Gravitational waves.

Linearized vacuum Einstein equations are:

- $\partial^c \partial_c \bar{h}_{ab} = 0$ with the gauge condition

$$\partial^a \bar{h}_{ab} = 0 .$$

- residual gauge freedom.

Can still transform to a new gauge

$x^a \to x^a + \eta^a$ such that $\partial^a \partial_a \eta_b = 0$

Gauge functions η_b can be chosen to

make $\bar{h} = h = 0$ and $h_{0\mu} = 0$, $\mu = 1, 2, 3$

- with this $\bar{h}_{ab} = h_{ab}$ and $\partial^a \bar{h}_{ab} = 0 \Rightarrow$

$\partial^a h_{ab} = 0$. since $h_{0\mu} = 0$ for $\mu = 1, 2, 3$

this implies $\partial^0 h_{00} = 0$

- Thus, time-time field equations reduce to

$$\partial^c \partial_c h_{00} = \nabla^2 h_{00} = 0$$

the only well-behaved solutions are

if $h_{00} = 0$.

- So only space-space components of h_{ab} are nonzero and given by:

$\partial^c \partial_c h_{ab} = 0$ with the soln.

$h_{ab} = H_{ab} e^{ik_\mu x^\mu}$ where H_{ab} is

a constant tensor field.

Field equations are

$$\partial^c \partial_c h_{ab} = \partial^c \partial_c \left(H_{ab} e^{ik_\mu x^\mu} \right)$$

$$= H_{ab} \, \partial^c \left(i k_\mu \delta^\mu_c \, e^{ik_\mu x^\mu} \right)$$

$$= H_{ab}\, ik_c \; ik\, \delta^{\mu c} k_\mu$$

$$= -H_{ab}\, k_c k^c$$

$$= 0 \implies \quad k_c k^c = 0$$

or k^a is null vector.

Furthermore, $\quad \partial^a h_{ab} = 0$

$$\Rightarrow \quad \partial^a \left(H_{ab}\, e^{ik_\mu x^\mu} \right) = H_{ab}\, ik_\mu \delta^{\mu a}$$

$$= i k^a H_{ab} = 0$$

Since k^a defines the direction of propagation the above eqn $\Rightarrow$ waves are transverse. In particular, if $k^a = (k^0, 0, 0, k^z)$ we have $k^z H_{zb} = 0$ for all b.

- In Summary: $h = \eta^{ab} h_{ab} = 0$

$h_{0\mu} = 0$ and $h_{z\mu} = 0$ giving:

$$h_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{Transverse traceless gauge.}$$

- geodesic deviation:

$$\frac{d^2 \xi^\mu}{dt^2} = R_{\nu 0 0}{}^{\mu} \xi^\nu$$

$$= \frac{1}{2} \frac{\partial^2 h^\mu{}_\nu}{\partial t^2} \xi^\nu$$

In TT gauge only x and y components of ξ will be "deformed".

So $d^2 \xi^x / dt^2 = \dfrac{1}{2} \dfrac{\partial^2 h^x{}_\nu}{\partial t^2} \xi^\nu$

and $\partial^2 \xi^y / \partial t^2 = \frac{1}{2} \frac{\partial^2 h^y{}_\nu}{\partial t^2} \xi^\nu$

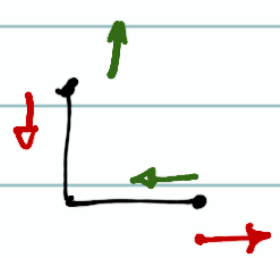
or $$\ddot{\xi}^x = \frac{1}{2} \left(\ddot{h}_{xx} \xi^x + \ddot{h}_{xy} \xi^y \right)$$

$$\ddot{\xi}^y = \frac{1}{2} \left(\ddot{h}_{yx} \xi^x + \ddot{h}_{yy} \xi^y \right).$$

Let $h_+ \neq 0$, $h_\times = 0$. then

$$\ddot{\xi}^x = \frac{1}{2} \ddot{h}_{xx} \xi^x, \quad \ddot{\xi}^y = -\frac{1}{2} \ddot{h}_{xx} \xi^y$$

- The force field is opposite in the two different directions.



- The force is prop. to separation

- This is typical of tidal forces

So GW cause a tidal defor

motion of free test masses.

- This could be used to detect gravitational waves.

Generation of gravitational waves

$$\partial^c \partial_c \bar{h}_{ab} = 8\pi T_{ab}.$$

Young's modulus of spacetime.

$$Y = \frac{\text{Stress}}{\text{Strain}} = \frac{|T_{ab}|}{|h|}$$

with c and G

$$\partial^c \partial_c \bar{h}_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

$\partial^c \partial_c \bar{h}_{ab} \simeq (\omega^2/c^2)\bar{h}_{ab}$ for a wave of frequency ω

So $$\bar{h}_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

hence $$\frac{|T_{ab}|}{|\bar{h}_{ab}|} = \frac{c^2 4\pi^2 f^2}{8\pi G} = \frac{\pi f^2 c^2}{G}$$

$\frac{3\times10^6 \times 9\times10^{16}}{6.67\times10^{-11}}$

if $f = 10^3$ Hz (LIGO band)

$Y_{st} \cong 10^{24}$ GPa $\qquad Y_{steel} \sim 200$ GPa

- what sort of objects can produce large strain amplitudes

- Green's function:

$$\bar{h}_{\mu\nu}(x) = 4 \int_{\Lambda} \frac{T_{\mu\nu}(x')}{|\vec{x} - \vec{x}'|} r^2 dr d\Omega$$

Since $\partial^a T_{ab} = 0$ we are assured of the gauge condn to hold good even in the presence of sources. However, we will not impose the radiation gauge conditions in the presence of sources.

- Assume that the source is localized and slow motion $v \ll 1$. (analogous to dipole approximation in E&M).

Let $\tilde{\bar{h}}(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{h}_{\mu\nu}(t, \vec{x}) e^{i\omega t} dt$

and the same with T_{ab}.

$$\tilde{\bar{h}}_{\mu\nu}(\omega, \vec{x}) = 4 \int \frac{\tilde{T}_{\mu\nu}(\omega, \vec{x}')}{|\vec{x} - \vec{x}'|} e^{i\omega(\vec{x}-\vec{x}')} d^3x'$$

extra factor of $e^{i\omega(\vec{x}-\vec{x}')}$ arises as the integral is over the past light cone.

- Only space-space components needed since $\frac{\partial \bar{h}_{0\mu}}{\partial x^\mu} = 0 \Rightarrow$

$$i\omega \tilde{\bar{h}}_{0\mu} = \sum_{\nu=1}^{3} \frac{\partial \tilde{\bar{h}}_{\nu\mu}}{\partial x^\nu} .$$

- In the far zone $R \gg 1/\omega$

so $e^{i\omega(\vec{x}-\vec{x}')}$ varies negligibly

so $e^{i\omega(\vec{x}-\vec{x}')}/|\vec{x}-\vec{x}'| \longrightarrow \frac{e^{i\omega R}}{R}$

- Thus inside the integral we have

$$\int \tilde{T}^{\mu\nu} d^3x = \sum_{\alpha=1}^{3} \left\{ \int \frac{\partial}{\partial x^\alpha} (\tilde{T}^{\alpha\nu} x^\mu) - \int \frac{\partial \tilde{T}^{\alpha\nu}}{\partial x^\alpha} x^\mu \right\} d^3x$$

$$= -i\omega \int \tilde{T}^{0\nu} x^\mu \, d^3x$$

$$= \frac{i\omega}{2} \int (\tilde{T}^{0\nu} x^\mu + \tilde{T}^{0\mu} x^\nu) d^3x$$

...

$$= -\frac{\omega^2}{2} \int \tilde{T}^{00} x^\mu x^\nu d^3x$$

Thus

$$\tilde{h}_{\mu\nu}(\omega, \vec{r}) = -\frac{2\omega^2}{3} \frac{e^{i\omega R}}{R} \tilde{q}_{\mu\nu}(\omega)$$

where $\tilde{q}$ is F.T. of q:

$$q_{\mu\nu} = 3 \int T^{00} x^{\mu} x^{\nu} d^3x$$

called the quadrupole moment tensor. Thus

$$\boxed{\bar{h}_{\mu\nu}(t, \vec{x}) = \frac{2}{3R} \frac{d^2 q_{\mu\nu}}{dt^2}\Big|_{ret}.}$$

der. evaluated at retarded time $t' = t - R$.

- Conservation of momentum $\Rightarrow$ no dipole radiation.
- So emission of radiation is less efficient.

Full non linear

* Gravitational radiation from gravitational collapse.

Consider an axisymmetric, non-spherical star. Its principle axes are of size a, b and c. Then first compute the quadrupole tensor assuming the density is const.

$$Q_{ij} = \int \rho\, x^i x^j \, d^3x .$$

The equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

This sugges reparemetrization

$$x = a\,r \sin\theta \cos\phi , \quad y = b\, r \sin\theta \sin\phi , \quad z = c\,r\cos\theta$$

This r is not the radial coordinate but a dimensionless variable

The Jacobian of the transformation is:

$$J = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \right|$$

$$= \text{Det} \begin{bmatrix} a \sin\theta \cos\phi & ar \cos\theta \cos\phi & -ar \sin\theta \sin\phi \\ b \sin\theta \sin\phi & br \cos\theta \sin\phi & br \sin\theta \cos\phi \\ c \cos\theta & -cr \sin\theta & 0 \end{bmatrix}$$

$$= c \cos\theta (abr^2 \sin\theta \cos\theta \cos^2\phi + abr^2 \sin\theta \cos\theta \sin^2\phi)$$

$$+ cr \sin\theta (abr \sin^2\theta \cos^2\phi + abr \sin^2\theta \sin^2\phi)$$

$$= abc \cos^2\theta \sin\theta \, r^2 + abc r^2 \sin^3\theta$$

$$= abc r^2 \sin\theta \; ; \quad r \in [0, 1], \; \theta \in [0, \pi], \; \phi \in [0, 2\pi]$$

$$\therefore \; Q_{ij} = \rho \int x^i(r, \theta, \phi) \, x^j(r, \theta, \phi) \, abc r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$\text{So } Q_{xx} = \rho \int a^2 r^2 \sin^2\theta \cos^2\phi \, abc r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho \frac{a^3 bc}{5} \int \sin^3\theta \cos^2\phi \, d\theta \, d\phi = \frac{4\pi}{15} \rho a^3 bc$$

$$\rho = 3M / 4\pi abc \quad \text{so} \quad \frac{4\pi}{3} abc \rho = M.$$

$$\text{Thus } Q_{xx} = \frac{M a^2}{5}$$

Similarly, $Q_{yy} = \frac{Mb^2}{5}$ and $Q_{zz} = \frac{Mc^2}{5}$

Due to the fact sin and cos are, respectively, odd and even functions, it turns out that $Q_{xy} = Q_{xz} = Q_{yz} = 0$. Thus,

$$Q = \frac{M}{5}\begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}.$$

The Moment of Inertia tensor defined by $I_{ij} = \int \rho (r^2 \delta_{ij} - x_i x_j) d^3x$

This r is the radial coordinate.

is given by

$$I_{ij} = \int r^2 \delta_{ij} d^3x - Q_{ij}$$

So off diagonals are the same

as Q_{ij} (except for a sign) and

diagonal ones differ just by the

trace for $r^2 = (x^2 + y^2 + z^2)$, so

$$\rho \int r^2 \, d^3x = \rho \int (x^2 + y^2 + z^2) \, d^3x$$

$$= Q_{xx} + Q_{yy} + Q_{zz} .$$

$$\therefore \quad I = -\frac{M}{5} \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

(This is not trace free).

Let $a = b = R$ so $I_3 . I_{zz}$ is

$$I_3 = -\frac{2MR^2}{5}$$

Let $I_1 = I_2 = I_3 (1 - e^2/2)$

so $-(R^2 + c^2)\frac{M}{5} = -\frac{2MR^2}{5}\left(1 - \frac{e^2}{2}\right)$

$$\frac{R^2+c^2}{R^2} = 2-e^2 \Rightarrow \frac{c^2}{R^2} = 1-e^2$$

or $e^2 = \left(1-\frac{c^2}{R^2}\right)$, $e = \sqrt{1-\frac{c^2}{a^2}}$

where we wrote $R=a$. ($c>a$ or oblate spheroid).

$$Q_{ij} = \frac{M}{5}\begin{bmatrix} a^2 & & 0 \\ & a^2 & \\ 0 & & c^2 \end{bmatrix}$$

$$Q = \text{Det}\, Q_{ij} = \left(2a^2+c^2\right) M/5$$

$$a^2 - \frac{1}{3}\left(2a^2+c^2\right) = a^2 - \frac{2a^2}{3} - \frac{c^2}{3}$$

$$= \frac{1}{3}a^2\left(1-\frac{c^2}{a^2}\right) = \frac{e^2a^2}{3}$$

and $c^2 - \frac{2a^2}{3} - \frac{c^2}{3} = -\frac{2a^2}{3}\left(1 - \frac{c^2}{a^2}\right)$

$$= -\frac{2e^2a^2}{3}$$

Thus $\tilde{Q}_{ij} = -\frac{e^2a^2}{3}\frac{M}{5}\begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 2 \end{bmatrix}$

Rotate by an angle i about y axis: ~~then~~

$$\tilde{Q}'_{ij} = R_{ik} R_{jm} \tilde{Q}_{km}$$

$$= R_{ik} \tilde{Q}_{km} R^T_{mj} = R \tilde{Q} R^T$$

$$= \begin{pmatrix} \cos i & 0 & \sin i \\ 0 & 1 & 0 \\ -\sin i & 0 & \cos i \end{pmatrix}\begin{pmatrix} \tilde{Q}_{xx} & 0 & 0 \\ 0 & \tilde{Q}_{yy} & 0 \\ 0 & 0 & \tilde{Q}_{zz} \end{pmatrix}\begin{pmatrix} ci & 0 & -si \\ 0 & 1 & 0 \\ si & 0 & ci \end{pmatrix}$$

$$= \begin{pmatrix} ci & 0 & si \\ 0 & 1 & 0 \\ -si & 0 & ci \end{pmatrix}\begin{pmatrix} ci\,\tilde{Q}_{xx} & 0 & -si\,\tilde{Q}_{xx} \\ 0 & \tilde{Q}_{yy} & 0 \\ si\,\tilde{Q}_{zz} & 0 & ci\,\tilde{Q}_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 i\, \tilde{Q}_{xx} + \sin^2 i\, \tilde{Q}_{zz} & 0 & -\cos i \sin i (\tilde{Q}_{xx} - \tilde{Q}_{zz}) \\ 0 & \tilde{Q}_{yy} & 0 \\ -\cos i \sin i (\tilde{Q}_{xx} - \tilde{Q}_{zz}) & 0 & \sin^2 i\, Q_{xx} + \cos^2 i\, Q_{zz} \end{pmatrix}$$

$$= \frac{e^2 M a^2}{15} \begin{pmatrix} 3\cos^2 i - 2 & 0 & -3\sin 2i/2 \\ 0 & 1 & 0 \\ -\frac{3\sin 2i}{2} & 0 & -3\cos^2 i + 1 \end{pmatrix}$$

$$3\cos^2 i - 2 = \frac{3}{2}(\cos 2i + 1) - 2 = \frac{3}{2}\cos 2i - \frac{1}{2}$$

$$= \frac{1}{2}(3\cos 2i - 1)$$

$$-3\cos^2 i + 1 = -\frac{3}{2}(\cos 2i + 1) + 1$$

$$= -\frac{3}{2}\cos 2i - \frac{1}{2} = \frac{1}{2}(-3\cos 2i - 1)$$

$$\therefore \tilde{Q}' = \frac{M e^2 a^2}{30} \begin{pmatrix} 3\cos 2i - 1 & 0 & -3\sin 2i \\ 0 & 2 & 0 \\ -3\sin 2i & 0 & -3\cos 2i - 1 \end{pmatrix}$$

Finally,

$$\tilde{Q}^{TT}_{ij} = \left(P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}\right) \tilde{Q}'_{kl}$$

$$P_{ik} \tilde{Q}'_{kl} P_{jl} = P Q' P^T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q'_{xx} & 0 & Q'_{xz} \\ 0 & Q'_{yy} & 0 \\ Q'_{zx} & 0 & Q'_{zz} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q'_{xx} & 0 & 0 \\ 0 & Q'_{yy} & 0 \\ Q'_{zx} & 0 & 0 \end{pmatrix} = \begin{pmatrix} Q'_{xx} & 0 & 0 \\ 0 & Q'_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_{kl} \tilde{Q}'_{kl} = Q'_{xx} + Q'_{yy},$$

$$\therefore \tilde{Q}^{TT} = \begin{pmatrix} \frac{Q'_{xx} - Q'_{yy}}{2} & 0 & 0 \\ 0 & -\frac{Q'_{xx} + Q'_{yy}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Q'_{xx} - Q'_{yy} = \frac{e^2 M a^2}{30}\left(3\cos 2i - 1 - 2 \right)$$

$$= \frac{M e^2 a^2}{30}\left(\cos 2i - 1 \right) 3$$

$$= \frac{M e^2 a^2}{10}\left(1 - 2\sin^2 i - 1 \right)$$

$$= -\frac{M e^2 a^2}{5} \sin^2 i$$

so $$Q^{TT} = \frac{-M e^2 a^2 \sin^2 i}{10}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$