Maybe

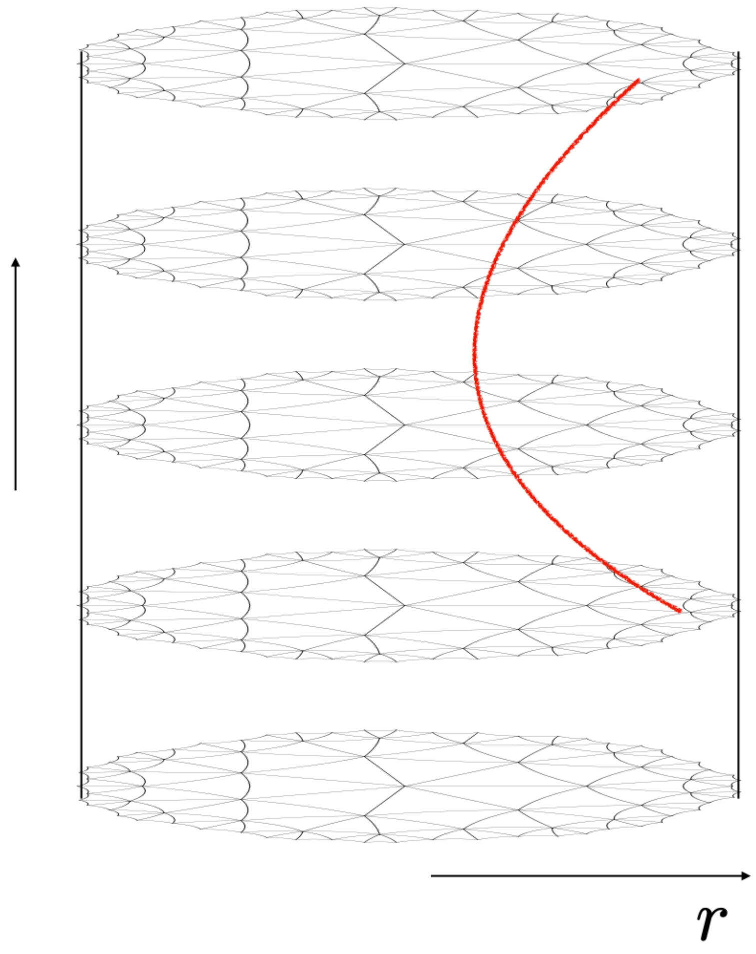
Regular convex 4-polytopes [hide]

Symmetry group	A_4	B_4		F_4	H_4	
Name	5-cell Hyper-tetrahedron 5-point	16-cell Hyper-octahedron 8-point	8-cell Hyper-cube 16-point	24-cell 24-point	600-cell Hyper-icosahedron 120-point	120-cell Hyper-dodecahedron 600-point
Schläfli symbol	{3, 3, 3}	{3, 3, 4}	{4, 3, 3}	{3, 4, 3}	{3, 3, 5}	{5, 3, 3}
Coxeter mirrors						
Mirror dihedrals	$\frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{2}\ \frac{\pi}{2}\ \frac{\pi}{2}$	$\frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{4}\ \frac{\pi}{2}\ \frac{\pi}{2}\ \frac{\pi}{2}$	$\frac{\pi}{4}\ \frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{2}\ \frac{\pi}{2}\ \frac{\pi}{2}$	$\frac{\pi}{3}\ \frac{\pi}{4}\ \frac{\pi}{3}\ \frac{\pi}{2}\ \frac{\pi}{2}\ \frac{\pi}{2}$	$\frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{5}\ \frac{\pi}{2}\ \frac{\pi}{2}\ \frac{\pi}{2}$	$\frac{\pi}{5}\ \frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{2}\ \frac{\pi}{2}\ \frac{\pi}{2}$
Graph						
Vertices	5 tetrahedral	8 octahedral	16 tetrahedral	24 cubical	120 icosahedral	600 tetrahedral
Edges	10 triangular	24 square	32 triangular	96 triangular	720 pentagonal	1200 triangular
Faces	10 triangles	32 triangles	24 squares	96 triangles	1200 triangles	720 pentagons
Cells	5 tetrahedra	16 tetrahedra	8 cubes	24 octahedra	600 tetrahedra	120 dodecahedra
Tori	1 5-tetrahedron	2 8-tetrahedron	2 4-cube	4 6-octahedron	20 30-tetrahedron	12 10-dodecahedron
Inscribed	120 in 120-cell	675 in 120-cell	2 16-cells	3 8-cells	25 24-cells	10 600-cells
Great polygons		2 squares x 3	4 rectangles x 4	4 hexagons x 4	12 decagons x 6	100 irregular hexagons x 4
Petrie polygons	1 pentagon x 2	1 octagon x 3	2 octagons x 4	2 dodecagons x 4	4 30-gons x 6	20 30-gons x 4
Long radius	1	1	1	1	1	1
Edge length	$\sqrt{\frac{5}{2}} \approx 1.581$	$\sqrt{2} \approx 1.414$	1	1	$\frac{1}{\phi} \approx 0.618$	$\frac{1}{\phi^2\sqrt{2}} \approx 0.270$
Short radius	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\frac{1}{2}} \approx 0.707$	$\sqrt{\frac{\phi^4}{8}} \approx 0.926$	$\sqrt{\frac{\phi^4}{8}} \approx 0.926$
Area	$10\left(\frac{5\sqrt{3}}{8}\right) \approx 10.825$	$32\left(\sqrt{\frac{3}{4}}\right) \approx 27.713$	24	$96\left(\sqrt{\frac{3}{16}}\right) \approx 41.569$	$1200\left(\frac{\sqrt{3}}{4\phi^2}\right) \approx 198.48$	$720\left(\frac{\sqrt{25+10\sqrt{5}}}{8\phi^4}\right) \approx 90.366$
Volume	$5\left(\frac{5\sqrt{5}}{24}\right) \approx 2.329$	$16\left(\frac{1}{3}\right) \approx 5.333$	8	$24\left(\frac{\sqrt{2}}{3}\right) \approx 11.314$	$600\left(\frac{\sqrt{2}}{12\phi^3}\right) \approx 16.693$	$120\left(\frac{15+7\sqrt{5}}{4\phi^6\sqrt{8}}\right) \approx 18.118$
4-Content	$\frac{\sqrt{5}}{24}\left(\frac{\sqrt{5}}{2}\right)^4 \approx 0.146$	$\frac{2}{3} \approx 0.667$	1	2	$\frac{\text{Short}\times\text{Vol}}{4} \approx 3.863$	$\frac{\text{Short}\times\text{Vol}}{4} \approx 4.193$

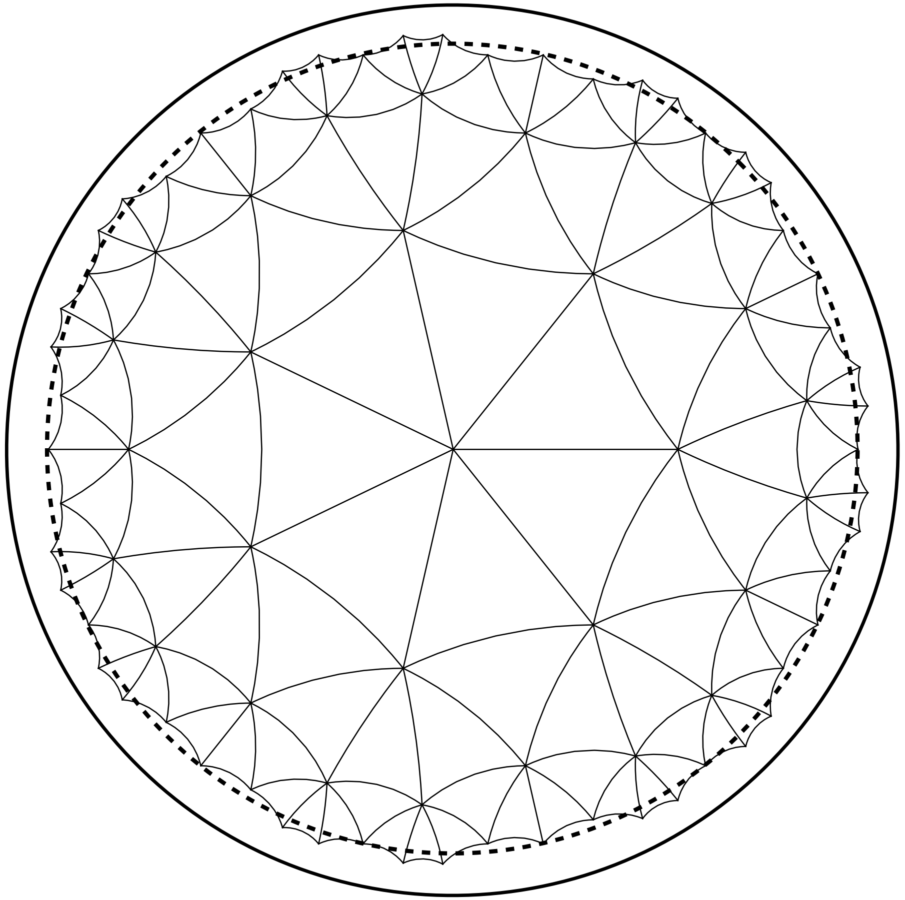
https://en.wikipedia.org/wiki/5-polytope#Regular_5-polytopes https://en.wikipedia.org/wiki/4-polytope

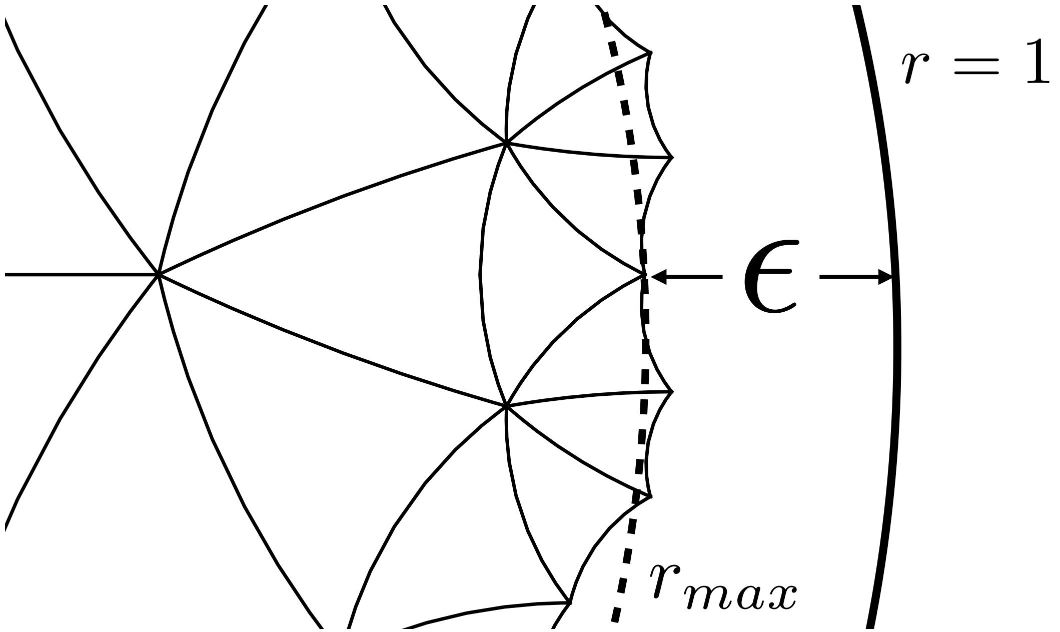
AdS3 Hamiltonian from



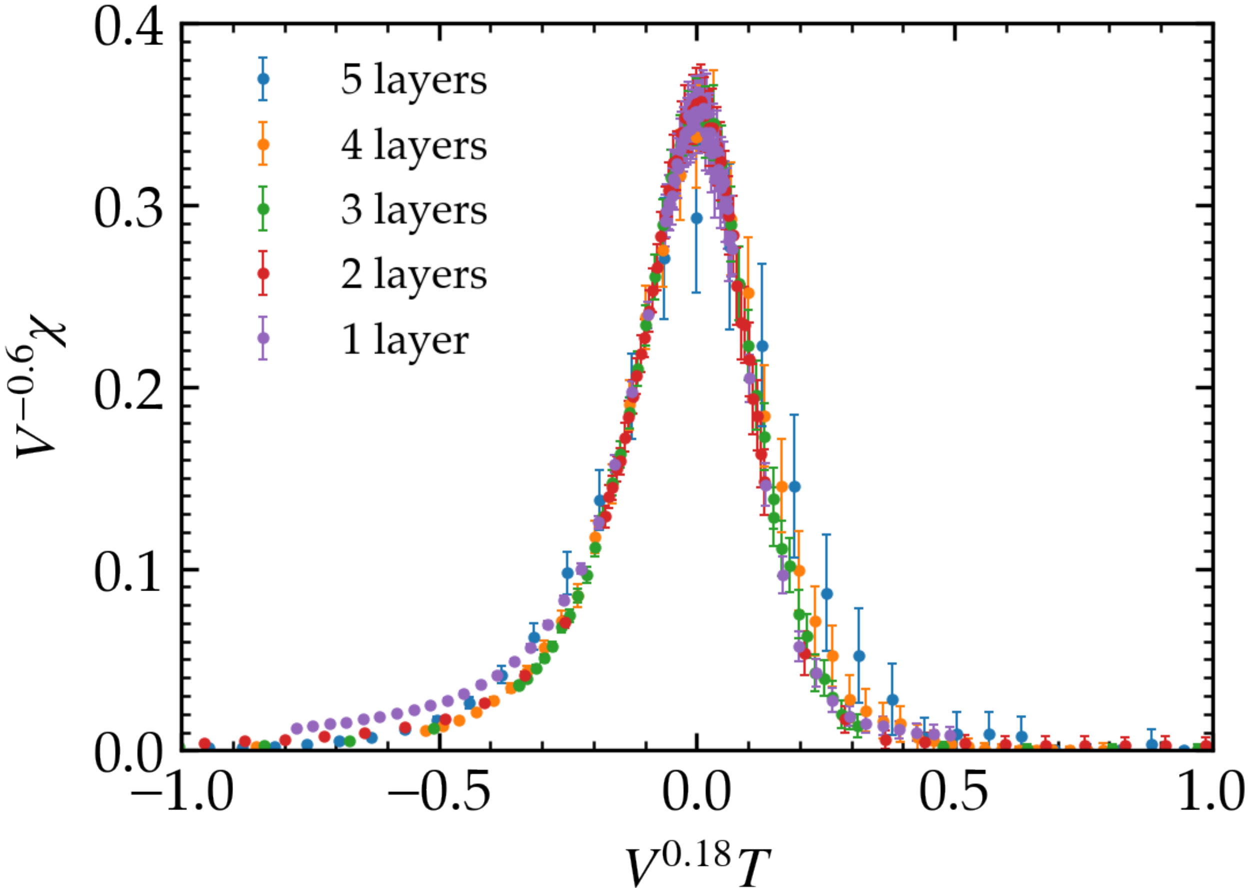


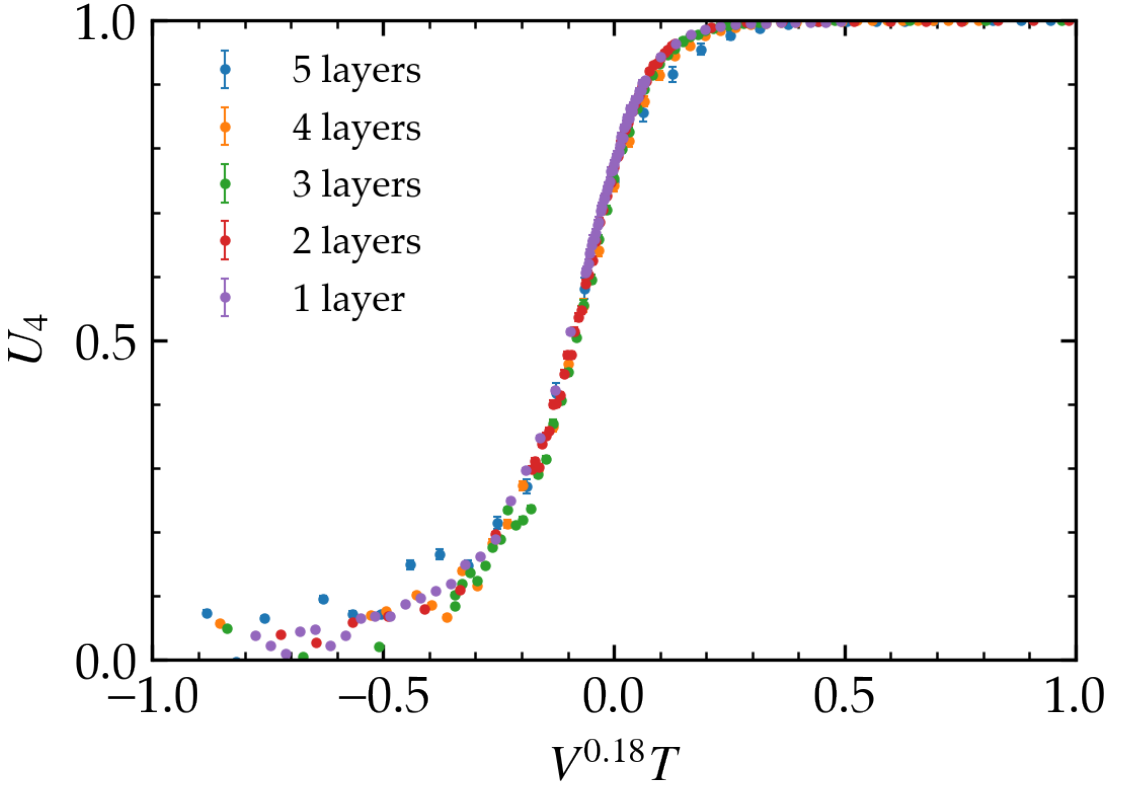
UV cut off problem





Bulk to Boundary Critical Phenomena





Making Steady Progress Going Backward!

- 2013: Lattice Radial Quantized: 3D Ising (R x S2)
- 2017: Lattice Dirac on S2 Simplicial Riemann Manifold (S2:Free CFT)
- 2018: phi^4 test of 2-d Ising CFT on S2 (S2)
- 2019: Lattice Setup for Quantum Field Theory in AdS2
- 2021: Radial Lattice Quantization of 3D phi^4 Field Theory (R x S2)
- 2022: Lattice AdS3 for Scalar Field Theory (w. C. Cogburn, E. Owen)
- 2022: Ising Model on the Affine Plane (w. E. Owen) (2D Torus!)
- 2023-4 "Exact" Ising Model on the the 2 sphere

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[3] Richard C. Brower, George T. Fleming, Andrew D. Gasbarro, Dean Howarth, Timothy G. Raben, Chung-I Tan, and Evan S. Weinberg.
Radial Lattice Quantization of 3D ϕ^4 Field Theory.
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