Islands and Page curve for sub-region complexity in Eternal AdS Black Holes

Cracow School of Theoretical Physics, 2024

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JAGIELLONIAN UNIVERSITY
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funded by
Narodowe
Centrum
Nauki
grant 2021/42/E/ST2/00234

Based on 2103.15852 (with
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P. Nandy (YITP, Kyoto) &
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(JHEP 05(2021)135)

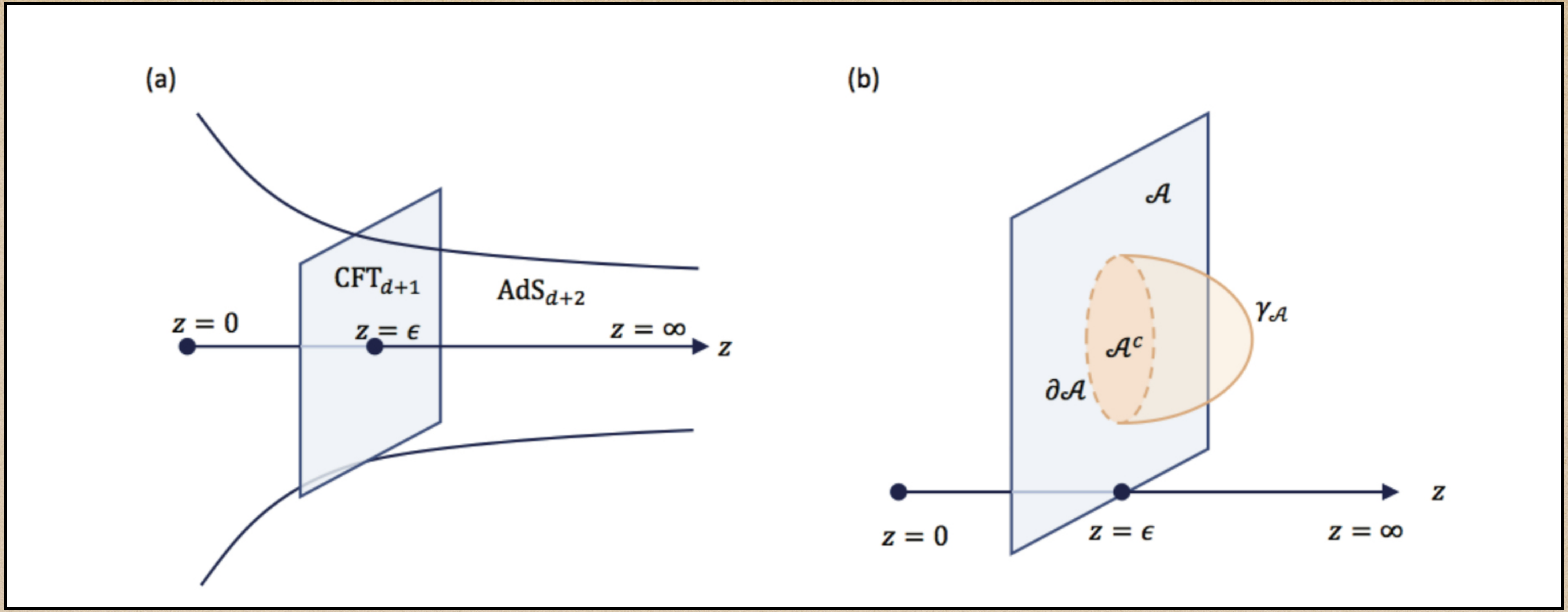
Introduction

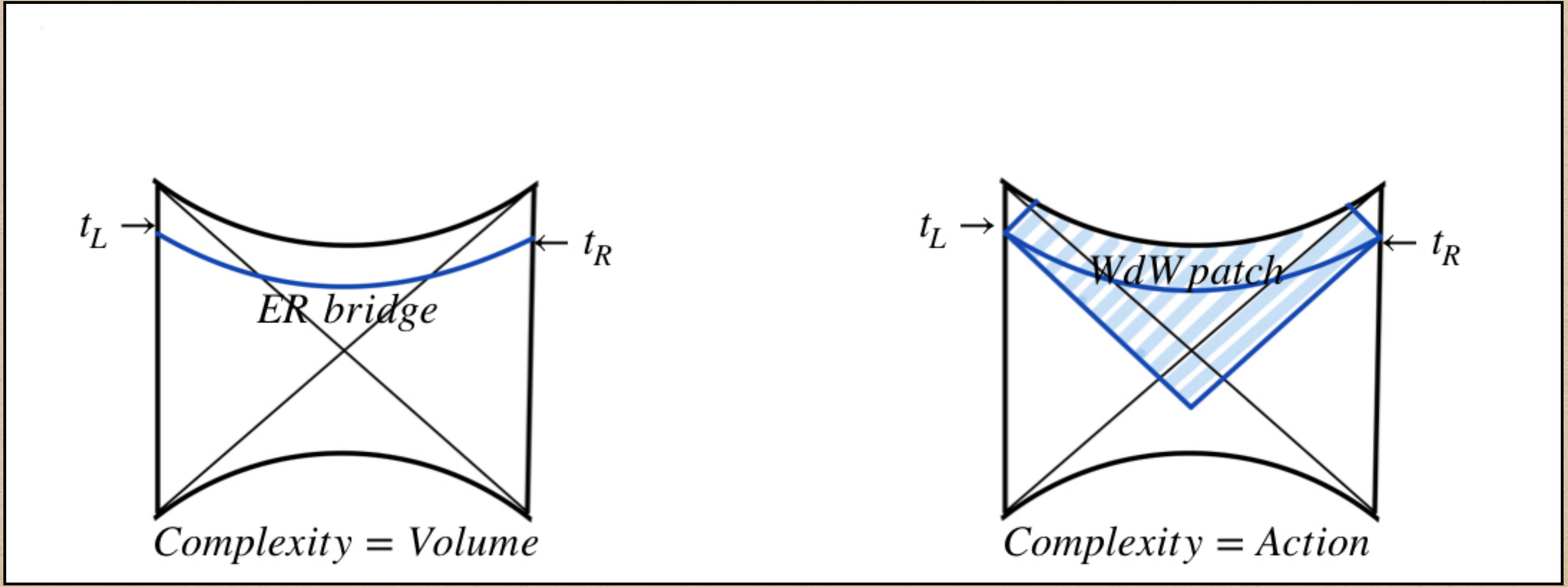
- Applications of QI in holography → HEE (RT, HRT,....)→ $\frac{A_{min/extr}}{4G} = Tr(-\rho_A log \rho_A) \implies$ RT = Reduced density matrix

- Entanglement builds geometry → QI builds geometry!

- Looking beyond BH horizon → Complexity = Volume (pure states). (Susskind, Brown,.......)

- Complexity→ difficulty of preparing a state (reference → target, pure). (Nielsen, Jefferson, Myers, Chapman, Heller, Caputa, Magan, Flory,.....)


(a)
(b)
CFT$_{d+1}$
AdS$_{d+2}$
$z = 0$
$z = \epsilon$
$z = \infty$
z
$\mathcal{A}$
$\gamma_{\mathcal{A}}$
$\mathcal{A}^c$
$\partial\mathcal{A}$
$z = 0$
$z = \epsilon$
$z = \infty$
z

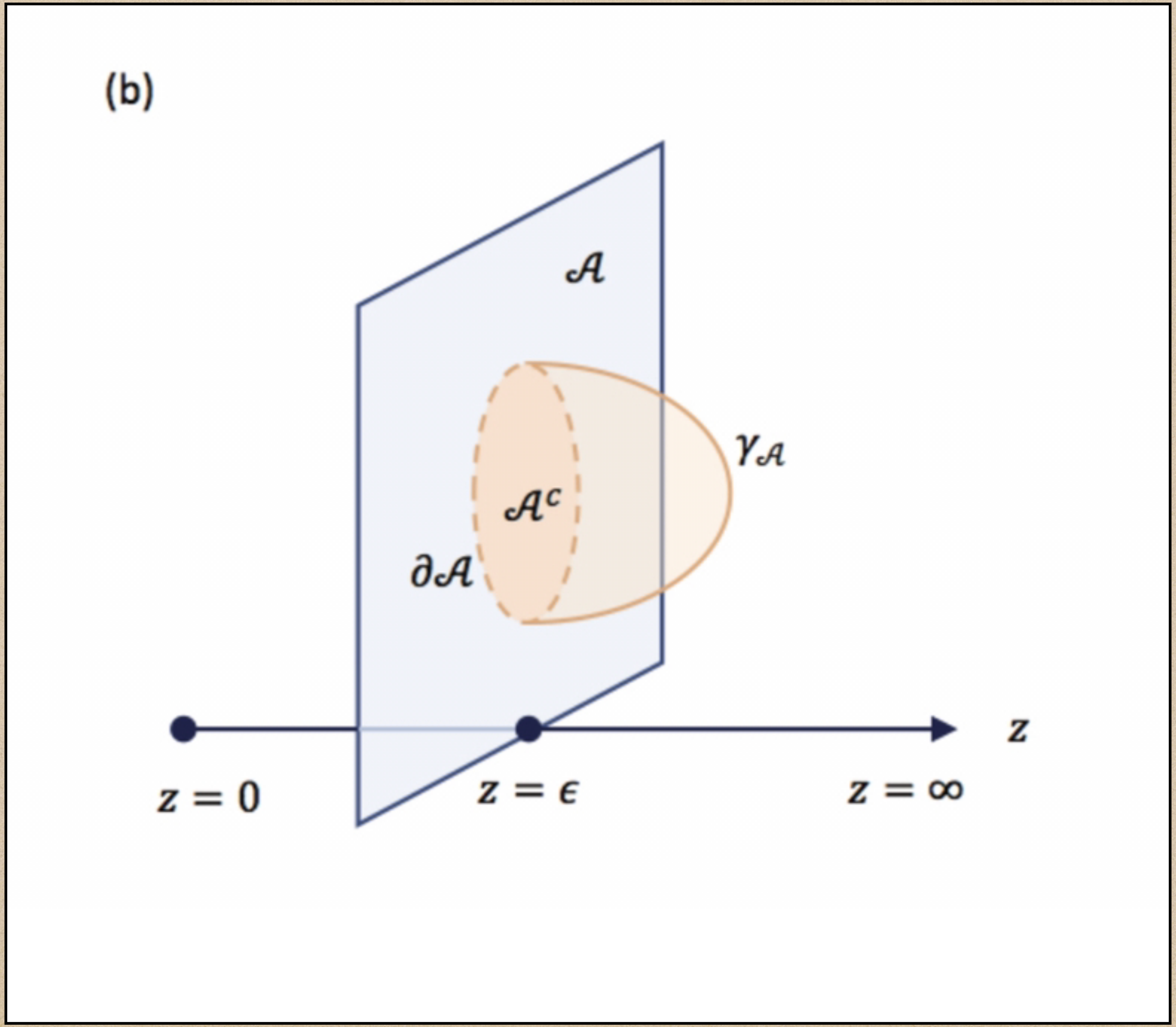

t_L
t_R
ER bridge
Complexity = Volume
t_L
t_R
WdW patch
Complexity = Action

- Mixed states → purification (add auxiliary system) + minimisation of complexity functional in terms of auxiliary system parameters. (Chapman, Myers, Heller, Caputa, Camargo, Jahn, Takayanagi,.....)

- Mixed states → holographic sub-region complexity $= \frac{V_{max}(subregion)}{8\pi LG} \Longrightarrow$ entanglement wedge =Full info of ρ_A. (Alishahiha, Swingle, Caceres, AB, Erdmenger, Abt, Northe,....)

- Extension of HRT → QES→ Page curve of EE between BH & non-gravitational bath (Engelhardt, Wall, Pennington, Almheiri, Maldacena, Hartman, Stanford, Mahajan,...)[2 and higher dimensions]

- Purely bulk regions → entanglement islands appear in the EW of radiation starting from Page time → path-shift/bending of growing EE curve. (Breakdown of bulk EFT description at page time)


(b)
$\mathcal{A}$
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z
$z = 0$
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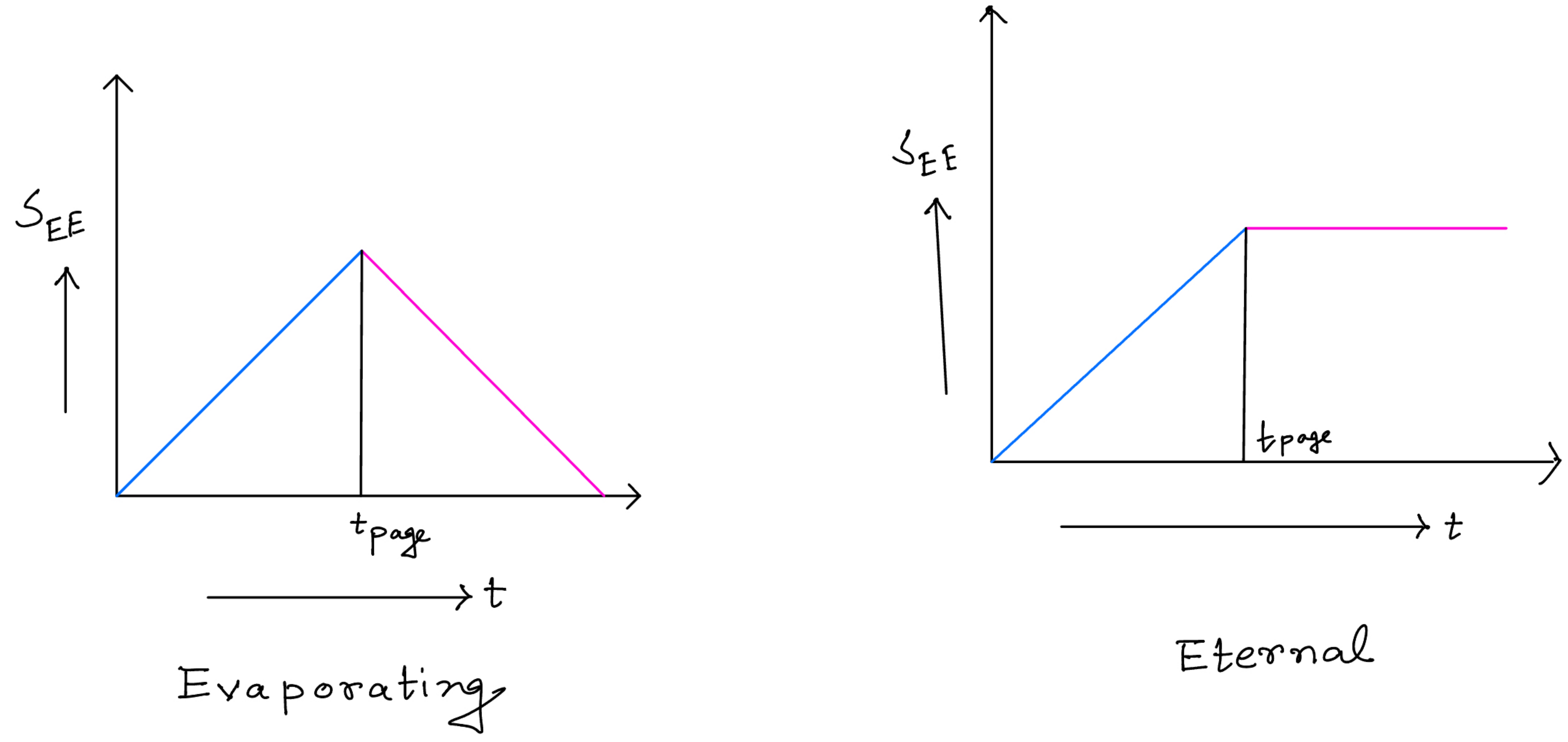


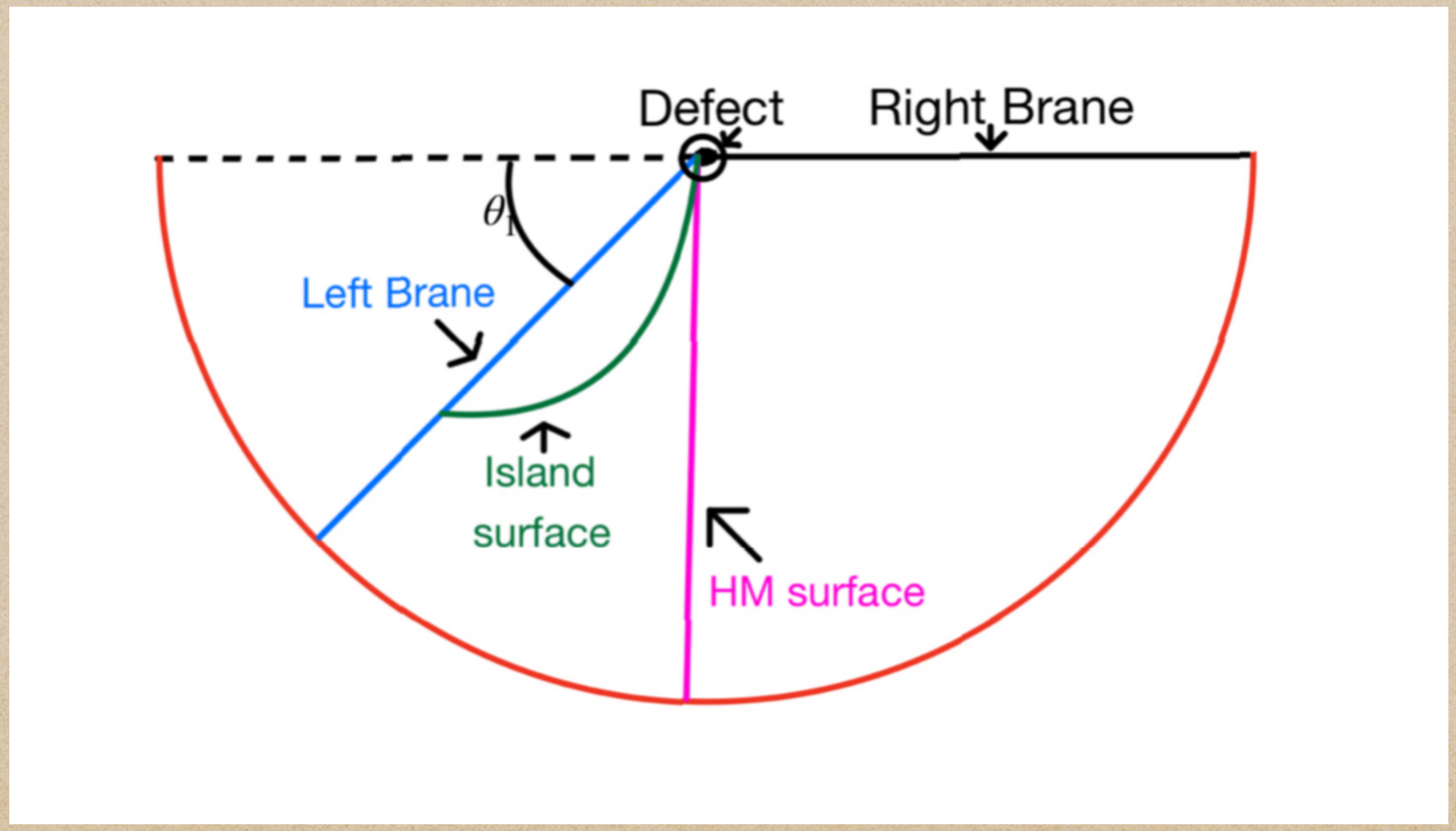
S_{EE}
t_{page}
t
Evaporating
S_{EE}
t_{page}
t
Eternal

- 2 versions of Page curve→ Evaporating & Eternal.
- What about sub-region complexity? (AB, Chanda, Maulik, Northe, Roy 2020, Myers et al, 2021)
- what happens at page time? Suddenly some beyond-the-horizon modes become available to the density matrix of the radiation, auto-purification of the partner modes.
- Naively: the complexity of one half of partner modes+ add the complexity of purification for the partner modes. Result??
- 2010.04134 → MbW wormhole model→ evaporating BH → jump in sub region complexity of radiation at Page time→ holographic complexity of Purification.

Braneworld model (2012.04671, Geng, Karch, Randall, Raju et al JHEP)

- Islands in higher dimensions → Randall-Sundrum braneworlds with subcritical tension branes →Karch-Randall braneworld.
- Bulk cosmological constant & tension of the brane→ fine tuning parameters.
- Weakly gravitating bath - introduce a second brane
- Brane angles dictate the strength of gravity on the brane (brane tension) - weak gravity -less angled right brane.

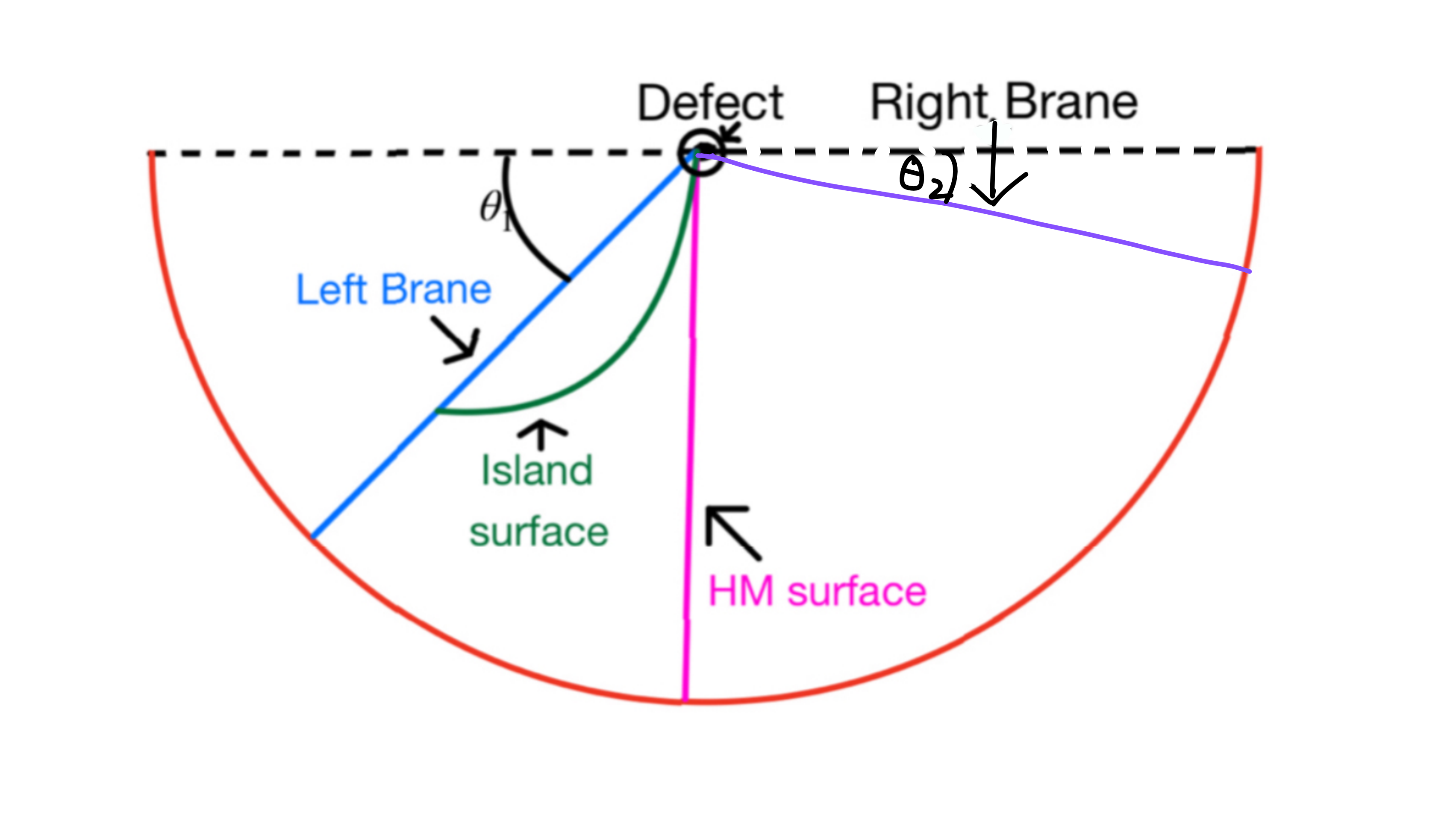
- One brane in bulk (non-gravitating bath)- i)BCFT$_d$. ii)asymp AdS$_d$ connected to d dim CFT on Half- Minkowski space iii) Einstein gravity on asymp AdS$_{d+1}$ containing d dim KR brane as EOW brane.
- Two branes (gravitating bath)- i) d-1 dim defect CFT ii) two d dim CFTs (asymp AdS$_d$ bulks) connected at defect iii) Einstein gravity on asymp AdS$_{d+1}$ containing two EOW branes connected at defect.
- Double holography- compute dynamical RT surface in bulk.

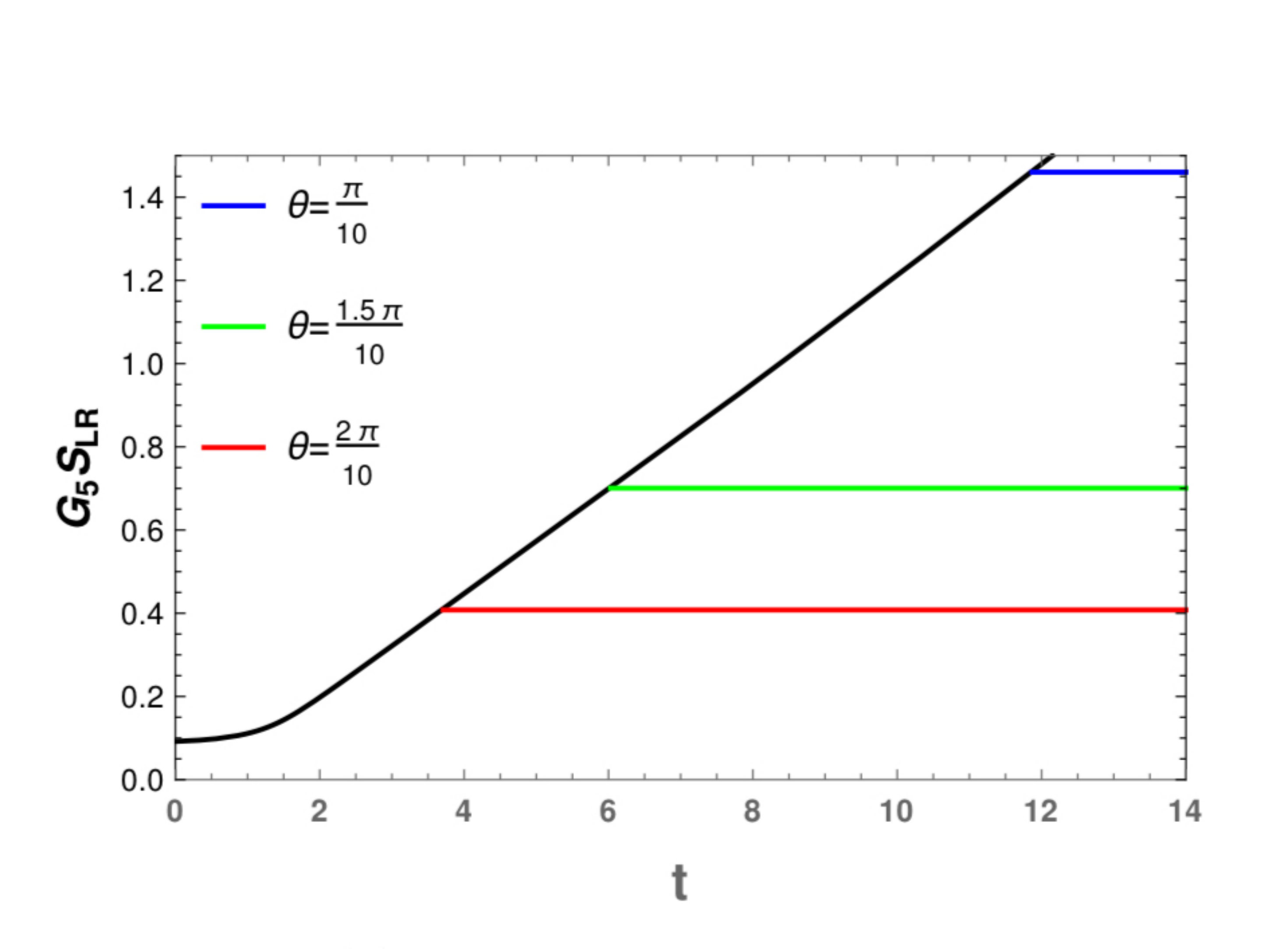


Left-right entanglement (Page curve)

- Eternal BH Page curve for the left and right modes of the thermofield double state of (d-1) dim defect CFT where the two branes meet with finite temperature.
- Left and right modes = left and right branes ??
- candidate RT surfaces have to end on the defect.

- Two candidates - i) Hartman-Maldacena-Mathur (grows indefinitely) starting from defect , goes beyond BH horizon in the bulk, ends on thermofield double partner defect, ii)Constant Surfaces starting from defect and shooting towards one of the branes (don't grow, don't go beyond horizon).
- Initially HMM is minimal- later Constant surface is minimal (HMM grows) (ISLANDS!!!)
- After minimal surface shift from HMM to Island surface, right mode d.o.f appear on left branes as well. Therefore left and right modes $\neq$ left and right branes.





HMM &Island surfaces

- Consider AdS$_{d+1}$black string metric in bulk.
- $ds^2 = \frac{1}{u^2 \sin^2 \mu}\left[-h(u)dt^2 + \frac{du^2}{h(u)} + d\vec{x}^2 + u^2 d\mu^2 \right], \qquad h(u) = 1 - \frac{u^{d-1}}{u_h^{d-1}}$
- Here $u > 0$ is the radial direction, $0 \leq \mu < 2\pi$ is the angular coordinate and $\vec{x}$ is (d-2) orthogonal directions.
- The Hartman-Maldacena surface is located at $\mu = \frac{\pi}{2}$ - only need to minimize the area functional there.
- $\mathscr{A} = \int dt \mathscr{L}$, with the Lagrangian , $\mathscr{L} = u^{d-1}\sqrt{h(u) + \frac{\dot{u}^2}{h(u)}}$.

- Now

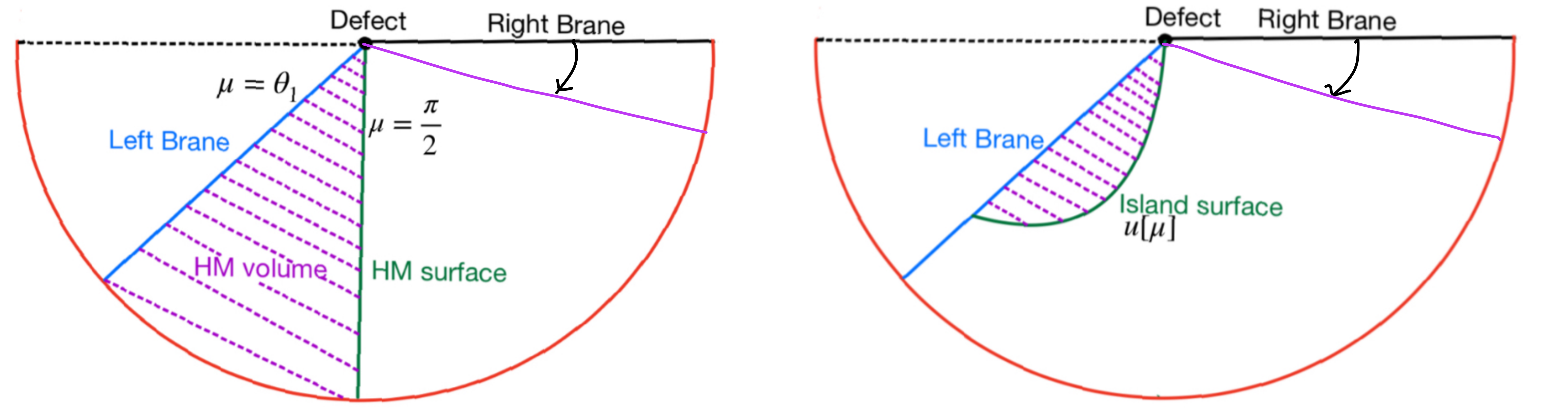
$$A^{\text{reg}}_{HM}(t_{\textbf{DIFF}}) = \lim_{\delta\to 0}\left[-\frac{1}{(d-2)\delta^{d-2}} + \int_{\delta}^{u_{\text{crit}}} \frac{du}{\dot{u}\ \ u^{d-1}}\sqrt{-h(u)+\frac{\dot{u}^2}{h(u)}}\right]$$

- For island surfaces (constant), the embedding function is found by solving with a timeslice and considering $u = u(\mu)$embedding.

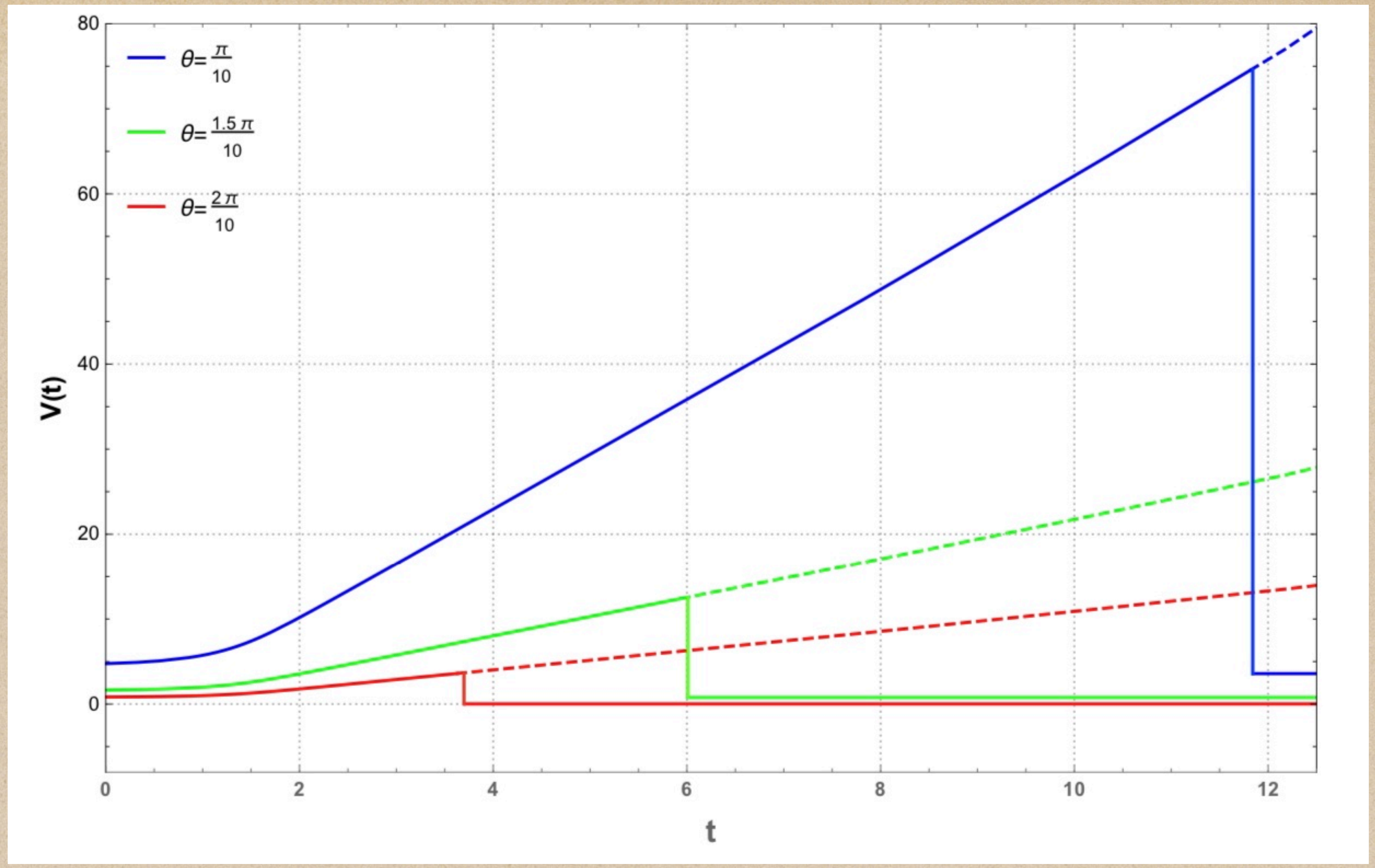
- The embedding action for the black string metric in $d = 4$ is given by $\mathscr{A} = \int_{\theta_1}^{\pi-\theta_2} \frac{d\mu}{(u\sin\mu)^3}\sqrt{u(\mu)^2 + \frac{u'(\mu)^2}{h(u)}}$

Sub-region complexity

- Compute Volumes below HMM until Page time + Volumes below island surface after Page time- argue maximal volumes follow the same qualitative behaviour.
- HMM surface - grows with time, independent of μ - foliate the HMM volume by HMM surfaces for all μ - V_{L-HM} (grows in time)
- Island surface - constant - compute volumes between Island and left brane from $u = \epsilon$ to critical anchor -V_{L-Is}.

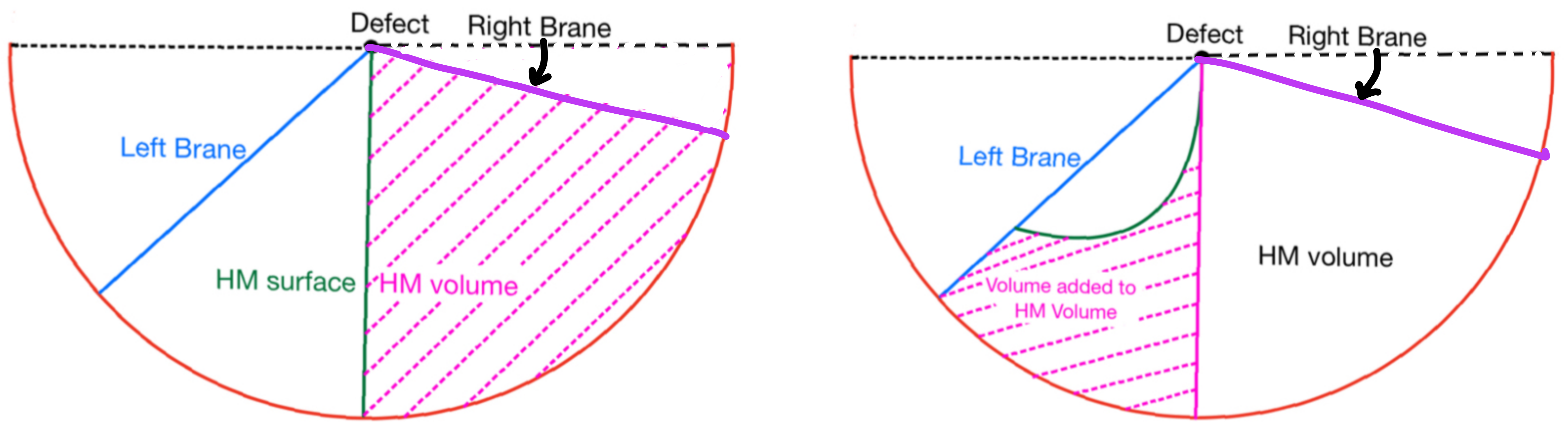


(a) Volume between physical brane and HM surface. (b) Volume under island on physical brane.



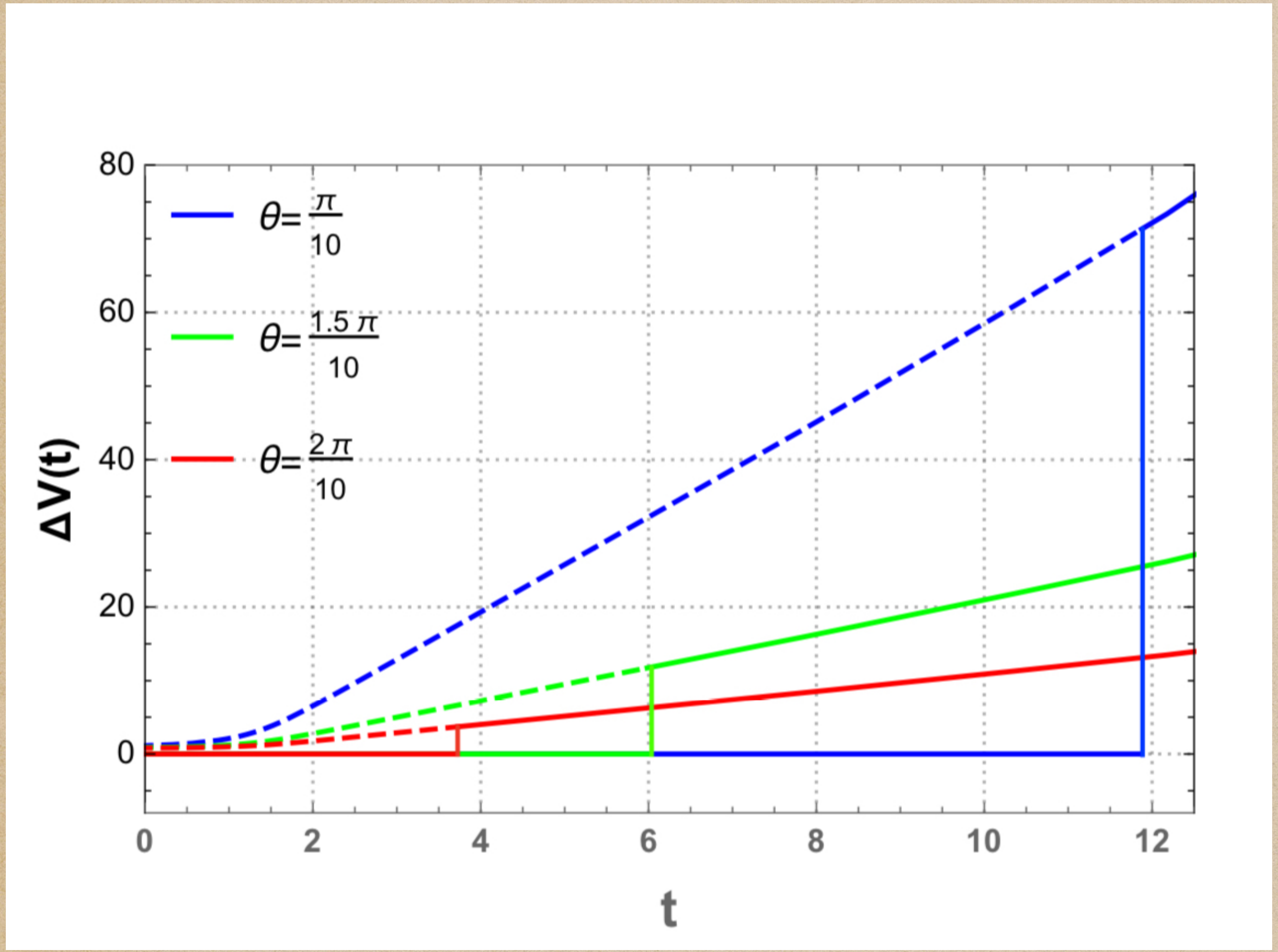
- $V_{L-HM(\text{MAX})}(t) \geq V_{L-HM}(t_0)$, Island surface is indep. of time
  $V_{L-Is}(t) = V_{L-Is}(t_0)$
- $V_{L-HM}(t=0) > V_{L-Is}(t=0)$.
- $V_{L-HM(\text{MAX})}(t) \geq V_{L-HM}(t_0) > V_{L-Is}$.

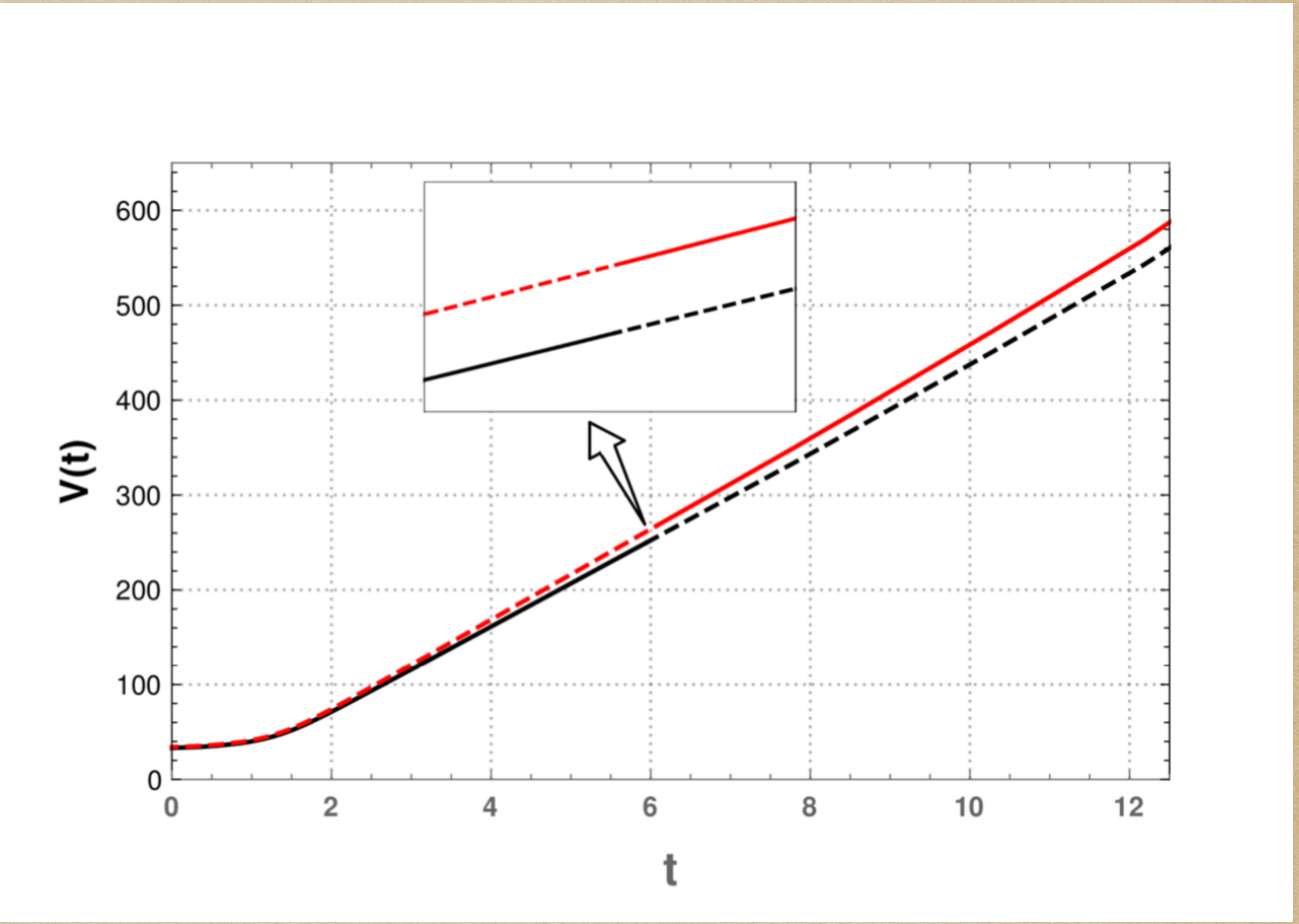
- Right brane - volume keeps growing after Page time.
- Jump at Page time .
- Newly added volume also grows with time as involves beyond horizon region- slope increases - similar qualitative arguments for covariant volumes hold.



(a) Volume between HM surface and right brane.

(b) Volume added to right brane after Page time.


ΔV(t)
t
0
2
4
6
8
10
12
0
20
40
60
80
θ=π/10
θ=1.5π/10
θ=2π/10

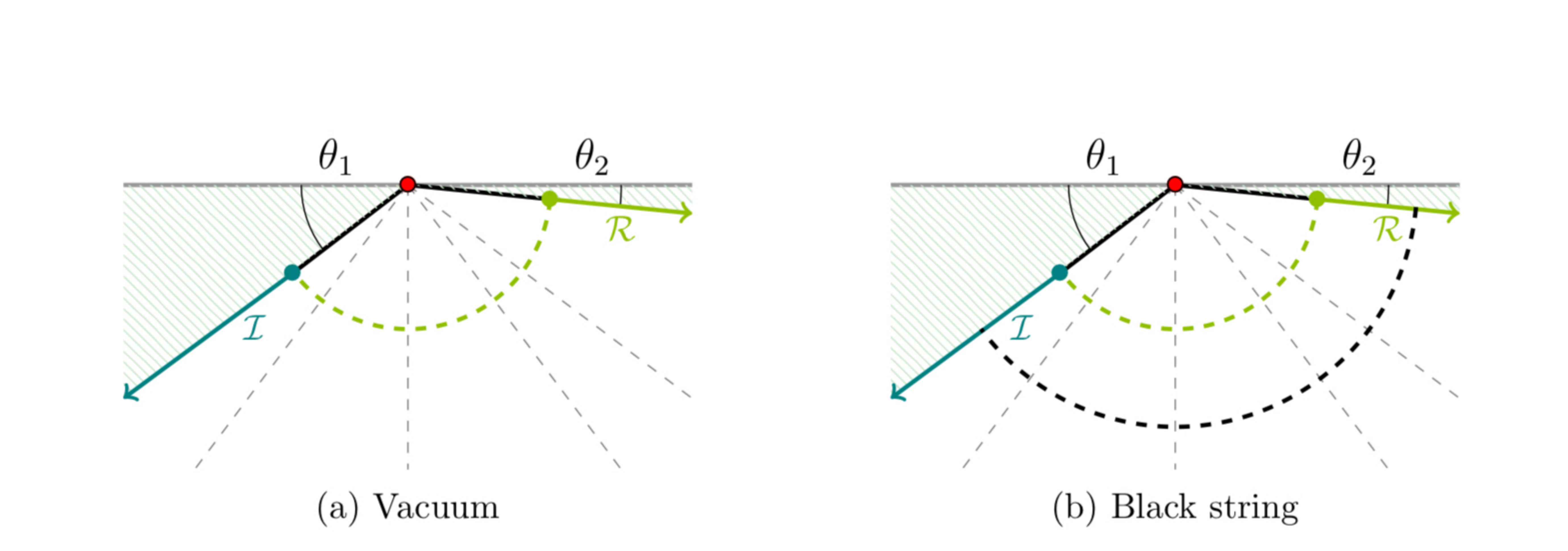

V(t)
t
0
2
4
6
8
10
12
0
100
200
300
400
500
600

Concluding Remarks

- At Page transition point, the subregion complexities go through a jump(right) or dip(left) (consistent with previous studies about evaporating BHs).
- The divergence-subtracted volumes corresponding to the left brane after Page time becomes constant, which is the vol. between the island surface and the left brane.
- For the right brane, the volume increases for ever with a jump and incremental growth rate after Page time.

- Extensions: i) To compute volumes for sub-systems of the radiation system to check how the quantum secret is shared between parts of radiation. (2109.07842, JHEP 2021)
- ii)To understand the event of auto-purification due to the appearance of islands in simple lattice models by mimicking the guess in a manipulative way.

Thank you



Source: 2012.04671

- Zero temperature as well as thermal black string configuration.
- Gravitational bath - no diffeo-invariant way to differentiate b/w local d.o.f→ dynamical radiation region (remember both branes contain gravity)→ no simple tensor factorisation of Hilbert space in gravity .
- One brane in bulk (non-gravitating bath)- i)BCFT$_d$. ii)asymp AdS$_d$ connected to d dim CFT on Half- Minkowski space iii) Einstein gravity on asymp AdS$_{d+1}$ containing d dim KR brane as EOW brane.

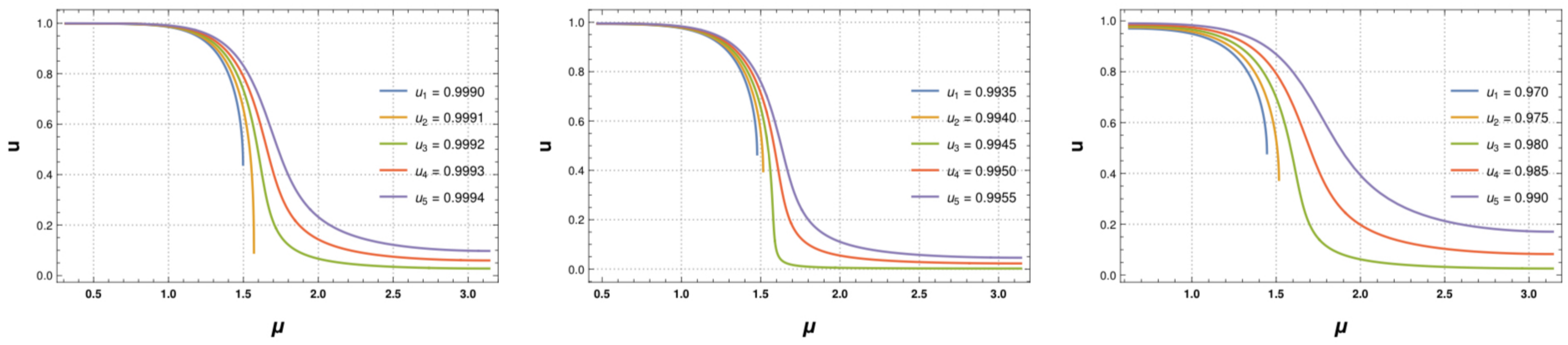
- no explicit time dependence in the Lagrangian thus we can write the conservation equation,
$$E = \dot{u}\frac{\partial \mathscr{L}}{\partial \dot{u}} - \mathscr{L} \implies \dot{u} = \pm \frac{h(u)}{E}\sqrt{E^2 + u^{-2(d-1)}h(u)},$$ where the sign is $+$ when $u < u_h$ and $-$ otherwise.

- The critical point u_{CRIT} upto which we should integrate the area is determined by the relation, $E^2 = -\, u_{\text{crit}}^{2(1-d)} h(u_{\text{crit}})$.

- Now $$A_{HM}^{\text{reg}}(t_{\text{DIFF}}) = \lim_{\delta \to 0}\left[-\frac{1}{(d-2)\delta^{d-2}} + \int_{\delta}^{u_{\text{crit}}} \frac{du}{\dot{u}\;\; u^{d-1}}\sqrt{-h(u) + \frac{\dot{u}^2}{h(u)}}\right]$$

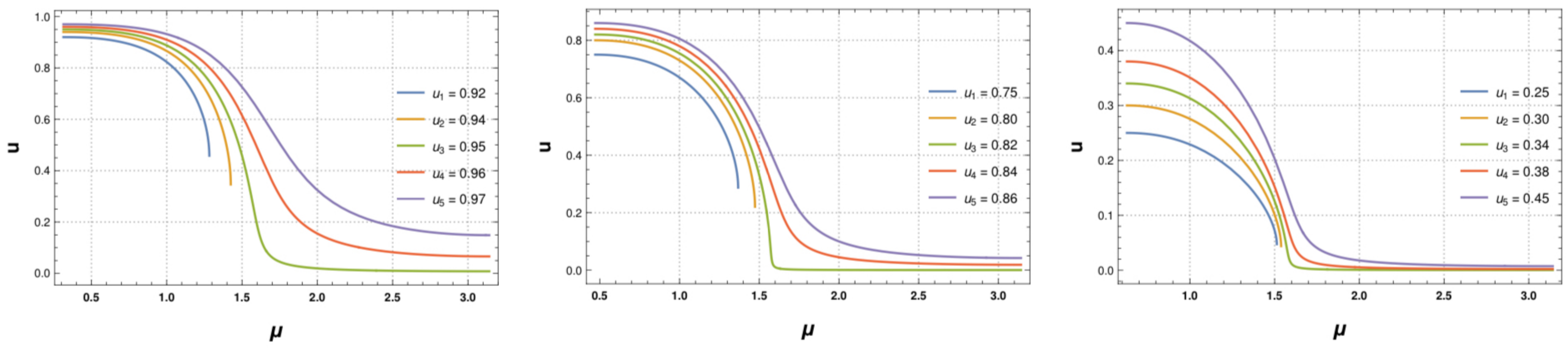
- For island surfaces (constant), the embedding function is found by solving with a timeslice and considering $u = u(\mu)$ embedding.

- **The embedding action for the black string metric in $d = 4$ is given by** $\mathscr{A} = \displaystyle\int_{\theta_1}^{\pi-\theta_2} \frac{d\mu}{(u\sin\mu)^3}\sqrt{u(\mu)^2 + \frac{u'(\mu)^2}{h(u)}}.$



(a) $\theta = \pi/10$. (b) $\theta = 1.5\,\pi/10$. (c) $\theta = 2\pi/10$.

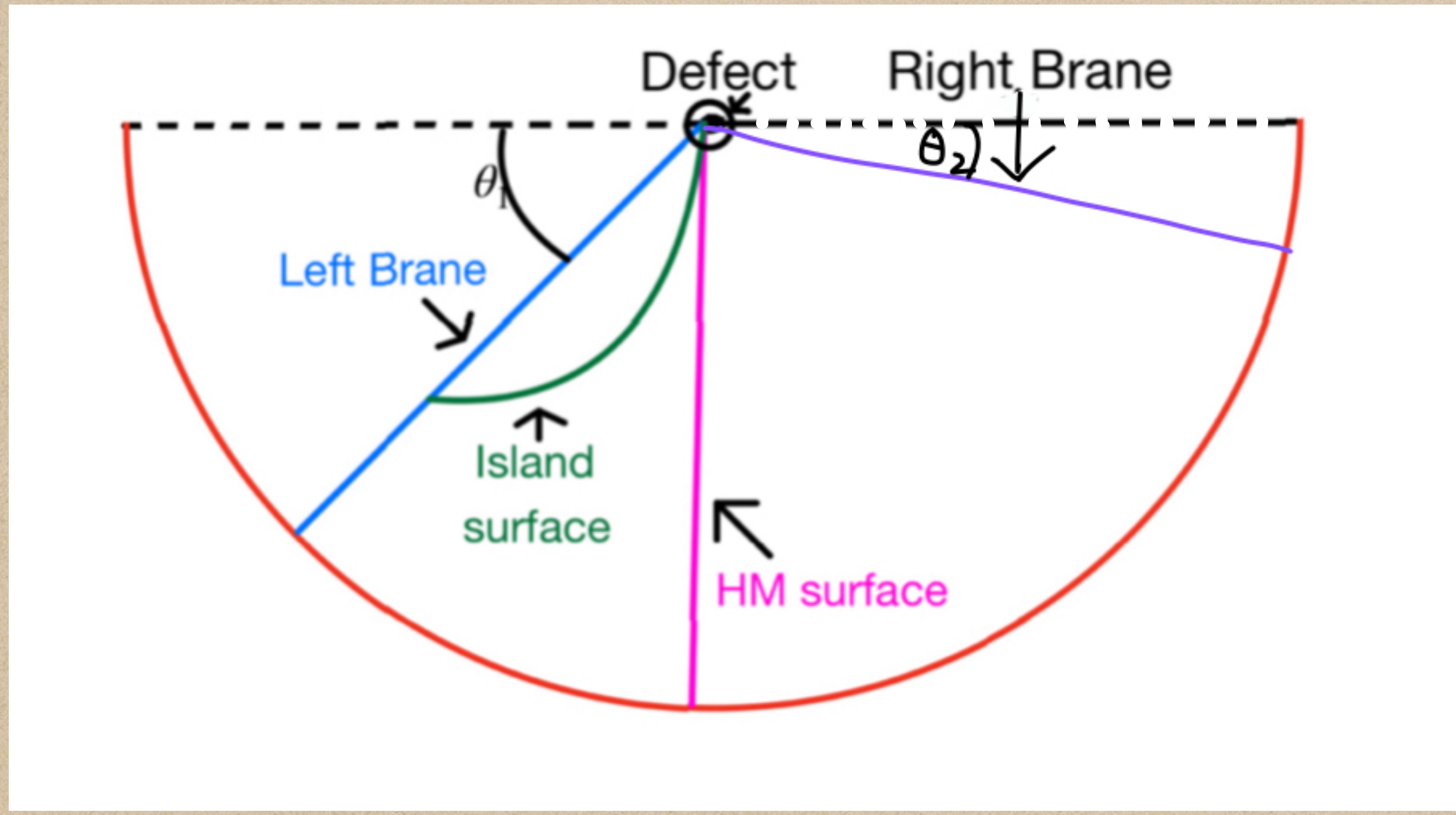
Figure 3: Critical anchor at $d = 4$ for different values of physical (left) brane angle.

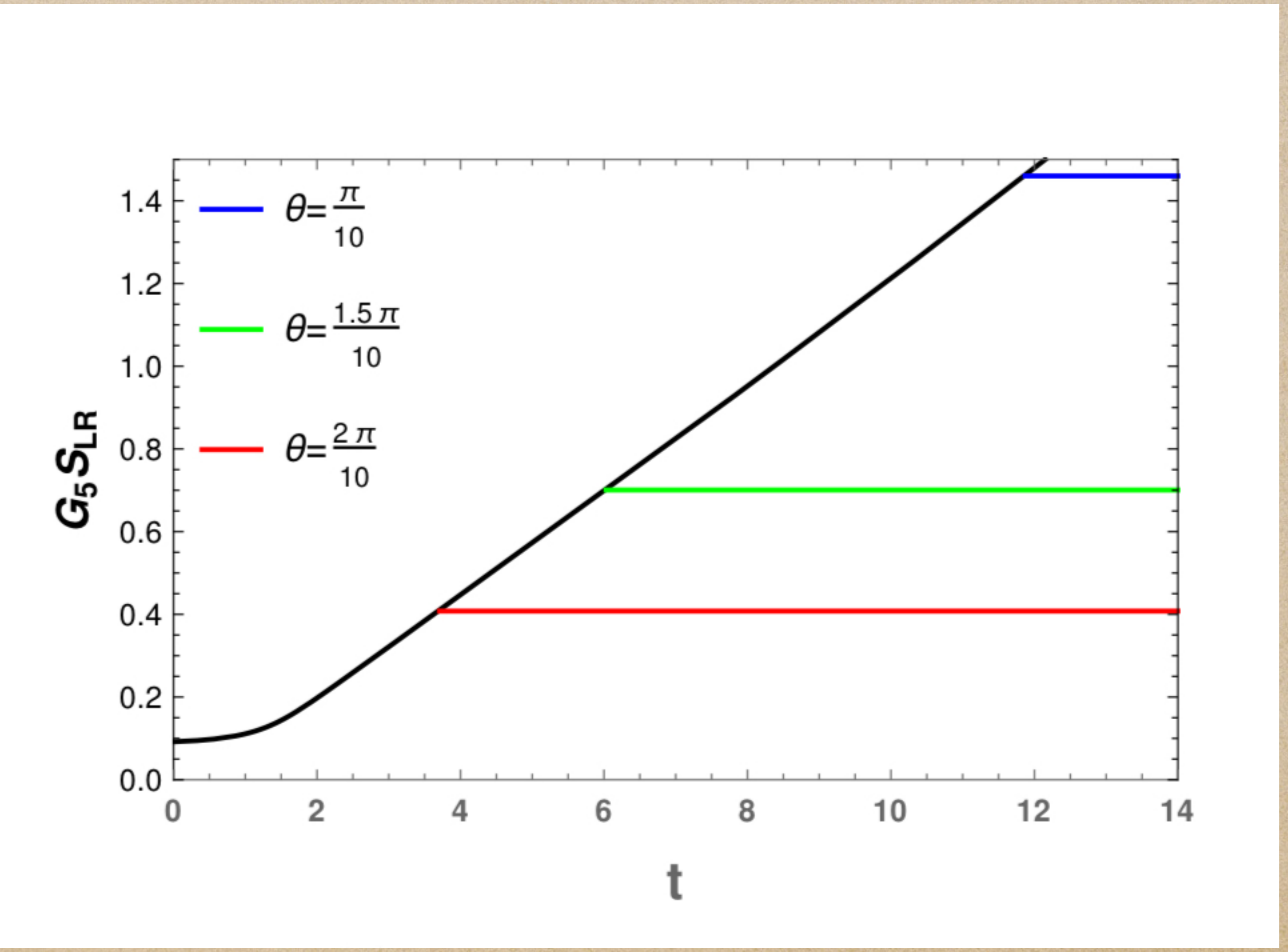


(a) $\theta = \pi/10$. (b) $\theta = 1.5\,\pi/10$. (c) $\theta = 2\pi/10$.

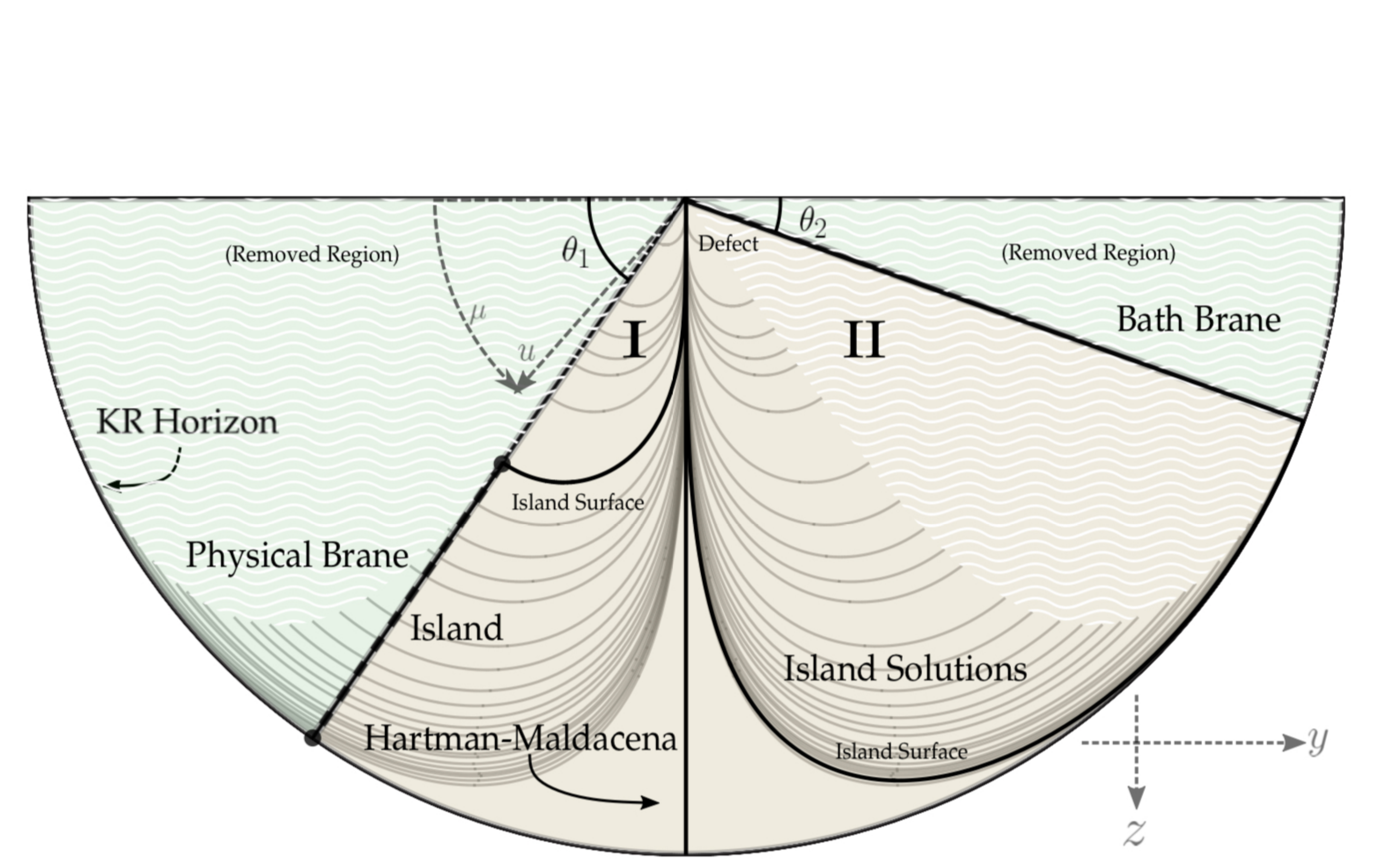
Figure 4: Critical anchor at $d = 3$ for different values of physical (left) brane angle.

- Non-trivial transition from one RT to another. Left modes - eternal BH w/ non-grav bath , Right modes - Radiation....
- This Page curve exists in both gravitating as well as non-grav. bath. So, the right brane can be at the conformal boundary or at a small angle (weakly gravitating.) in the bulk with the defect.





- Island surfaces can in principle be found on both the branes.
- Operationally, we shoot geodesics from the physical left brane towards the conformal boundary from various points on the brane. The minimal one reaching the defect point is the island geodesic, corresponding u value is known as critical anchor.
- Critical anchor is monotonically decreasing function of brane angle.
- Physical brane being at a higher angle will have minimal islands.



Source: 2012.04671

Sub-region complexity

- Static case - volume below the RT surface - $C_A = \frac{V\left(\gamma_{RT(AA^C)}\right)}{8\pi\ell G_N}$.

  (Alishahiha, 2015)

- Covariant def. - for time-dependent cases is a combination of the ``Complexity= Volume'' proposal and static proposal . (Swingle et al ,2018)

- Covariant proposal of subregion volume complexity - look for co-dimension one bulk slices $\Sigma_A(t_0, t)$ w/ boundary $\partial\Sigma_A(t_0, t) = A(t_0) \cup \gamma(t_0, t)$

- Among infinite number of such HRT slices Σ_A, take the one with the maximal volume- maximal volume Cauchy slice of the EW
- EW- bulk domain of dependence, bounded by $\partial\Sigma_A(t_0, t)$ - boils down to Alishahiha's proposal in a time-independent scenario - the t dependence of $\Sigma_A(t_0, t)$ is not there.
- $$C_{A_{\text{cov}}}(t) = \text{Max}_{\Sigma_A(t_0,t)}\left[\frac{V\left(\Sigma_A\left(t_0, t\right)\right)}{8\pi\ell G_N}\right]$$