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Kinetic theory of massive spin-1/2 particles from the Wigner-function formalism

N. Weickgenannt, X.-l. Sheng, E. Speranza, Q. Wang, and D.H. Rischke,
arXiv:1902.06513

Cracow School of Theoretical Physics, LIX Course, 2019

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Why magneto-hydrodynamics (MHD)?

- Early stage of non-central heavy-ion collisions: large orbital angular momenta and strong electromagnetic fields.

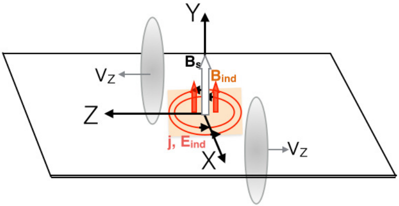


Figure from V. Roy, S. Pu, L. Rezzolla, and D. H. Rischke, PRC96 (2017) 054909

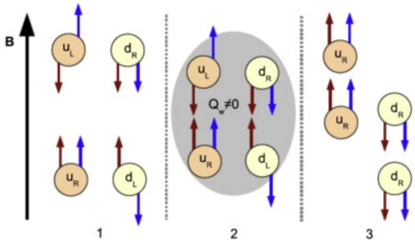
- Strong electromagnetic fields in early universe and compact stars.
- For massive spin-0 particles, second-order dissipative MHD has already been studied.

  G.S. Denicol, X.-G. Huang, E. Molnar, G.M. Monteiro, H. Niemi, J. Noronha, D.H. Rischke, and Q. Wang, PRD 98 (2018) 076009

  G.S. Denicol, E. Molnar, H. Niemi, and D.H. Rischke, PRD 99 (2019) 056017
- But all elementary matter particles are fermions...

Spin effects in heavy-ion collisions

- Chiral vortical effect (CVE): charge currents induced by vorticity.
- Chiral magnetic effect (CME): charge currents induced by magnetic fields.



Dmitri E. Kharzeev, Larry D. McLerran, and Harmen J. Warringa, NPA 803 (2008)

- Has been studied in massless case.

  J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601;

  Y. Hidaka, S. Pu, and D-L. Yang, PRD95 (2017) 091901;

  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010
- Similar effects for massive particles?

Towards MHD with spin

- What we want: kinetic theory and fluid dynamics for massive spin-1/2 particles in inhomogeneous electromagnetic fields.

  J.-H. Gao, and Z.-T. Liang, arXiv:1902.06510 (2019)

  K. Hattori, Y. Hidaka, and D.-L. Yang, arXiv:1903.01653 (2019)

  Z. Wang, X. Guo, S. Shi, and P. Zhuang, arXiv:1903.03461 (2019)
- Starting point: quantum field theory, Dirac equation.
- Strategy: use Wigner functions to derive kinetic theory.
- Goal: determine fluid-dynamical equations of motion from resulting Boltzmann equation.

Wigner functions

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- Quantum analogue of classical distribution function.
- Contains information about quantum state of system.
- Off-equilibrium: two-point function depends not only on relative coordinate y, but also on central coordinate x.
- Wigner transformation of two-point function:
  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$W(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p\cdot y} \langle : \bar{\Psi}(x+\frac{y}{2}) \Psi(x-\frac{y}{2}) : \rangle \,,$$

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H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$W(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p\cdot y} \langle : \bar{\Psi}(x+\frac{y}{2}) U(x+\frac{y}{2}, x) U(x, x-\frac{y}{2}) \Psi(x-\frac{y}{2}) : \rangle \,,$$

with gauge link

$$U(b,a) \equiv P \exp\left(-\frac{i}{\hbar}\int_a^b dz^\mu\, A_\mu(z)\right)$$

to ensure gauge invariance.

Calculating the Wigner function

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- In general: result of calculation of Wigner function directly from definition is not on-shell.
- Momentum variable of directly calculated Wigner function is physical (kinetic) momentum only at zeroth order in $\hbar$/gradients.
- Dirac equation implies transport equation for Wigner function.
- Idea: Find general solutions for this transport equation by expanding in powers of $\hbar$.
- Decompose W in transport equation into generators of Clifford algebra:

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right).$$

- Equations for $\mathcal{F}$ (scalar, "distribution function") and $\mathcal{S}_{\mu\nu}$ (tensor, "dipole moment") decouple from rest.
- Determine $\mathcal{V}_\mu$ ("vector current"), $\mathcal{A}_\mu$ ("polarization"), $\mathcal{P}$ from $\mathcal{S}_{\mu\nu}, \mathcal{F}$.
- Results will hold up to order $\mathcal{O}(\hbar)$.
- Notation: $W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2)$.

General results

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$$\mathcal{F} = m\left[V\,\delta(p^2-m^2) - \frac{\hbar}{2}F^{\mu\nu}\Sigma^{(0)}_{\mu\nu}A^{(0)}\,\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,$$

$$\mathcal{S}_{\mu\nu} = m\left[\bar{\Sigma}_{\mu\nu}\delta(p^2-m^2) - \hbar F_{\mu\nu}V^{(0)}\delta'(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,$$

$$\mathcal{P} = \frac{\hbar}{4m}\epsilon^{\mu\nu\alpha\beta}\nabla_\mu\left[p_\nu\Sigma^{(0)}_{\alpha\beta}A^{(0)}\,\delta(p^2-m^2)\right] + \mathcal{O}(\hbar^2)\,,$$

$$\mathcal{V}_\mu = \delta(p^2-m^2)\left[p_\mu V + \frac{\hbar}{2}\nabla^\nu\Sigma^{(0)}_{\mu\nu}A^{(0)}\right] - \hbar\left[\frac{1}{2}p_\mu F^{\alpha\beta}\Sigma^{(0)}_{\alpha\beta} + \Sigma^{(0)}_{\mu\nu}F^{\nu\alpha}p_\alpha\right]A^{(0)}\,\delta'(p^2-m^2) + \mathcal{O}(\hbar^2)\,,$$

$$\mathcal{A}_\mu = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}p^\nu\bar{\Sigma}^{\alpha\beta}\,\delta(p^2-m^2) + \hbar\tilde{F}_{\mu\nu}p^\nu V^{(0)}\,\delta'(p^2-m^2) + \mathcal{O}(\hbar^2)\,,$$

with

$$\bar{\Sigma}^{(0)\mu\nu} = \Sigma^{(0)\mu\nu}A^{(0)},$$

$$p_\nu\bar{\Sigma}^{\mu\nu} = \hbar\nabla^\mu V^{(0)}$$

$$\nabla^\mu \equiv \partial^\mu_x - F^{\mu\nu}\partial_{p\nu}$$

Boltzmann equation for massive spin-1/2 particles

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- Unknown $V = f_+ + f_-$ and $A = f_+ - f_-$ are determined by generalized Boltzmann equation

$$\sum_s \delta\left(p^2 - m^2 - \frac{s}{2}\hbar F^{\alpha\beta}\Sigma^{(0)}_{\alpha\beta}\right)\left\{p^\mu \partial_{x\mu} f_s + \partial_{p\mu}\left[F^{\mu\nu}p_\nu + \frac{\hbar}{4}s\Sigma^{(0)\nu\rho}(\partial^\mu F_{\nu\rho})\right] f_s\right\} = 0$$

- f_s distribution functions for spin-up $(s = +)$ and spin-down $(s = -)$ particles.
- Modified on-shell condition!
- Force on particle: first Mathisson-Papapetrou-Dixon (MPD) equation
  $\rightarrow$ Particle with classical dipole moment $\Sigma^{(0)\mu\nu}$ in electromagnetic field:

  W. Israel, General Relativity and Gravitation 9 (1978) 451

$$m\frac{d}{d\tau}p^\mu = F^{\mu\nu}p_\nu + \frac{\hbar}{4}s\Sigma^{(0)\nu\rho}(\partial^\mu F_{\nu\rho}) .$$

τ: worldline parameter, $\frac{d}{d\tau} = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{p}^\mu \frac{\partial}{\partial p^\mu}$.

Dipole moment and classical limit

- $\bar{\Sigma}_{\mu\nu}$ determined by kinetic equation for dipole moment.
- To zeroth order:

$$m\frac{d}{d\tau}\Sigma^{(0)\mu\nu} = \Sigma^{(0)\lambda\nu}F^{\mu}_{\ \lambda} - \Sigma^{(0)\lambda\mu}F^{\nu}_{\ \lambda},$$

- Recover second MPD equation for dipole-moment tensor $\Sigma^{(0)}_{\mu\nu}$!

  W. Israel, General Relativity and Gravitation 9 (1978) 451

- Equivalent to Bargmann-Michel-Telegdi (BMT) equation

  V. Bargmann, L. Michel, and V.L. Telegdi, PRL 2 (1959) 435

$$m\frac{d}{d\tau}n^{(0)\mu} = F^{\mu\nu}n^{(0)}_{\nu},$$

with classical spin vector

$$n^{(0)\mu} = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu\Sigma^{(0)}_{\alpha\beta}$$

Massless limit

- Non-relativistic dipole-moment tensor connected to spin three-vector n^k:

$$\Sigma^{ij} = \epsilon^{ijk} n^k .$$

- For massive particles: define spin in rest frame.

  U. Heinz, PLB 144 (1984) 228

$$\Sigma^{\mu\nu} = -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta .$$

- For massless particles: define spin in arbitrary frame with four-velocity u^μ. Spin vector n^μ is always parallel to momentum.

  J-Y. Chen, D.T. Son, and M. Stephanov, PRL 115 (2015) 021601

$$\Sigma_u^{\mu\nu} = -\frac{1}{p \cdot u}\epsilon^{\mu\nu\alpha\beta} u_\alpha p_\beta .$$

- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \to \Sigma_u^{\mu\nu}$.
- Result agrees with previously known massless solution!

  Y. Hidaka, S. Pu, and D-L. Yang, PRD 95 (2017) 091901

  A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010

  J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, PRD 98 (2018) 036019

Vector current in global equilibrium

- In global equilibrium: Analytic solution for Boltzmann equation.
- Vector current is explicitly calculated as:

$$\mathcal{V}^\mu = \frac{2}{(2\pi\hbar)^3}\sum_s \Big[\delta(p^2-m^2)\left(p^\mu - m\hbar\frac{s}{2}\tilde{\omega}^{\mu\nu}n_\nu^{(0)}\partial_{\beta\cdot\pi}\right) + \hbar s\tilde{F}^{\mu\nu}n_\nu^{(0)}\delta'(p^2-m^2) + \hbar\frac{s}{2m}\delta(p^2-m^2)\epsilon^{\nu\mu\alpha\beta}p_\alpha\nabla_\nu n_\beta^{(0)}\Big] f_s^{(0)},$$

with zeroth-order equilibrium distribution function

$$f_s^{(0)} = [\exp(\beta\cdot\pi - \beta\mu_s)+1]^{-1},$$

where π^μ canonical momentum, β^μ thermal fluid velocity, β inverse temperature, μ_s chemical potential.

- Analogue of chiral vortical effect (CVE) for massive particles.

  D. T. Son and P. Surowka, PRL 103 (2009) 0906.5044
- Analogue of chiral magnetic effect (CME).

  D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, NPA 803 (2008) 0711.0950
- Thermal vorticity tensor: $\omega_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$.
- Dual thermal vorticity tensor: $\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta}$.

Axial-vector current in global equilibrium

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- Obtain expression for axial-vector current:

$$\mathcal{A}^\mu = \frac{2}{(2\pi\hbar)^3}\sum_s \left[\delta(p^2-m^2)\left(s\,m\,n^{(0)\mu} - \frac{\hbar}{2}\tilde{\omega}^{\mu\nu}p_\nu\partial_{\beta\cdot\pi}\right)\right.$$
$$\left. + \hbar\tilde{F}^{\mu\nu}p_\nu\delta'(p^2-m^2)\right] f_s^{(0)} - \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}p_\nu\Xi_{\alpha\beta}\,\delta(p^2-m^2)\,.$$

- Classical spin precession.
- Analogue of axial chiral vortical effect (ACVE).
- Analogue of chiral separation effect (CSE).

D. E. Kharzeev, J. Liao, S. A. Voloshin, and G. Wang, Prog. Part. NP 88 (2016), 1511.04050

Fluid-dynamical equations with spin I

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- Particle current:

$$J^\mu = \int d^4p\, \mathcal{V}^\mu.$$

Not parallel to the fluid velocity!

$$\partial_\mu J^\mu = 0.$$

Conserved!

- Canonical energy-momentum tensor (matter part):

$$T^{\mu\nu}_{mat} = \int d^4p\, p^\nu \mathcal{V}^\mu.$$

Not symmetric!

$$\partial_\mu T^{\mu\nu}_{mat} = F^{\nu\mu} J_\mu.$$

Conserved in combination with electromagnetic and interaction part.

Fluid-dynamical equations with spin II

- Canonical spin tensor (matter part):

$$S_{mat}^{\lambda,\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\int d^4p\,\mathcal{A}_\rho$$

$$\hbar\partial_\lambda S_{mat}^{\lambda,\mu\nu} = T_{mat}^{\nu\mu} - T_{mat}^{\mu\nu}$$

Not conserved!
Spin angular momentum and orbital angular momentum are converted into one another.
$\rightarrow$ Consideration of spin leads to additional fluid-dynamical equation of motion.

W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, PRC 97 (2018) 041901;

W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, PRD 97 (2017) 116017;

W. Florkowski, F. Becattini, and E. Speranza, APB 49 (2018) 1409;

F. Becattini, W. Florkowski, and E. Speranza, PLB 789 (2019) 419-425

Conclusions

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- Derived transport equation for distribution function and polarization for massive spin-1/2 particles in inhomogeneous electromagnetic fields.
- Recovered classical equations of motion.
- Solution agrees with previously known massless solution in massless limit.
- Derived explicit expressions for currents in global equilibrium.
- Found analogues of CVE, CME, ACVE, and CSE for massive particles.
- Derived fluid-dynamical equations of motion.

Outlook

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- Solve generalized Boltzmann equation.
- Include collisions.
  $\rightarrow$ Boltzmann equation without assuming equilibrium.
- Derive equations of motion for dissipative quantities.
  $\rightarrow$ Method of moments.

Back-up

Conventions and Definitions

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- Natural units, $c = k_B = 1$, but keep $\hbar$ explicitly.
- To simplify notation: only write positive-energy parts of solutions.
- Polarization direction n^μ: space-like unit vector parallel to axial-vector current.
- Spin quantization direction: unit vector, purely spatial in particle rest frame.
  "spin up", $s = +$: projection of spin onto quantization direction positive.
  "spin down", $s = -$: projection of spin onto quantization direction negative.
  Here: chosen to be identical to polarization direction.

$$\bar{u}_s \gamma^\mu \gamma^5 u_s = 2ms\, n^\mu$$

Spin tensor vs. dipole-moment tensor

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- Dipole-moment tensor:

$$s\Sigma^{\mu\nu} = \frac{1}{2m}\bar{u}_s \frac{i}{2}[\gamma^\mu, \gamma^\nu] u_s$$
$$= -s\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta$$

called "spin tensor" in

U. Heinz, PLB 144 (1984) 228,

J-Y. Chen, D. T. Son, and M. Stephanov, PRL 115 (2015), 021601

Y. Hidaka, S. Pu, D-L. Yang, PRD 95 (2017) 091901

S.R. De Groot, Relativistic Kinetic Theory. Principles and Applications (1980)

- Spin tensor:
rank-3 tensor $S^{\lambda,\mu\nu}$ such that total angular momentum

$$J^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\mu T^{\lambda\mu} + \hbar S^{\lambda,\mu\nu}$$

Diagonal spin basis

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- Distribution function f_{rs} is Hermitian matrix in spin space.
- Can be diagonalized by Unitary transformation:

$$f_{rs} = D_{rr'} \tilde{f}_{s'} \delta_{r's'} D^{\dagger}_{s's} .$$

- Redefine spinors

$$\tilde{u}_s \equiv \sum_{s'} u_{s'} D_{s's} .$$

- Define

$$s n^{\mu} \equiv \bar{\tilde{u}}_s \gamma^{\mu} \gamma^5 u_s .$$

- Only diagonal part contributes!

Transport equation

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- From Dirac equation: transport equation for Wigner function:
  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462

$$(\gamma_\mu K^\mu - m) W(X,p) = 0,$$

with

$$\begin{aligned} K^\mu &\equiv \Pi^\mu + \frac{1}{2} i\hbar \nabla^\mu, \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta) F^{\mu\nu} \partial_{p\nu}, \\ \Pi^\mu &\equiv p^\mu - \hbar \frac{1}{2} j_1(\Delta) F^{\mu\nu} \partial_{p\nu}, \end{aligned}$$

$\Delta = \frac{1}{2}\hbar \partial_p \cdot \partial_x$ with ∂_x only acting on $F^{\mu\nu}$ and $j_0(r) = \sin(r)/r$, $j_1(r) = [\sin(r) - r\cos(r)]/r^2$ spherical Bessel functions.

- Exact quantum kinetic equation for Wigner function for massive spin 1/2-particles and inhomogeneous fields!
- Only assumption: vanishing collision kernel, external classical gauge fields.

Massless limit: details

- $p \cdot u$ related to rest-frame energy $\sqrt{p^2} \to \delta$-function!
- Massless limit: replace massive by massless dipole-moment tensor $\Sigma^{\mu\nu} \to \Sigma_u^{\mu\nu}$.
- Attention: $\delta(p^2 - m^2)/m \to \delta(p^2)/(p \cdot u)$.
- Find general solution for constraint on $\bar{\Sigma}^{\mu\nu}$.
- Define right- and left-handed currents $J_\mu^\pm \equiv \frac{1}{2}(\mathcal{V}_\mu^{m=0} \pm \mathcal{A}_\mu^{m=0})$
- Result

$$J_\mu^\pm = \left[p_\mu \delta(p^2) \pm \frac{1}{2} \hbar \epsilon_{\mu\nu\alpha\beta} p^\nu F^{\alpha\beta} \delta'(p^2) \pm \frac{1}{2} \hbar \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha u^\beta}{p \cdot u} \delta(p^2) \nabla^\nu \right] f_\pm .$$

agrees with previously known massless solution!

Y. Hidaka, S. Pu, D-L. Yang, PRD 95 (2017) 091901; A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, PRD 98 (2018) 036010; J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, PRD 98 (2018) 036019

Distribution functions in global equilibrium

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- Equilibrium distribution function:

$$f_s^{eq} = \left(e^{g_s} + 1\right)^{-1},$$

with g linear combination of conserved quantities charge, momentum, and angular momentum:

$$g_s = \beta\pi \cdot U - \beta\mu_s + \frac{\hbar}{4} s\Sigma^{\mu\nu}\partial_\mu(\beta U_\nu).$$

Here, $\pi_\mu \equiv p_\mu + A_\mu$ is canonical momentum, U_μ is fluid velocity, $\beta \equiv \frac{1}{T}$ is inverse temperature, and μ_s is chemical potential.

- To zeroth order

$$f_s^{(0)} = \left(e^{g_{s0}} + 1\right)^{-1},$$

with

$$g_{s0} = \beta(\pi \cdot U - \mu_s).$$

Equilibrium conditions

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- "Homogeneous" part of the Boltzmann equation fulfilled if:

$$\begin{aligned} \mu_s &= \text{const}, \\ \partial_\nu \beta_\mu + \partial_\mu \beta_\nu &= 0, \end{aligned}$$

- "Inhomogeneous" part of Boltzmann equation:
  additional conditions to make global equilibrium possible, e.g.
  $\mu_{s=+} - \mu_{s=-} = 0$ or
  $\partial_{x\alpha} F_{\mu\nu} = 0$.