The Schrödinger-Newton-Hooke equation

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Derivation

of nonrelativistic limit for spacetime perturbations

Equations

Einstein equation
$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Klein-Gordon equation
$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{m^2c^2}{\hbar^2}\phi = 0$$

$$T_{\mu\nu} = \frac{\hbar^2}{2m}\left[\partial_\mu\phi\partial_\nu\bar{\phi} + \partial_\mu\bar{\phi}\partial_\nu\phi - g_{\mu\nu}\left(\partial_\lambda\phi\partial^\lambda\bar{\phi} + \frac{m^2c^2}{\hbar^2}|\phi|^2\right)\right]$$

Derivation

Ansatz

$$\phi(t,x) = e^{-\frac{imc^2}{\hbar}t}\, u(t,x)$$

$$ds^2 = -c^2\left(1 + \frac{2A(t,x)}{c^2}\right)dt^2 + \left(1 + \frac{2B(t,x)}{c^2}\right)\sum_{j=1}^{d}(dx^j)^2$$

Derivation

Nonrelativistic Limit

$$\lim_{c\to\infty} \Lambda c^2 = \varepsilon \frac{d(d-1)}{2}\omega^2$$

$$\varepsilon = \begin{cases} +1 & \text{de Sitter} \\ -1 & \text{anti de Sitter} \end{cases}$$

Derivation

Highest order

KG equation:

$$i\hbar\,\partial_t u = -\frac{\hbar^2}{2m}\,\Delta u + mAu$$

Einstein equation (tt):

$$-(d-1)\Delta B - \varepsilon\frac{d(d-1)}{2}\omega^2 = 8\pi G m|u|^2$$

Einstein equation (jj):

$$\Delta A - \partial_j^2 A + (d-2)(\Delta B - \partial_j^2 B) + \varepsilon\frac{d(d-1)}{2}\omega^2 = 0$$

Derivation

Result

$$\begin{cases} i\hbar\partial_t u = -\frac{\hbar^2}{2m}\Delta u + mAu \\ \Delta A = \frac{8\pi G(d-2)}{d-1}|u|^2 - \varepsilon d\omega^2 \end{cases}$$

$$\begin{cases} i\hbar\partial_t u = -\frac{\hbar^2}{2m}\Delta u - \frac{\varepsilon}{2}m\omega^2|x|^2 u + Vu \\ \Delta V = \frac{8\pi G(d-2)}{d-1}|u|^2 \end{cases}$$

Derivation

$$\begin{cases} i\partial_t \psi & = -\Delta\psi - \varepsilon|x|^2\psi + v\psi \\ \Delta v & = |\psi|^2 \end{cases}$$

$$i\hbar\partial_t\psi = -\Delta\psi - \varepsilon|x|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}} dy\right)$$

We focus on the anti de Sitter case ($\varepsilon = -1$).

Schrödinger-Newton-Hooke (SNH) equation

Motivation

$$\begin{cases} i\partial_t \psi &= -\Delta\psi + |\mathbf{x}|^2\psi + v\psi \\ \Delta v &= |\psi|^2 \end{cases}$$

- Atomic physics
- Plasma physics
- Solid state physics

- Investigations of the AdS stability
- Interesting example of a spatially confined system

Motivation to investigate this system in higher dimensions!

Symmetries

Lens transform

$$u : I \times \mathbb{R}^d \to \mathbb{C}$$

$$Łu(t, \mathbf{x}) = \frac{1}{\cos^{d/2} t}\, u\left(\tan t, \frac{\mathbf{x}}{\cos t}\right) e^{-\frac{i}{2}|\mathbf{x}|^2 \tan t}$$

$$Łu : \tan^{-1}(I) \times \mathbb{R}^d \to \mathbb{C}$$

Symmetries

Lens transform

$$u : I \times \mathbb{R}^d \to \mathbb{C}$$

$$\text{Ł}^{-1}u(t, \mathbf{x}) = \frac{1}{(1+t^2)^{d/4}}\, u\left(\tan^{-1} t, \frac{\mathbf{x}}{\sqrt{1+t^2}}\right) e^{\frac{i}{2}|\mathbf{x}|^2 \frac{t}{1+t^2}}$$

$$\text{Ł}u : \tan^{-1}(I) \times \mathbb{R}^d \to \mathbb{C}$$

$$\text{for } I \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Symmetries

Lens transform

$$\textbf{SN} \begin{cases} i\partial_t \psi & = -\frac{1}{2}\Delta\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}} dy\right)\psi \\ \psi(0,\mathbf{x}) & = \phi(\mathbf{x}) \end{cases}$$

$$\Big\downarrow \text{Ł}$$

$$\textbf{SNH} \begin{cases} i\partial_t \text{Ł}\psi & = -\frac{1}{2}\Delta \text{Ł}\psi + \frac{1}{2}|\mathbf{x}|^2 \text{Ł}\psi - (\cos t)^{d-4} \left(\int_{\mathbb{R}^d} \frac{|\text{Ł}\psi(t,y)|^2}{|x-y|^{d-2}} dy\right) \text{Ł}\psi \\ \text{Ł}\psi(0,\mathbf{x}) & = \phi(\mathbf{x}) \end{cases}$$

Symmetry enhancement in 4 spatial dimensions

Resonant approximation

$$\psi(t,r) = \sum_{n=0}^{\infty} \alpha_n(t) e^{-i\omega_n t} e_n(r)$$

$$e_n(r) = A_n \, L_n^{\left(\frac{d}{2}-1\right)} e^{-r^2/2}$$

$$\omega_n = d + 4n$$

$$i\dot{\alpha}_n = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} S_{nj(kl)} \bar{\alpha}_j \alpha_k \alpha_l \; e^{i(\omega_n + \omega_j - \omega_k - \omega_l)t}$$

$$S_{njkl} = \int_0^\infty \int_0^\infty \frac{r\, s\, e_n(r) e_j(s) e_k(s) e_l(r)}{\max\{r^{2-d}, s^{2-d}\}} \, dr \, ds$$

Resonant approximation

$$\omega_n + \omega_j = \omega_k + \omega_l$$

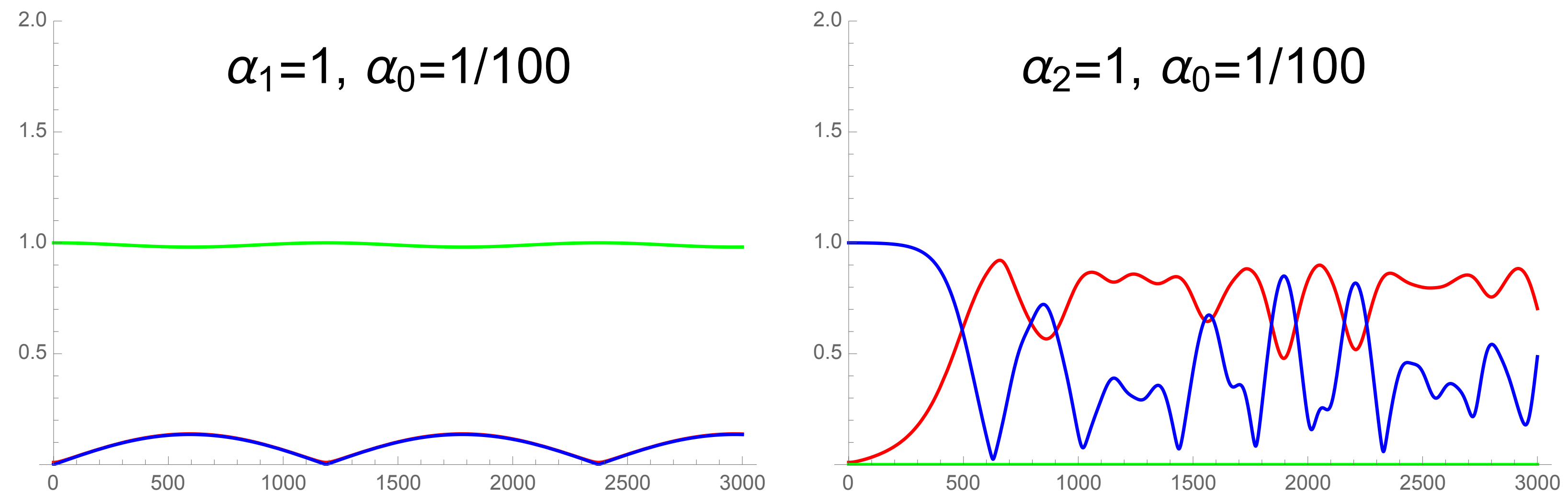
$$i\dot{\alpha}_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{nj(k,n+j-k)} \bar{\alpha}_j \alpha_k \alpha_l$$

Resonant approximation

$$i\dot{\alpha}_n = \sum_{j=0}^{\infty} \sum_{k=0}^{n+j} S_{nj(k,n+j-k)} \bar{\alpha}_j \alpha_k \alpha_l$$

$$\alpha_n(t) = \sqrt{n+1} \left(b(t) + \frac{a(t)}{p(t)} n \right) (p(t))^n$$

Finite-dimensional invariant manifolds



Stability of single-mode solutions

Stationary solutions

$$i\partial_t \psi = -\Delta\psi + |\mathbf{x}|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}} dy\right)\psi$$

$$\Big\downarrow \quad \psi(t,\mathbf{x}) = e^{-i\omega t}u(\mathbf{x})$$

$$-\Delta\psi + |\mathbf{x}|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}} dy\right)\psi = \omega\psi$$

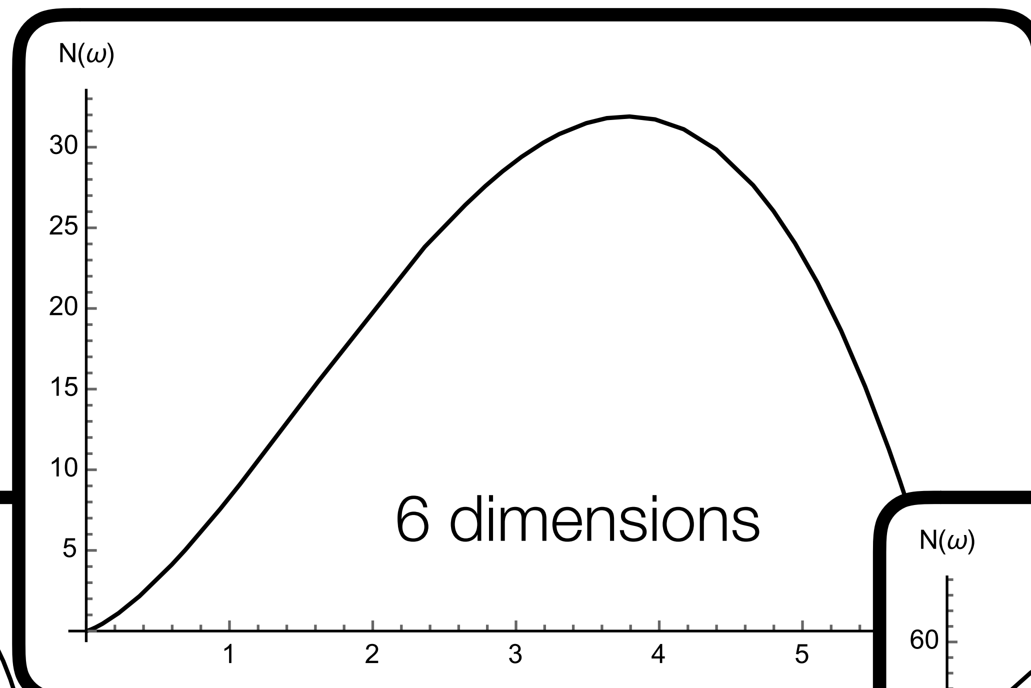
Stationary solutions

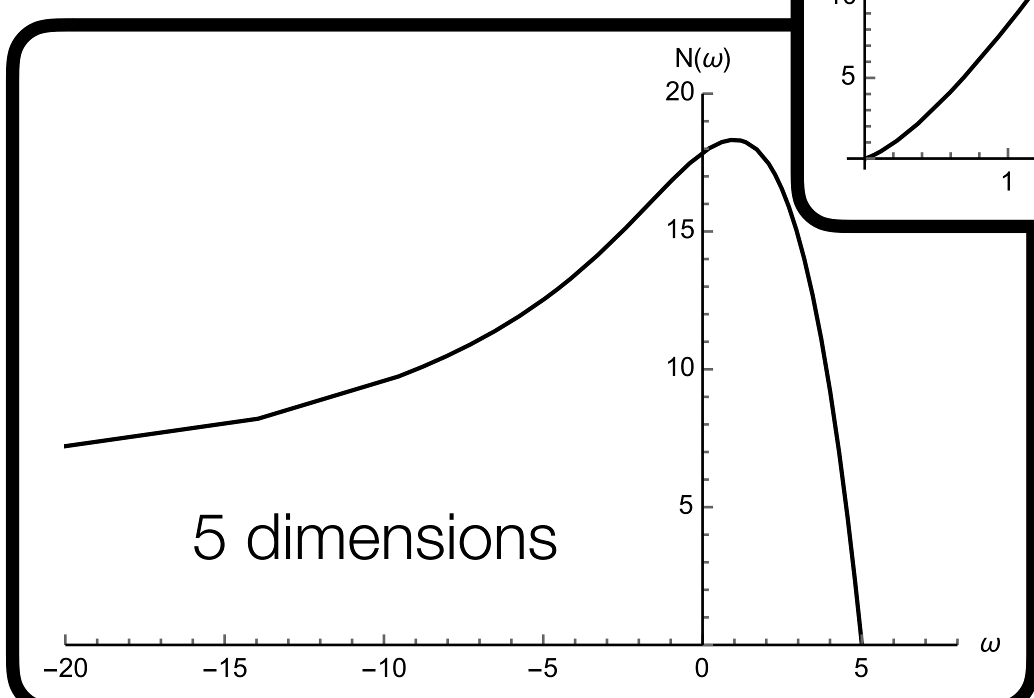
$$-\Delta\psi + |\mathbf{x}|^2\psi - \left(\int_{\mathbb{R}^d} \frac{|\psi(t,y)|^2}{|x-y|^{d-2}}dy\right)\psi = \omega\psi$$

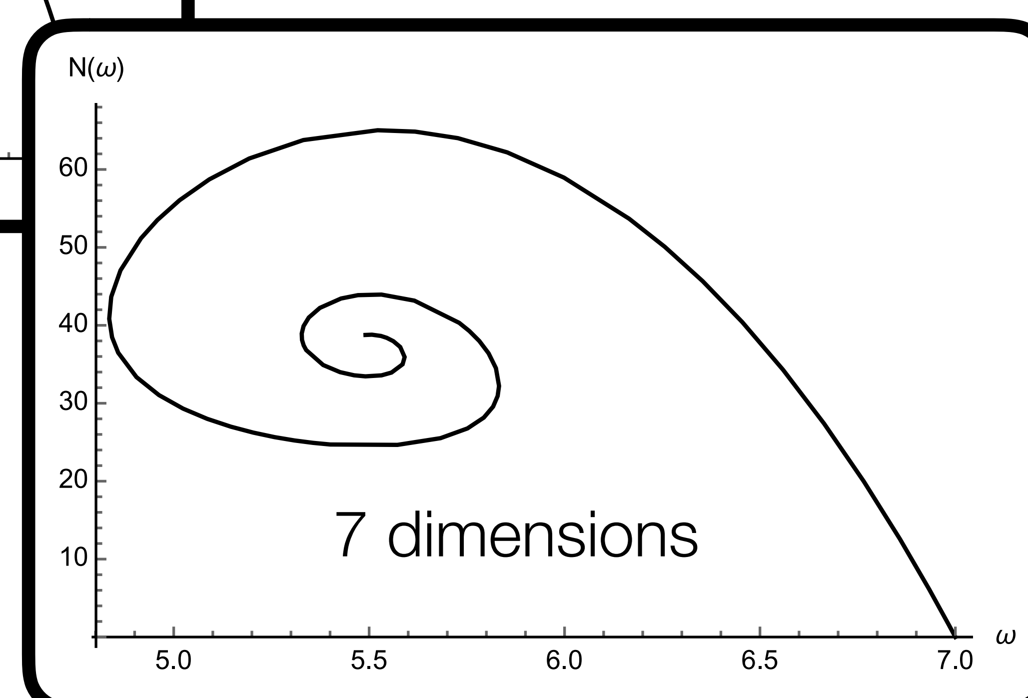
So far we focus on ground solutions (radial and positive).

Existence?
Pohožaev identities

Uniqueness?







Stability?
VK criterion

Summary

- The Schrödinger-Newton-Hooke equation can be obtained as a nonrelativistic limit of the AdS perturbations.

- It is an interesting system to investigate in a context of AdS stability, but also on its own, as a spatially confined system with a non-local potential.

- Different approaches to SNH equations show various interesting behaviour in higher dimensions.

Thank you for your attention!

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