A new perspective on metric gravitational perturbations

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Problem

To solve vacuum Einstein equations

$$R_{\mu\nu}[g] - \kappa \frac{d}{\ell^2} g_{\mu\nu} = 0, \quad \kappa = 0, +1, -1, \quad \Lambda = \kappa \frac{d(d-1)}{2\ell^2},$$

in the form of a perturbation of some known exact solution $\bar{g}_{\mu\nu}$, i.e.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

Once we expand

$$\delta g_{\mu\nu} = \sum_{1 \le i} {}^{(i)}h_{\mu\nu}\, \varepsilon^i$$

we describe linear ($i = 1$) and nonlinear ($i > 1$) gravitational waves

Key assumption: the background metric $\bar{g}_{\mu\nu}$ can be put in the form $d\bar{s}^2 = \bar{g}_{ab}(y) dy^a dy^b + F^2(y) d\bar{\sigma}^2$, where $d\bar{\sigma}^2$ is a metric of n-dimensional maximally symmetric space (spherical/euclidian/hyperbolic) and $n = d - 1$ (does not apply to Kerr); harmonic functions in this maximally symmetric space define modes (and polarizations) of **linear** gravitational waves.

Problems that can be addressed with perturbation framework

- in linear approximation: linear stability of solutions, ringdown phase of black holes merging (**QNMs**), imprint of cosmological perturbations in the CMB fluctuations
- beyond linear approximation:
  establishing full stability of solutions: relaxation
  Schwarzschild $\to$ Schwarzschild with shifted mass parameter or
  Schwarzschild $\to$ Kerr,
  my personal motivation: AdS instability vs existence of AdS geons.

Metric perturbations in vacuum - general setup

Setting

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

and expanding

$$\delta g_{\mu\nu} = \sum_i \varepsilon^i h^{(i)}_{\mu\nu}$$

we get a hierarchy of **linear** PDEs

$$\Delta_L h^{(i)}_{\mu\nu} = S^{(i)}_{\mu\nu} \quad \Longleftrightarrow \quad \Delta_L h^{(i)}_{\ell\,\mu\nu} = S^{(i)}_{\ell\,\mu\nu}$$

$$\Delta_L h_{\mu\nu} = \frac{1}{2}\left(-\bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h - 2\bar{R}_{\mu\alpha\nu\beta} h^{\alpha\beta} + \bar{\nabla}_\mu \bar{\nabla}^\alpha h_{\nu\alpha} + \bar{\nabla}_\nu \bar{\nabla}^\alpha h_{\mu\alpha}\right), \qquad h = \bar{g}^{\alpha\beta} h_{\alpha\beta},\ h^{\alpha\beta} = \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} h_{\mu\nu}$$

Thus, we trade **nonlinearities** of Einstein equations for an **infinite system of linear inhomogeneous equations** (the sources $S^{(i)}_{\ell\,\mu\nu}$ are constructed from metric perturbations $h^{(j)}_{\ell'\,\mu\nu}$, with $j < i$). To solve it one needs:

1. a general solution of the principal (homogeneous) part
   - inherited from the linear approximation
2. a particular solution of an inhomogeneous part

Solution in three steps:

1. Separate the "the angular dependence" and organise metric perturbations into two/three different sectors.
   Done with scalar-vector-tensor ($\mathscr{S}-\mathscr{V}-\mathscr{T}$) decomposition (there is no tensor sector for $n=2$ ($D=4$))
2. Then, **for each multipole**, rewrite perturbations into gauge-invariant combinations (or equivalently fix the gauge uniquely with Regge-Wheeler or Detweiler (easy) gauge)
   This is necessary because of the gauge freedom:

$$x^{\mu} \to x^{\mu} - \varepsilon\zeta^{\mu}, \qquad \delta g_{\mu\nu} \to \delta g_{\mu\nu} + \varepsilon\mathscr{L}_{\zeta}\bar{g}_{\mu\nu} + \mathscr{O}\left(\varepsilon^{2}\right)$$

3. Solve the resulting system of PDEs for gauge invariants (or RW/D gauge invariant characteristics of perturbations)
   (after separating "the angular dependence"/expanding into multipoles, this is $1+1$ dimensional problem)

scalar-vector-tensor (S-V-T) decomposition in $n-$dimensional maximally-symmetric space

$$d\bar{s}^2 = \bar{g}_{ab}(y)dy^a dy^b + F^2(y)d\bar{\sigma}^2$$

- scalar S
- vector

$$V_i = \underbrace{\partial_i V}_{scalar} + \underbrace{\hat{V}_i}_{vector}, \text{ where } \nabla^i \hat{V}_i = 0$$

- rank-2 symmetric tensor

$$T_{ij} = \underbrace{Q\bar{g}_{ij} + 2\left(\bar{\nabla}_i\bar{\nabla}_j - \frac{1}{n}\bar{g}_{ij}\bar{\nabla}^2\right)T}_{scalar} + \underbrace{2\bar{\nabla}_{(i}\hat{T}_{j)}}_{vector} + \underbrace{\hat{T}_{ij}}_{tensor},$$

where

$$\bar{\nabla}^i\hat{T}_i = 0,$$

$$\bar{\nabla}^i\hat{T}_{ij} = 0, \quad \bar{g}^{ij}\hat{T}_{ij} = 0$$

The key result of BH linear perturbations:

Linear metric perturbations (of maximally symmetric BHs) were studied in full generality (i.e. allowing for any value of cosmological constant and different horizon geomeries) by [Kodama&Ishibashi, 03]

Outcome: perturbations split into two (in $D = 4$) or three (in higher dimensions) decoupled sectors; in each sector gauge invariant characteristics of perturbations are given in terms of **master scalar variables** (one type of scalar for each sector) satisfying homogeneous wave equations on the zero order solution, with suitably chosen potential. (The master scalars appear in a few copies (polarizations) in vector/tensor sectors for $D > 4$ and in a single copy (polarization) in a scalar sector).

The same structure emerges for Einstein equations coupled to matter in the form of some fundamental fields (Maxwell, scalar field, etc.) but derivations become more and more involved. These results are conventionally obtained by a kind of *massage* of linearized Einstein equations (no guiding principle known a priory).

I will discuss a new perspective on perturbation expansion which provides a guiding principle (algorithm) for such derivations, can be easily extended beyond linear level and works also for an effective matter model (perfect fluid) used in cosmological models (thus time dependent backgrounds).

PHYSICAL REVIEW VOLUME 108, NUMBER 4 NOVEMBER 15, 1957

Stability of a Schwarzschild Singularity

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AND

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(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

VOLUME 24, NUMBER 13 PHYSICAL REVIEW LETTERS 30 MARCH 1970

EFFECTIVE POTENTIAL FOR EVEN-PARITY REGGE-WHEELER GRAVITATIONAL PERTURBATION EQUATIONS*

Frank J. Zerilli

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(Received 29 January 1970)

The Schrödinger-type equation for odd-parity perturbations on a background geometry has been extended to the even-parity perturbations. This should greatly simplify the analysis for calculations of gravitational radiation from stars and from objects falling into black holes.

Progress of Theoretical Physics, Vol. 110, No. 4, October 2003

A Master Equation for Gravitational Perturbations of Maximally Symmetric Black Holes in Higher Dimensions

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We show that in four or more spacetime dimensions, the Einstein equations for gravitational perturbations of maximally symmetric vacuum black holes can be reduced to a single second-order wave equation in a two-dimensional static spacetime, irrespective of the mode of perturbations. Our starting point is the gauge-invariant formalism for perturbations in an arbitrary number of dimensions developed by the present authors, and the variable for the final second-order master equation is given by a simple combination of gauge-invariant variables in this formalism. Our formulation applies to the case of non-vanishing as well as vanishing cosmological constant Λ. The sign of the sectional curvature K of each spatial section of equipotential surfaces is also kept general. In the four-dimensional Schwarzschild background with $\Lambda = 0$ and $K = 1$, the master equation for a scalar perturbation is identical to the Zerilli equation for the polar mode and the master equation for a vector perturbation is identical to the Regge-Wheeler equation for the axial mode. Furthermore, in the four-dimensional Schwarzschild-anti-de Sitter background with $\Lambda < 0$ and $K = 0, 1$, our equation coincides with those recently derived by Cardoso and Lemos. As a simple application, we prove the perturbative stability and uniqueness of four-dimensional non-extremal spherically symmetric black holes for any Λ. We also point out that there exists no simple relation between scalar-type and vector-type perturbations in higher dimensions, unlike in four dimension. Although in the present paper we treat only the case in which the horizon geometry is maximally symmetric, the final master equations are valid even when the horizon geometry is described by a generic Einstein manifold, if we employ an appropriate reinterpretation of the curvature K and the eigenvalues for harmonic tensors.

Homogeneous part - inherited from the linear approximation

Guiding principle to solve the homogeneous part:

to express gauge invariant characteristics of perturbations for each mode as **linear combinations of master scalar and its derivatives** with **the master scalar satisfying the homogeneous wave equation with a potential** (to be determined). In present days, the (function) coefficients of such linear combinations and the form of the potential in the wave equation can be easily found by substitution of such **ansatz** into (the homogeneous part of) perturbative Einstein equations and use of computer algebra packages.

Solving a system of linear algebraic equations we express the potential and all but one coefficient in those linear combinations in terms of a single coefficient for which we have to solve a simple differential equation - its solution introduces a multiplicative constant, as should be expected.

Particular solutions at higher orders

It seems that the particular solutions can be found in the form of linear combinations of the sources $S^{(i)}_{\ell\,\mu\nu}$ and their derivatives once **the master scalar variables** satisfy **inhomogeneous** wave equation with a suitably defined **scalar source** $\tilde{S}^{(i)}_{\ell}$. Again, (function) coefficients of this linear combinations can be easily found by plugging the **ansatz** into perturbative Einstein equations and using a computer algebra package.

Remark: to obtain the scalar source for the inhomogeneous wave equation master scalars must be given in terms of gauge invariant characteristics of perturbations (also needed to relate initial data for scalar wave equations with the solution of the initial value problem for Einstein equations of physical interest, cf, [Moncrief, 74]).

Key issues at higher orders

- Identities for the sources $S^{(i)}_{\ell\,\mu\nu}$ coming from $\bar\nabla^{\mu}\left(\Delta_L h^{(i)}_{\ell\,\mu\nu} - S^{(i)}_{\ell\,\mu\nu}\right) = 0$.
  They are **crucial** for the consistency at higher orders.
- Gauge issues with $\delta g_{\mu\nu} = \sum_i \varepsilon^i h^{(i)}_{\mu\nu}$:
  once we perform a gauge transformation

$$x^{\mu} \to x^{\mu} - \varepsilon\zeta^{\mu}, \qquad \delta g_{\mu\nu} \to \delta g_{\mu\nu} + \varepsilon \mathscr{L}_{\zeta}\bar g_{\mu\nu} + \mathscr{O}\left(\varepsilon^2\right)$$

  At higher orders, gauge issues can become a nuisance [Bruni et al., 97] and there is no need to mess them up (cf. [Garat&Price, 00], [Brizuela et al., 09]). Thus, we do not use fully gauge invariant approach at higher orders of perturbation expansion!
  In fact, while $\Delta_L h^{(i)}_{\ell\,\mu\nu}$ given in terms of $(D(D+1)/2 - D)$ Regge-Wheeler gauge invariants only, $S^{(i)}_{\ell\,\mu\nu}$ depend on the gauge choices made at lower orders $j < i$.
  However, since we solve perturbations iteratively, gauge invariance in the Regge-Wheeler sense seems sufficient! (Moreover RW gauge is neither asymptotically flat (in $\Lambda = 0$ case) nor asymptotically AdS (in $\Lambda < 0$ case) thus suitable gauge transformation are needed.)

Perturbations in vacuum - general strategy - summary

$$\Delta_L h^{(i)}_{\ell\,\mu\nu} = S^{(i)}_{\ell\,\mu\nu} \qquad \longrightarrow \tilde{\Box}\Phi^{(i)}_{\ell} = \tilde{S}^{(i)}_{\ell}\,, \quad \Phi^{(i)}_{\ell} \longleftrightarrow h^{(i)}_{\ell\,\mu\nu}$$

1. A general strategy to solve the homogeneous part:
   Linearised gravity is about gravitational waves. Thus gauge invariant characteristics of metric perturbations $h^{(i)}_{\ell\,\mu\nu}$ should be given in terms of **master scalar variables** satisfying the homogeneous scalar wave equation (with some potential) on the zero order solution $\bar{g}_{\mu\nu}$; we make an **ansatz** that the gauge invariant characteristics of $h^{(i)}_{\ell\,\mu\nu}$ **are given in terms of linear combinations of master scalar variables and their derivatives**
2. A general strategy to find a particular solution of an inhomogeneous part:
   It follows from a few case studies that particular solutions of $\Delta_L h^{(i)}_{\ell\,\mu\nu} = S^{(i)}_{\ell\,\mu\nu}$ are given in terms of linear combinations of the sources $S^{(i)}_{\ell\,\mu\nu}$ and their derivatives. Thus, we take it as the general **ansatz** for particular solutions. We will also have to introduce source terms $\tilde{S}^{(i)}_{\ell}$ into scalar wave equations for master scalar variables

Partial summary

Thus, in $D = 4$, at each order of perturbation expansion, the problem of solving the system of **10** (linear but highly coupled) PDEs of mixed (**hyperbolic and elliptic**) type is reduced to solving **only 2** scalar wave (**hyperbolic**) equations and **some linear algebra** (i.e. the only integration to be done is at the level of scalar wave equation!)
For scalar wave equations one can set the initial data **freely**!

In $4 < D$, there exist three (instead of two) **master scalar variables** (coming in some number of copies/polarizations).

There exists robust and conceptually simple guiding principle to deal with perturbation expansion.

Two worked examples:

1. Schwarzschild BH perturbations
   (up to arbitrarily high order of perturbation expansion)
   Phys. Rev. D96, 124026
2. cosmological perturbations in $2+2$ splitting
   (at linear level)
   [arXiv:1902.05090]

Example1: Schwarzschild BH perturbations

Phys. Rev. D96, 124026

- We will limit ourselves to axial symmetry (stepping beyond axial symmetry is a technical, not a conceptual, issue). Then we can limit ourselves to scalar (scalar / polar / parity-even) perturbations only (vector/axial/parity-odd perturbations can be treated analogously)
- We use multipole expansion. At nonlinear orders of perturbation expansion the $\ell = 0, 1$ parts need special treatment

Polar perturbations at axial symmetry (on concrete example)

Schwarzschild in static coordinates:

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega_2^2, \quad A = 1 - \kappa r^2/\ell^2 - 2M/r$$

$$h^{(i)}_{\alpha\beta} = \begin{pmatrix} h^{(i)}_{tt} & h^{(i)}_{tr} & h^{(i)}_{t\theta} & 0 \\ h^{(i)}_{tr} & h^{(i)}_{rr} & h^{(i)}_{r\theta} & 0 \\ h^{(i)}_{t\theta} & h^{(i)}_{r\theta} & h^{(i)}_{\theta\theta} & 0 \\ 0 & 0 & 0 & h^{(i)}_{\phi\phi} \end{pmatrix},$$

$$h^{(i)}_{rr}(t,r,\theta) = \sum_\ell h^{(i)}_{\ell\, rr}(t,r) P_\ell(\cos\theta)$$

$$h^{(i)}_{r\theta}(t,r,\theta) = \sum_\ell h^{(i)}_{\ell\, r\theta}(t,r) \partial_\theta P_\ell(\cos\theta)$$

Similarly, the sources $S^{(i)}_{\mu\nu}$ and perturbative Einstein equations expanded into multipoles

RW gauge: only $h^{(i)}_{\ell\, tt}, h^{(i)}_{\ell\, rr}, h^{(i)}_{\ell\, tr}, h^{(i)}_{\ell\, +} = \left(h^{(i)}_{\ell\, \theta\theta} + h^{(i)}_{\ell\, \phi\phi} / \sin^2\theta\right)/2$ non zero,

or out of **seven** polar metric components **four** RW gauge invariant functions $f^{(i)}_{\ell\, tt}, f^{(i)}_{\ell\, rr}, f^{(i)}_{\ell\, tr}, f^{(i)}_{\ell\, +}$ can be constructed

$$\left(f^{(i)}_{\ell\, tt} = h^{(i)}_{\ell\, tt} - 2\partial_t h^{(i)}_{\ell\, t\theta} + AA' h^{(i)}_{\ell\, r\theta} + \tfrac{2}{r} AA' h^{(i)}_{\ell\, -} - AA' \partial_r h^{(i)}_{\ell\, -} + 2\partial_{tt} h^{(i)}_{\ell\, -}, \ \ldots\right)$$

$f^{(i)}_{\ell\,tt}(t,r), f^{(i)}_{\ell\,rr}(t,r), f^{(i)}_{\ell\,tr}(t,r)$ and $f^{(i)}_{\ell\,+}(t,r)$ are Regge-Wheeler (gauge invariant) variables

$\zeta^{(j)}_{\ell\,t}(t,r)$, $\zeta^{(j)}_{\ell\,r}(t,r)$, $\zeta^{(j)}_{\ell\,\theta}(t,r)$ define the j-th order polar gauge vector

$\zeta^{(j)}_{\alpha} = \sum_\ell \left(\zeta^{(j)}_{\ell\,t} P_\ell(\cos\theta),\, \zeta^{(j)}_{\ell\,r} P_\ell(\cos\theta),\, \zeta^{(j)}_{\ell\,\theta}\partial_\theta P_\ell(\cos\theta),\, 0\right)$

and the corresponding gauge transformation $x^\mu \longrightarrow x^\mu - \varepsilon^j \zeta^{(j)\mu}$

$$\sum_{1\leq i} \varepsilon^i h^{(i)}_{\mu\nu} \to \sum_{1\leq i} \varepsilon^i h^{(i)}_{\mu\nu} + \varepsilon^j \mathscr{L}_{\zeta^{(j)}} \bar{g}_{\mu\nu} + \mathscr{O}\left(\varepsilon^{j+1}\right).$$

At each order:

- in axial sector: **three** equations for **two** RW gauge invariant variables
- in polar sector: **seven** equations for **four** RW gauge invariant variables

$$\Delta_L h^{(i)}_{\ell\,\mu\nu}\left[f^{(i)}_{\ell\,tt}, f^{(i)}_{\ell\,rr}, f^{(i)}_{\ell\,tr}, f^{(i)}_{\ell\,+}\right] = S^{(i)}_{\ell\,\mu\nu}$$

$$\Delta_L{}^{(i)}h_{\ell\, tt} = \left[\frac{(2A+rA')^2-2(rA')^2+2(\ell-1)(\ell+2)A}{4r^2A} + \left(\frac{A'}{4}-\frac{A}{r}\right)\partial_r - \frac{A}{2}\partial_{rr}\right]{}^{(i)}f_{\ell\, tt} + \left[\frac{AA'}{2}A\partial_r - \partial_{tt}\right]{}^{(i)}f_{\ell\, +}$$

$$-A\left[\frac{(2A+rA')^2-4A}{4r^2} + \frac{AA'}{4}\partial_r + \frac{1}{2}\partial_{tt}\right]{}^{(i)}f_{\ell\, rr} + \left[\left(\frac{A'}{2}+\frac{2A}{r}\right)\partial_t + A\partial_{tr}\right]{}^{(i)}f_{\ell\, tr}\,,$$

$$\Delta_L{}^{(i)}h_{\ell\, rr} = \left[\frac{4A(1-A)+(rA')^2}{4r^2A^3} - \frac{A'}{4A^2}\partial_r + \frac{1}{2A}\partial_{rr}\right]{}^{(i)}f_{\ell\, tt} - \left[\left(\frac{A'}{2A}+\frac{2}{r}\right)\partial_r + \partial_{rr}\right]{}^{(i)}f_{\ell\, +}$$

$$+\left[\frac{(2A+rA')^2+2A\left(2rA'+(\ell-1)(\ell+2)\right)}{4r^2A} + \left(\frac{A'}{4}+\frac{A}{r}\right)\partial_r + \frac{1}{2A}\partial_{tt}\right]{}^{(i)}f_{\ell\, rr} - \left(\frac{A'}{2A^2}\partial_t + \frac{1}{A}\partial_{tr}\right){}^{(i)}f_{\ell\, tr}\,,$$

$$\Delta_L{}^{(i)}h_{\ell\, tr} = \frac{A}{r}\partial_t{}^{(i)}f_{\ell\, 11} + \left[\left(\frac{A'}{2A}-\frac{1}{r}\right)\partial_t - \partial_{tr}\right]{}^{(i)}f_{\ell\, +} + \frac{\ell(\ell+1)}{2r^2}{}^{(i)}f_{\ell\, tr}\,,$$

$$\Delta_L{}^{(i)}h_{\ell\, +} = \left[-\frac{2rA'+\ell(\ell+1)}{4A} + \frac{r}{2}\partial_r\right]{}^{(i)}f_{\ell\, tt} + \frac{A}{2}\left[\left(2A+3rA'+\frac{\ell(\ell+1)}{2}\right) + rA\partial_r\right]{}^{(i)}f_{\ell\, rr}$$

$$+\frac{1}{2}\left[(\ell-1)(\ell+2) - r(4A+rA')\partial_r - r^2A\partial_{rr} + \frac{r^2}{A}\partial_{tt}\right]{}^{(i)}f_{\ell\, +} - r\partial_t{}^{(i)}f_{\ell\, tr}\,,$$

$$\Delta_L{}^{(i)}h_{\ell\, t\theta} = \frac{1}{2}\left[\left(A'+A\partial_r\right){}^{(i)}f_{\ell\, tr} - A\partial_t{}^{(i)}f_{\ell\, rr} - \partial_t{}^{(i)}f_{\ell\, +}\right],$$

$$\Delta_L{}^{(i)}h_{\ell\, r\theta} = \frac{2A+rA'}{4r}{}^{(i)}f_{\ell\, rr} - \frac{1}{2}\partial_r{}^{(i)}f_{\ell\, +} - \frac{1}{2A}\partial_t{}^{(i)}f_{\ell\, tr} + \frac{1}{2A}\left(-\frac{2A+rA'}{2rA} + \partial_r\right){}^{(i)}f_{\ell\, tt}\,,$$

$$\Delta_L{}^{(i)}h_{\ell\, -} = \frac{1}{4}\left(\frac{1}{A}{}^{(i)}f_{\ell\, tt} - A\,{}^{(i)}f_{\ell\, rr}\right).$$

General approach to gravitational perturbations (2)

1. At each order there is only **one scalar gravitational degree of freedom** (for polar/axial perturbations, and for a given multipole ℓ) satisfying (in)homogeneous linear wave equation with a potential (to be determined)

$$\tilde{\Box}_\ell \Phi_\ell^{(i)}(t,r) := r\left(-\bar{\Box} + V_\ell\right)\frac{\Phi_\ell^{(i)}(t,r)}{r} = \tilde{S}_\ell^{(i)} \tag{1}$$

2. RW variables $f_{\ell\,+}^{(i)}, f_{\ell\,rr}^{(i)}, f_{\ell\,tr}^{(i)}, f_{\ell\,tt}^{(i)}$ are given as linear combinations of $\Phi_\ell^{(i)}$ and its derivatives (+ source functions at nonlinear orders):

$$f_{\ell\,+}^{(i)} = B\Phi_\ell^{(i)} + C\partial_t\Phi_\ell^{(i)} + D\partial_r\Phi_\ell^{(i)} + E\partial_{tr}\Phi_\ell^{(i)} + F\partial_{rr}\Phi_\ell^{(i)} + \alpha_\ell^{(i)}(t,r)\,, \tag{2}$$

$$f_{\ell\,rr}^{(i)} = \cdots + \beta_\ell^{(i)}(t,r)\,, \qquad f_{\ell\,tr}^{(i)} = \cdots + \gamma_\ell^{(i)}(t,r)$$

3. Satisfying (perturbative) Einstein equations fixes the potential V_ℓ and the coefficient functions in the equations above **uniquely (!)**
4. The relations (2) can be inverted for $\Phi_\ell^{(i)}$. There is a **unique (!)** way compatible with the ADM initial problem formulation. This also gives the source $\tilde{S}_\ell^{(i)}$ in (1) **uniquely (!)**

Perturbations of spherically symmetric spaces, $A = 1 + \kappa r^2/\ell^2 - 2M/r$ (an easy way to the Zerilli equation)

master wave equation:

$$\tilde{\Box}_\ell \Phi_\ell^{(i)} := \frac{1}{A}\partial_{tt}\Phi_\ell^{(i)} - A\,\partial_{rr}\Phi_\ell^{(i)} - A'\,\partial_r\Phi_\ell^{(i)} + \left(\frac{A'}{r} + V_\ell\right)\Phi_\ell^{(i)} = \tilde{S}_\ell^{(i)}$$

potential (the celebrated Zerilli potential in the Schwarzschild case):

$$V_\ell = \frac{\ell(\ell+1)}{r^2} - \frac{A'}{r} + \underbrace{\left(2A - rA' - 2\right)}_{-6M/r} \frac{2A\left(rA'-2\right) - \left(rA'\right)^2 + \ell^2(\ell+1)^2}{r^2\left(2A - rA' - \ell(\ell+1)\right)^2}$$

and RW variables in terms of the master scalar variable (and source functions at nonlinear orders):

$$f_{\ell\,+}^{(i)} = A\partial_r\Phi_\ell^{(i)} + \frac{1}{r}\left(\frac{\ell(\ell+1)}{2} - \frac{2A - rA' - 2}{2A - rA' - \ell(\ell+1)}A\right)\partial_t\Phi_\ell^{(i)} + \alpha_\ell^{(i)}(t,r)$$

$$f_{\ell\,rr}^{(i)} = \cdots + \beta_\ell^{(i)}(t,r)$$

$$f_{\ell\,tr}^{(i)} = \cdots + \gamma_\ell^{(i)}(t,r)$$

Perturbations of spherically symmetric spaces, $A = 1 + \kappa r^2/\ell^2 - 2M/r$

The master variable in terms of RW potentials - the **unique** form compatible with the ADM initial problem formulation:

$$\Phi_\ell^{(i)} = \frac{2r}{\ell(\ell+1)} \left(f_{\ell\,+}^{(i)} + 2A \frac{A f_{\ell\,rr}^{(i)} - r\partial_r f_{\ell\,+}^{(i)}}{\ell(\ell+1) - 2A + rA'} \right)$$

Then the sources $\tilde{S}_\ell^{(i)}$ at higher order scalar wave equations can be read off from

$$\tilde{\Box}_\ell \Phi_\ell^{(i)} := \frac{1}{A}\partial_{tt}\Phi_\ell^{(i)} - A\,\partial_{rr}\Phi_\ell^{(i)} - A'\,\partial_r\Phi_\ell^{(i)} + \left(\frac{A'}{r} + V_\ell\right)\Phi_\ell^{(i)} = \tilde{S}_\ell^{(i)}$$

$$\begin{aligned}
\tilde{S}_\ell^{(i)} = &\frac{4r^2}{(\ell-1)\ell(\ell+1)(\ell+2)} \left(\frac{A}{r}\left(A\,{}^{(i)}S_{\ell\,rr} - \frac{1}{A}\,{}^{(i)}S_{\ell\,tt} \right) + \frac{(\ell-1)(\ell+2) - 2(3A-2)}{r^3}\,{}^{(i)}S_{\ell\,+} - 2A\partial_r\left({}^{(i)}S_{\ell\,+} \big/ r^2 \right) \right. \\
&- \frac{2\ell(\ell+1)}{r^2} A\,{}^{(i)}S_{\ell\,r\theta} + \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{r^3}\,{}^{(i)}S_{\ell\,-} \\
&- \frac{2A - rA' - 2}{2A - rA' - \ell(\ell+1)} \left(\frac{A}{r}\left(A\,{}^{(i)}S_{\ell\,rr} - \frac{1}{A}\,{}^{(i)}S_{\ell\,tt} \right) - \frac{A(\ell-1)(\ell+2)}{r\,(2A - rA' - \ell(\ell+1))}\left(A\,{}^{(i)}S_{\ell\,rr} + \frac{1}{A}\,{}^{(i)}S_{\ell\,tt} \right) \right. \\
&\left.\left. -2\,\frac{3A\left(2A - rA' - 2\right) - \ell(\ell+1)\left(2A - rA' - \ell(\ell+1)\right) - 2(\ell-1)(\ell+1)A}{r^3\left(2A - rA' - \ell(\ell+1)\right)}\,{}^{(i)}S_{\ell\,+} - 2A\partial_r\left({}^{(i)}S_{\ell\,+} \big/ r^2 \right) - \frac{2\ell(\ell+1)}{r^2} A\,{}^{(i)}S_{\ell\,r\theta} \right) \right)
\end{aligned}$$

To fix the source functions $\alpha_\ell^{(i)}$, $\beta_\ell^{(i)}$ and $\gamma_\ell^{(i)}$ we write them down as linear combinations of the sources $S_{\ell\,\mu\nu}^{(i)}$ and their first derivatives. Fixing $3 \times 7 \times 3 = 63$ function coefficients of these linear combinations is a technical task. It turns out that 54 functions (out of 63) are fixed in terms of 9 free functions. Moreover, in the resulting expressions, coefficients of these 9 free functions are identically zero due to the identities for the sources, thus the final expressions are **uniquely** defined:

$$\alpha_\ell^{(i)} = -\frac{2A\left(r^2\left(A^{-1}S_{\ell\,tt}^{(i)} - AS_{\ell\,rr}^{(i)}\right) + 2S_{\ell\,+}^{(i)}\right)}{\ell(\ell+1)\left(\ell(\ell+1) - 2A + rA'\right)}$$

$$\beta_\ell^{(i)} = \frac{1}{A}\left(r\partial_r\alpha_\ell^{(i)} - \frac{\ell(\ell+1) - 2A + rA'}{2A}\alpha_\ell^{(i)}\right)$$

$$\gamma_\ell^{(i)} = \frac{r}{A}\partial_t\alpha_\ell^{(i)} + \frac{2r^2}{\ell(\ell+1)}S_{\ell\,tr}^{(i)}$$

Example2: Cosmological perturbations in the $2+2$ splitting

[arXiv:1902.05090]

Motivation

Schwarzschild black hole perturbations studied in 2+2 splitting resulting in scalar/vector sectors (after expansion into suitably chosen scalar/vector spherical harmonics) [Regge&Wheeler, 57]

FLRW perturbations studied in 1+3 splitting resulting in scalar/vector/tensor sectors [Lifshitz, 46]

The key result: the general linear perturbation can be given in terms of only two (scalar/vector) master scalars satisfying scalar wave equation on the Schwarzschild background with Regge-Wheeler/Zerilli potentials for scalar/vector sectors; this can be extended beyond linear level

Does the same structure emerge for FLRW?
(time-dependent background)

$$E_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - T_{\mu\nu} = 0, \quad (8\pi G = 1)$$

Setup

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h^{(RW)}_{\mu\nu} + \mathscr{O}\left(\varepsilon^2\right)$$

$$d\bar{s}^2_{FLRW} = a^2(\tau)\left[-d\tau^2 + dq^2 + f^2(q)d\Omega_2^2\right], \qquad f(q) = q, \sin q, \sinh q$$

$$d\Omega_2^2 = du^2/(1-u^2) + (1-u^2)d\phi^2, \qquad -1 \le u = \cos\theta \le 1$$

Axial symmetry:

$h^{(RW)}_{\mu\nu}$ [Clarkson, Clifton & February, 2009]

$$= \sum_{0\le\ell} \begin{pmatrix} (\varphi_\ell + \chi_\ell + \psi_\ell) P_\ell(u) & \sigma_\ell P_\ell(u) & 0 & m_\ell \left(1-u^2\right) P'_\ell(u) \\ * & (\varphi_\ell + \chi_\ell) P_\ell(u) & 0 & n_\ell \left(1-u^2\right) P'_\ell(u) \\ 0 & 0 & \varphi_\ell \frac{P_\ell(u)}{1-u^2} & 0 \\ * & * & 0 & \varphi_\ell \left(1-u^2\right) P_\ell(u) \end{pmatrix}$$

where $\varphi_\ell, \chi_\ell, \psi_\ell, \sigma_\ell, m_\ell, n_\ell$ are functions of (τ, q)

gauge issues: if $x^\mu \to x^\mu - \varepsilon\xi^\mu$ then $g_{\mu\nu} \to g_{\mu\nu} + \varepsilon \mathscr{L}_\xi g_{\mu\nu} + \mathscr{O}\left(\varepsilon^2\right)$

Matter content: a single component perfect fluid

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \,,$$

with

$$\rho(\tau, q, u) = \rho_0(\tau) + \varepsilon \sum_{0 \le \ell} \delta\rho_\ell P_\ell(u) + \mathscr{O}\left(\varepsilon^2\right) \,,$$

$$p(\tau, q, u) = p_0(\tau) + \varepsilon \sum_{0 \le \ell} \delta p_\ell P_\ell(u) + \mathscr{O}\left(\varepsilon^2\right) \,,$$

$$\begin{aligned} u_\mu(\tau, q, u) &= (-a(\tau), 0, 0, 0) \\ &+ \varepsilon \sum_{0 \le \ell} \left(\frac{(\varphi_\ell + \chi_\ell + \psi_\ell)}{2a(\tau)} P_\ell(u), \delta w_\ell P_\ell(u), \delta v_\ell P'_\ell(u), \delta z_\ell \left(1 - u^2\right) P'_\ell(u) \right) \\ &+ \mathscr{O}\left(\varepsilon^2\right) \,, \end{aligned}$$

where $\delta\rho_\ell, \delta p_\ell, \delta w_\ell, \delta v_\ell, \delta z_\ell$ are functions of (τ, q)

Background equations

$$\rho_0 = \frac{3}{a^2}\left(\mathcal{H}^2 + \frac{1-f'^2}{f^2}\right) - \Lambda\,, \tag{3}$$

$$p_0 = \Lambda - \frac{1}{a^2}\left(\mathcal{H}^2 + \frac{1-f'^2}{f^2} + 2\dot{\mathcal{H}}\right),\;\; \mathcal{H} = \frac{\dot{a}}{a} \text{ \textit{conformal} Hubble constant} \tag{4}$$

Note that $\left[\left(1-f'^2\right)/f^2\right]' = 0$, thus $f'' = -\left(1-f'^2\right)/f$
The system (3,4) is closed by the equation of state of the fluid $p = p(\rho)$.
Differentiating (3,4) with respect to τ and defining the speed of sound as

$$c_s^2 = dp/d\rho|_{\rho=\rho_0}\,,$$

we get

$$\ddot{\mathcal{H}} = \left(1+3c_s^2\right)\mathcal{H}\left(\mathcal{H}^2 + \frac{1-f'^2}{f^2}\right) + \left(1-3c_s^2\right)\mathcal{H}\dot{\mathcal{H}}\,.$$

Linear perturbations: equations [Kulczycki&Malec, 2017]

$$E_- : \psi_\ell = 0\,,$$
$$E_{qu} : \chi'_\ell - \dot\sigma_\ell = 0\,,$$
$$E_+ : \ddot\chi_\ell - \chi''_\ell + \frac{2f'}{f}\chi'_\ell - 2\dot{\mathcal{H}}\chi_\ell - 2\mathcal{H}\dot\chi_\ell + \frac{\ell(\ell+1)-2}{f^2}\chi_\ell = 0\,,$$

$$E_{\tau\tau} : 2a^4\,\delta\rho_\ell = \left(\frac{\ell(\ell+1)+6f'^2-4}{f^2} + 2\mathcal{H}^2\right)\chi_\ell + 2\left(\frac{\ell(\ell+1)+3f'^2-3}{f^2} - 3\mathcal{H}^2\right)\varphi_\ell + 2\frac{f'}{f}\left(\chi'_\ell + 2\varphi'_\ell - 4\mathcal{H}\sigma_\ell\right) + 2\mathcal{H}\left(\dot\chi_\ell + 3\dot\varphi_\ell - 2\sigma'_\ell\right) - 2\varphi''_\ell \tag{5}$$

$$E_{qq} : 2a^4\,\delta p_\ell = \left(\frac{\ell(\ell+1)-2f'^2}{f^2} + 2\mathcal{H}^2 - 4\dot{\mathcal{H}}\right)\chi_\ell + 2\left(\frac{1-f'^2}{f^2} + \mathcal{H}^2\right)\varphi_\ell + 2\frac{f'}{f}\chi'_\ell + 2\mathcal{H}\left(\dot\varphi_\ell - \dot\chi_\ell\right) - 2\ddot\varphi_\ell\,, \tag{6}$$

$$E_{\tau q} : 2a^3\left(\rho_0 + p_0\right)\delta w_\ell = -\frac{\ell(\ell+1)+4f'^2-4}{f^2}\sigma_\ell + 2\frac{f'}{f}\left(2\mathcal{H}\chi_\ell - \dot\chi_\ell\right) + 2\mathcal{H}\left(\chi'_\ell - \varphi'_\ell\right) + 2\dot\varphi'_\ell\,,$$

$$E_{\tau u} : 2a^3\left(\rho_0 + p_0\right)\delta v_\ell = -2\mathcal{H}\varphi_\ell - \sigma'_\ell + 2\dot\varphi_\ell + \dot\chi_\ell\,.$$

Using eq. of state eqs. (5,6) are combined to yield

$$-\ddot\varphi_\ell + c_s^2\varphi''_\ell + 2\frac{f'}{f}c_s^2\varphi'_\ell + \left(1-3c_s^2\right)\mathcal{H}\dot\varphi_\ell + \frac{\left(1+3c_s^2\right)\left(1+f^2\mathcal{H}^2-f'^2\right) - c_s^2\ell(\ell+1)}{f^2}\varphi_\ell$$
$$= \left[\frac{\left(1+3c_s^2\right)f'^2-2}{f^2} + \left(c_s^2-1\right)\left(\frac{\ell(\ell+1)}{2f^2} + \mathcal{H}^2\right) + 2\dot{\mathcal{H}}\right]\chi_\ell - 2c_s^2\mathcal{H}\left(\sigma'_\ell + 2\frac{f'}{f}\sigma_\ell\right) + \frac{f'}{f}\left(c_s^2-1\right)\chi'_\ell + \mathcal{H}\left(c_s^2+1\right)\dot\chi_\ell \tag{7}$$

Linear perturbations: solution in terms of a master scalar

The simplest solution: $\chi_\ell = 0 = \sigma_\ell$ and $\varphi_\ell = \tilde{\varphi}_\ell$, with $\tilde{\varphi}_\ell$ being a solution of the homogeneous part of (7).
In the general case we resort to the guiding principle introduced in [R., 2017]:

$$\begin{aligned}
\sigma_\ell &= \quad \alpha_{0,0}\Phi_\ell + \alpha_{1,0}\dot{\Phi}_\ell + \alpha_{0,1}\Phi'_\ell + \dots \\
\chi_\ell &= \quad \beta_{0,0}\Phi_\ell + \beta_{1,0}\dot{\Phi}_\ell + \beta_{0,1}\Phi'_\ell + \dots \\
\varphi_\ell &= \tilde{\varphi}_\ell + \gamma_{0,0}\Phi_\ell + \gamma_{1,0}\dot{\Phi}_\ell + \gamma_{0,1}\Phi'_\ell + \dots
\end{aligned}$$

where coefficients $\alpha_{i,j}, \beta_{k,m}, \gamma_{n,p}$ are (τ, q) dependent functions and the **master scalar** $\Phi_\ell = \Phi_\ell(\tau, q)$ solves the **scalar wave equation** on the FLRW background, with a potential $V_\ell = V_\ell(\tau, q)$

$$\left(-\Box_{\bar{g}} + V_\ell\right)\frac{\Phi_\ell}{f} = 0$$

Linear perturbations: solution in terms of a master scalar (2)

$$\sigma_\ell = a^2 \left[f\Phi_\ell'' - \frac{\ell(\ell+1)}{f}\Phi_\ell \right]',$$

$$\chi_\ell = a^2 \left[f\left(\dot{\Phi}_\ell'' + 2\mathcal{H}\Phi_\ell''\right) - \frac{\ell(\ell+1)}{f}\left(\dot{\Phi}_\ell + 2\mathcal{H}\Phi_\ell\right) \right],$$

$$\varphi_\ell = a^2 \left[-f\,\mathcal{H}\Phi_\ell'' + f'\left(\dot{\Phi}_\ell' + \mathcal{H}\Phi_\ell'\right) + \frac{\ell(\ell+1)}{2f}\left(\dot{\Phi}_\ell + 3\mathcal{H}\Phi_\ell\right) \right] + \tilde{\varphi}_\ell\,,$$

where

$$\left(-\Box_{\bar{g}} + V_\ell\right)\frac{\Phi_\ell}{f} = 0\,,$$

$$V_\ell = \frac{1}{a^2}\left(\frac{\ell(\ell+1)}{f^2} + 2\dot{\mathcal{H}}\right)$$

(an analogue of the Zerilli potential)

Final conclusions

It is rather remarkable that gravitational perturbations of exact solutions of Einstein equations are in fact governed by two scalar functions, satisfying scalar wave equations (with a potential) on the background solution, corresponding to two polarizations of gravitational waves. In our opinion a deeper understanding of this fact is an interesting mathematical problem.

- **The hard part of solving perturbative Einstein equations (PDEs) can be reduced to only one scalar wave equation (for each polarization mode) and some linear algebra (!)**
- Crucial ingredients:
    - gauge invariance - implemented iteratively, thus Regge-Wheeler definitions of gauge invariants are sufficient
    - **ansatze** for the form of solution (for RW gauge invariants and source functions (particular solutions of the linear inhomogeneous system))
    - identities for the sources (inhomogeneous terms) in perturbative Einstein equations
- Although the scheme is conceptually simple, its actual realization was rather unthinkable in pre- computer algebra era