Spin tensor and its role in non-equilibrium thermodynamics (Hydrodynamics with spin)

Enrico Speranza

Based on a work done with **F. Becattini** and **W. Florkowski**

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GOETHE UNIVERSITÄT FRANKFURT AM MAIN

CRC–TR 211

59 Cracow School of Theoretical Physics

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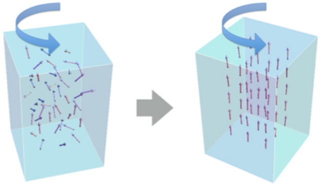
Outline

- Physical motivation
- What is the spin tensor?
- Why do we need the spin tensor in hydrodynamics?
- Is hydrodynamics invariant under different choices of energy-momentum and spin tensor?
- Can we find observables which are sensitive to different choices of energy-momentum and spin tensor?

Goal: Relativistic hydrodynamics (classical) + Spin (quantum)

Rotation and polarization

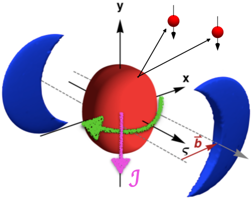
- Condensed matter: **Barnett effect**



Picture by Mamoru Matsuo

Ferromagnet gets magnetized when it rotates

- Heavy-ion collisions

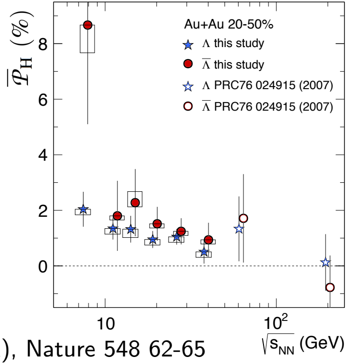


Picture by Radoslaw Ryblewski

Can something like the Barnett effect happen in heavy-ion collisions? Yes!

Experimental observation - Λ polarization

- Non-central nuclear collisions $\Rightarrow$ Large global angular momentum
  $\Rightarrow$ May generate spin polarization of hot and dense matter
- Connection between spin polarization and vorticity
- Measurement of Λ hyperon polarization along angular momentum direction



L. Adamczyk et al. (STAR), Nature 548 62-65

- Quark-gluon plasma is the "most vortical fluid ever observed"

$$\omega \approx (9+1) \times 10^{21} \mathrm{s}^{-1}$$

Great Red Spot of Jupiter $10^{-4}\,\mathrm{s}^{-1}$,
Turbulent flow superfluid He-II $150\,\mathrm{s}^{-1}$, Superfluid nanodroplets $10^{7}\,\mathrm{s}^{-1}$

Canonical energy-momentum and spin tensors

Lagrangian $\Rightarrow$ Poincaré symmetry $\Rightarrow$ Noether's th. $\Rightarrow$ Conservation laws

- **Conservation of energy and momentum:**
  Canonical energy-momentum tensor $\widehat{T}_C^{\mu\nu}(x)$

$$\partial_\mu \widehat{T}_C^{\mu\nu}(x) = 0$$

- **Conservation of total angular momentum:**
  Canonical total angular momentum tensor ("orbital"+"spin")

$$\widehat{J}_C^{\lambda,\mu\nu}(x) = x^\mu \widehat{T}_C^{\lambda\nu}(x) - x^\nu \widehat{T}_C^{\lambda\mu}(x) + \widehat{S}_C^{\lambda,\mu\nu}(x)$$

$$\partial_\lambda \widehat{J}_C^{\lambda,\mu\nu}(x) = 0 \Longrightarrow \partial_\lambda \widehat{S}_C^{\lambda,\mu\nu}(x) = \widehat{T}_C^{\nu\mu}(x) - \widehat{T}_C^{\mu\nu}(x)$$

Pseudo-gauge transformations

- Total energy-momentum and angular momentum must be fixed

$$\widehat{P}^{\mu} = \int d^3\Sigma_\lambda \, \widehat{T}^{\lambda\mu}(x) \qquad \widehat{J}^{\mu\nu} = \int d^3\Sigma_\lambda \, \widehat{J}^{\lambda,\,\mu\nu}(x)$$

- Densities are not uniquely defined
  $\Longrightarrow$ **Pseudo-gauge transformations:**
  (F. W. Hehl, Rep. Mat. Phys. 9, 55 (1976))

$$\widehat{T}'^{\,\mu\nu}(x) = \widehat{T}^{\mu\nu}(x) + \frac{1}{2}\partial_\lambda\left[\widehat{\Phi}^{\lambda,\,\mu\nu}(x) + \widehat{\Phi}^{\mu,\,\nu\lambda}(x) + \widehat{\Phi}^{\nu,\,\mu\lambda}(x)\right]$$

$$\widehat{S}'^{\,\lambda,\,\mu\nu} = \widehat{S}^{\lambda,\,\mu\nu}(x) - \widehat{\Phi}^{\lambda,\,\mu\nu}(x)$$

Leave $\widehat{P}^{\mu}$ and $\widehat{J}^{\mu\nu}$ invariant

- Belinfante's case $(\widehat{\Phi}^{\lambda,\,\mu\nu}(x) = \widehat{S}_C^{\lambda,\,\mu\nu}(x))$

$$\widehat{T}_B^{\mu\nu}(x) = \widehat{T}_C^{\mu\nu}(x) + \frac{1}{2}\partial_\lambda\left[\widehat{S}_C^{\lambda,\,\mu\nu}(x) + \widehat{S}_C^{\mu,\,\nu\lambda}(x) + \widehat{S}_C^{\nu,\,\mu\lambda}(x)\right]$$

$$\widehat{S}_B^{\lambda,\,\mu\nu}(x) = 0$$

Local equilibrium - Belinfante

- Maximization of entropy

$$S = -\mathrm{tr}(\widehat{\rho}_B \log \widehat{\rho}_B)$$

- Constraints on energy, momentum and charge

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \, \widehat{T}_B^{\mu\nu}(x)\right] = n_\mu T_B^{\mu\nu}(x), \qquad n_\mu \mathrm{tr}\left[\widehat{\rho}_B \, \widehat{j}^\mu(x)\right] = n_\mu j^\mu(x)$$

n^μ - vector orthogonal to hypersurface Σ

- Density operator

$$\widehat{\rho}_B = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_B^{\mu\nu}(x)\beta_\nu(x) - \zeta(x)\widehat{j}^\mu(x)\right)\right],$$

β_μ - Lagrange multiplier for conservation of energy and momentum
ζ - Lagrange multiplier for conservation of charge

Do we need a constraint for the conservation of total angular momentum?

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_B \, \widehat{J}_B^{\mu,\lambda\nu}(x)\right] = n_\mu \mathrm{tr}\left[\widehat{\rho}_B \left(x^\lambda \, \widehat{T}_B^{\mu\nu}(x) - x^\nu \, \widehat{T}_B^{\mu\lambda}(x)\right)\right] = n_\mu J_B^{\mu,\lambda\nu}(x)$$

No, it is redundant (it follows from constraint on energy and momentum)

Global equilibrium - Belinfante

- Density operator must be stationary

$$\partial_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) = \widehat{T}_B^{\mu\nu} (\partial_\mu \beta_\nu) - (\partial_\mu \zeta) \widehat{j}^\mu = 0$$

- Global equilibrium conditions

$$\zeta = \text{constant}$$
$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$
$$\beta_\nu = b_\nu + \Omega_{\nu\lambda} x^\lambda$$
$$\Omega_{\mu\nu} = \text{constant}$$

Local equilibrium - Canonical

- Maximization of entropy

$$S = -\mathrm{tr}(\widehat{\rho}_C \log \widehat{\rho}_C)$$

- Constraints on energy, momentum and charge

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C\, \widehat{T}_C^{\mu\nu}(x)\right] = n_\mu T_C^{\mu\nu}(x), \qquad n_\mu \mathrm{tr}\left[\widehat{\rho}_C\, \widehat{j}^{\mu}(x)\right] = n_\mu j^{\mu}(x)$$

Spin tensor $\Longrightarrow$ Constraint on total angular momentum

$$n_\mu \mathrm{tr}\left(\widehat{\rho}_C\, \widehat{J}_C^{\mu,\lambda\nu}\right) = n_\mu \mathrm{tr}\left[\widehat{\rho}_C \left(x^\lambda \widehat{T}_C^{\mu\nu} - x^\nu \widehat{T}_C^{\mu\lambda} + S_C^{\mu,\lambda\nu}\right)\right] = n_\mu J_C^{\mu,\lambda\nu}$$

- Density operator

$$\begin{aligned}\widehat{\rho}_C =& \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_C^{\mu\nu}(x) b_\nu(x) - \frac{1}{2}\widehat{J}_C^{\mu,\lambda\nu}(x)\Omega_{\lambda\nu}(x) - \zeta(x)\widehat{j}^{\mu}(x)\right)\right] \\ =& \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_C(x)^{\mu\nu}\beta_\nu(x) - \frac{1}{2}\widehat{S}_C^{\mu,\lambda\nu}(x)\Omega_{\lambda\nu}(x) - \zeta(x)\widehat{j}^{\mu}(x)\right)\right]\end{aligned}$$

$\Omega_{\lambda\nu}$ - (Antisymmetric) Lagrange multiplier for conservation of total angular momentum

Global equilibrium - Canonical

- Asymmetric EM tensor $\widehat{T}_C^{\mu\nu} = \widehat{T}_S^{\mu\nu} + \widehat{T}_A^{\mu\nu}$ with $\widehat{T}_S^{\mu\nu} = \widehat{T}_S^{\nu\mu}$, $\widehat{T}_A^{\mu\nu} = -\widehat{T}_A^{\nu\mu}$
- Density operator must be stationary

$$\frac{1}{2}\widehat{T}_S^{\mu\nu}(\partial_\mu\beta_\nu+\partial_\nu\beta_\mu)+\frac{1}{2}\widehat{T}_A^{\mu\nu}(\partial_\mu\beta_\nu-\partial_\nu\beta_\mu)-\frac{1}{2}(\partial_\mu\widehat{S}_C^{\mu,\lambda\nu})\Omega_{\lambda\nu}-\frac{1}{2}\widehat{S}_C^{\mu,\lambda\nu}(\partial_\mu\Omega_{\lambda\nu})=0$$

- Global equilibrium conditions:

$$\zeta = \text{constant}$$
$$\Omega_{\mu\nu} = \text{constant}$$
$$\partial_\mu\beta_\nu+\partial_\nu\beta_\mu = 0$$
$$\Omega_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) \quad \text{thermal vorticity}$$
$$\beta_\nu = b_\nu + \Omega_{\nu\lambda}x^\lambda$$

We used $\partial_\mu\widehat{S}_C^{\mu,\lambda\nu} = -2\widehat{T}_A^{\lambda\nu}$

Canonical vs Belinfante

- Consequences on some measurable quantities?
- Observable: single-particle polarization matrix

$$\Theta(p)_{\sigma,\sigma'} = \mathrm{tr}(\widehat{\rho}\, a^{\dagger}(p)_{\sigma} a(p)_{\sigma'})$$

Only $\widehat{\rho}$ can depend on pseudo-gauge!

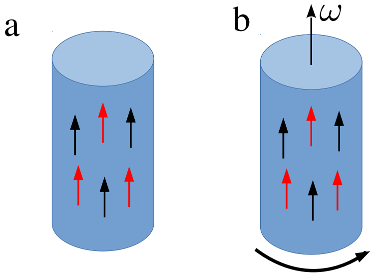
$$\Theta(p)_C = \Theta(p)_B?$$

$$\widehat{\rho}_B = \widehat{\rho}_C?$$

- $\widehat{\rho}_B = \widehat{\rho}_C$ only if:
  1. β_μ is the same in both cases
  2. $\Omega_{\lambda\nu}$ coincides with thermal vorticity, $\Omega_{\lambda\nu} = \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu)$
  3. $\xi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda + \partial_\lambda\beta_\nu) = 0$ or $\widehat{S}_C^{\lambda,\mu\nu} + \widehat{S}_C^{\nu,\mu\lambda} = 0$

Equivalence only in global equilibrium

Polarized neutral system - Physical meaning of $\Omega_{\mu\nu}$



- Polarization in global equilibrium $\Pi^\mu \sim \epsilon^{\mu\rho\sigma\tau}\partial_\rho\beta_\sigma p_\tau$
- a) Fluid at rest with constant temperature with particles and antiparticles polarized in the same direction $\beta^\mu = (1/T)(1,\mathbf{0}) \Rightarrow \partial_\mu\beta_\nu = 0$

Belinfante's "gauge" does not imply that polarization vanishes, but rather it is locked to thermal vorticity

In the Canonical "gauge" one uses the spin potential to describe hydro evolution, but only if spin density relaxes "slowly" to equilibrium

- In general in local equilibrium $\Omega_{\lambda\nu} = \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu)$

Hydrodynamics with spin tensor

- 11 equations of motion:
  5 usual hydro + 6 due to total angular momentum conservation

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu j^\mu \qquad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- 11 unknowns

$$\beta^\mu = \frac{u^\mu}{T} \qquad \zeta = \frac{\mu}{T} \qquad \Omega^{\mu\nu} \quad \text{(6 additional independent fields)}$$

$\Omega^{\mu\nu}$ and β^μ evolve separately

Summary

- Local equilibrium or non-equilibrium thermodynamics is sensitive to the choice of different sets of energy-momentum and spin tensors
- Particle polarization may be sensitive to different choices of energy-momentum and spin tensors
- Hydrodynamics with spin tensor is needed if spin density relaxes "slowly" to equilibrium
- Hydrodynamics with spin tensor: 6 additional fields $\Omega_{\mu\nu}$ (spin potential) to be evolved

BACKUP

Example - Dirac theory

- Dirac Lagrangian

$$\mathcal{L}(x) = \frac{i}{2}\bar{\widehat{\psi}}(x)\gamma^\mu \overleftrightarrow{\partial}_\mu \widehat{\psi}(x) - m\bar{\widehat{\psi}}(x)\widehat{\psi}(x)$$

- Canonical case

$$\widehat{T}_C^{\mu\nu}(x) = \frac{i}{2}\bar{\widehat{\psi}}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \widehat{\psi}(x) - g^{\mu\nu}\mathcal{L}(x)$$

$$\widehat{S}_C^{\lambda,\,\mu\nu}(x) = \frac{1}{4}\bar{\widehat{\psi}}(x)(\gamma^\lambda \sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\lambda)\widehat{\psi}(x)$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

- Belinfante case

$$\widehat{T}_B^{\mu\nu}(x) = \frac{i}{4}\bar{\widehat{\psi}}(x)(\gamma^\mu \overleftrightarrow{\partial}^\nu + \gamma^\nu \overleftrightarrow{\partial}^\mu)\widehat{\psi}(x) - g^{\mu\nu}\mathcal{L}(x)$$

$$\widehat{S}_B^{\lambda,\,\mu\nu}(x) = 0$$

Local equilibrium - Canonical vs Belinfante

- Start with Canonical

$$\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_C \beta_\nu - \frac{1}{2}\widehat{S}^{\mu,\lambda\nu}_C \Omega_{\lambda\nu} - \zeta \widehat{j}^\mu\right)\right]$$

- PS to Belinfante: $\widehat{T}^{\mu\nu}_B = \widehat{T}^{\mu\nu}_C + \frac{1}{2}\partial_\lambda(\widehat{S}^{\lambda,\,\mu\nu}_C + \widehat{S}^{\mu,\,\nu\lambda}_C + \widehat{S}^{\nu,\,\mu\lambda}_C)$, $\quad \widehat{S}^{\lambda,\,\mu\nu}_B = 0$
- Canonical density operator becomes

$$\widehat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}^{\mu\nu}_B \beta_\nu - \frac{1}{2}(\Omega_{\lambda\nu} - \varpi_{\lambda\nu})\widehat{S}^{\mu,\lambda\nu}_C + \frac{1}{2}\xi_{\lambda\nu}\left(\widehat{S}^{\lambda,\mu\nu}_C + \widehat{S}^{\nu,\mu\lambda}_C\right) - \zeta \widehat{j}^\mu\right)\right]$$

with $\quad \varpi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu), \qquad \xi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda + \partial_\lambda\beta_\nu)$

- When is $\rho_C = \rho_B$?

$$\widehat{\rho}_B = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma n_\mu \left(\widehat{T}^{\mu\nu}_B \beta_\nu - \zeta \widehat{j}^\mu\right)\right]$$

1. β_μ is the same in both cases
2. $\Omega_{\lambda\nu}$ coincides with thermal vorticity, $\Omega_{\lambda\nu} = \varpi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu)$
3. $\xi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda + \partial_\lambda\beta_\nu) = 0$ or $\widehat{S}^{\lambda,\mu\nu}_C + \widehat{S}^{\nu,\mu\lambda}_C = 0$

Equivalence in global equilibrium!