Numerical GR in the context of AdS/CFT

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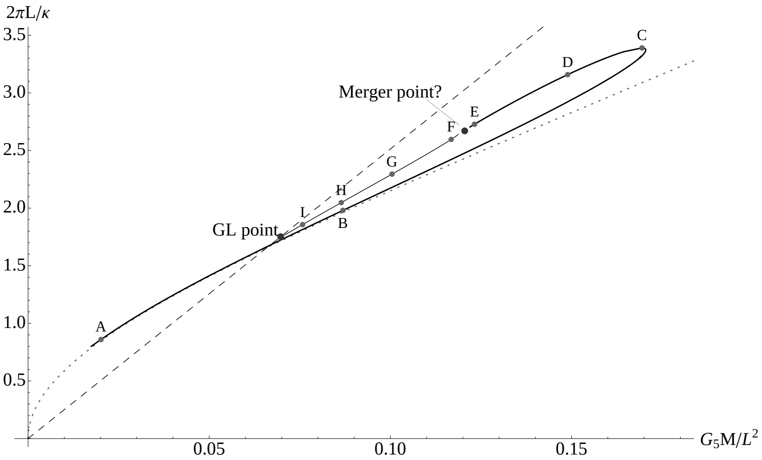
Introduction

- Numerical methods for exotic static/stationary solutions.
- AdS/CFT example: gravity dual to CFT on Schwarzschild background

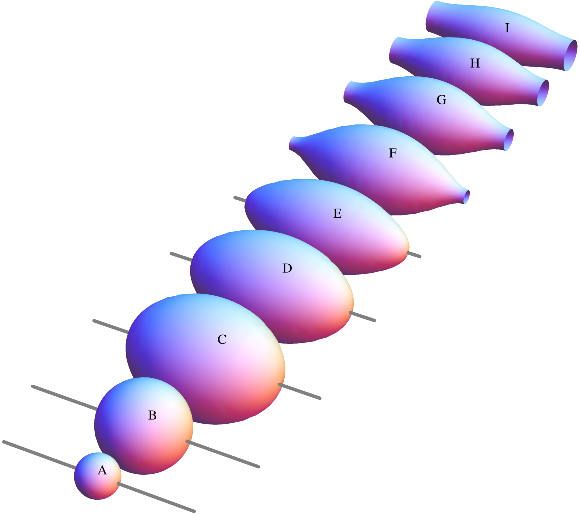
For example Mathematica notebook; download 'Static numerical example' from;
`http://www3.imperial.ac.uk/people/t.wiseman/teaching/`

Motivation

Kaluza-Klein black holes



Motivation



Motivation

Many exotic black holes (vacuum or otherwise) that we are likely to never know analytically but are still of physical interest.

- Asymptotically flat vacuum solutions; black rings, Saturns, etc... in $D > 5$
- Black holes in compact extra dimensions (eg. Kaluza-Klein theory)
- Black holes on branes
- Black holes in modified theories of gravity (eg. Einstein-Aether theory)
- **Black holes in AdS/CFT with a gauge theory interpretation**
- Also related problem of finding exotic Ricci flat Riemannian geometries such as Calabi-Yaus.

Static/Stationary solutions in AdS/CFT

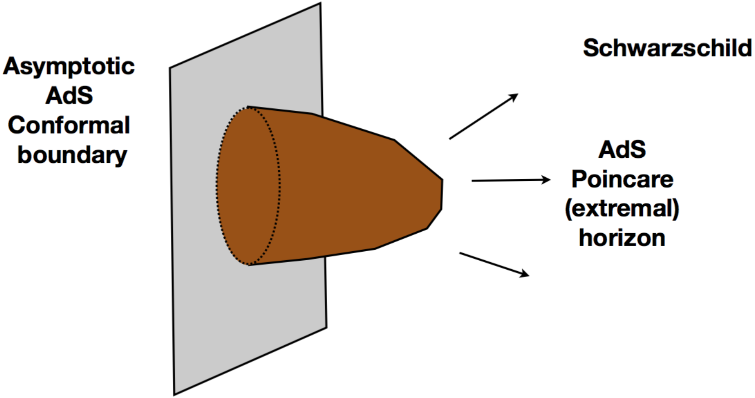
- In the context of AdS/CFT many areas where numerical GR is required.

Original method in *hep-th/0606086* with Headrick and *arXiv:0905.1822* with Headrick and Kitchen.

Further work in *arXiv:1104.4489* (with Figueras and Lucietti) and *arXiv:1105.6347* (with Adam and Kitchen)
Review article *arXiv:1107.5513*

AdS/CFT example

Consider CFT on a Schwarzschild background, rather than flat space. Solve $R_{ab} = -4/l^2\, g_{ab}$ to find the dual geometry?



AdS/CFT example

The 'UV' geometry must be asymptotic to;

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + h_{\mu\nu}(z,x)dx^\mu dx^\nu\right)$$

for $z \to 0$ with

$$\begin{aligned} h_{\mu\nu}dx^\mu dx^\nu &= \left(-(1-\frac{r_0}{r})dt^2 + (1-\frac{r_0}{r})^{-1}dr^2 + r^2 d\Omega^2\right) \\ &\quad + z^4 t_{\mu\nu}dx^\mu dx^\nu + \dots \end{aligned}$$

In the 'IR' the geometry must have an extremal horizon, with near horizon geometry given by that of the AdS Poincare horizon.

- Physical question: does the solution Hawking radiate at $O(N^2)$?
- Rephrased: is there a static black hole dual? Or must it be stationary/dynamic?

Plan

- General framework for static black holes:
- Harmonic Einstein equations (c.f. harmonic coordinates)
- Phrase the problem as an elliptic one
- Methods of solution: Ricci flow and Newton method
- Our AdS/CFT example
- Extension to stationary black holes

Dynamical approach

Could we use a full dynamical evolution to find a static/stationary solution as the end state? Yes, but ....

- Too much work!
- Difficult to achieve accuracy (must wait for radiation to dissipate)
- Unstable solutions are problematic

Static problem

Static problem should be elliptic; specify asymptotics and horizon regularity.

But $R_{\mu\nu}$ is not elliptic; perturb metric $g_{\mu\nu}$ by $h_{\mu\nu}$; then principal part;

$$\delta R_{\mu\nu} =_P \frac{1}{2}\left(g^{\alpha\beta}\partial_\mu\partial_\alpha h_{\beta\nu} + g^{\alpha\beta}\partial_\nu\partial_\alpha h_{\beta\mu} - g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - g^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta}\right)$$

Anhilates gauge perturbations $h_{\mu\nu} = \partial_{(\mu}u_{\nu)}$. So no definite character.

Harmonic Einstein equation

Take $R_{\mu\nu} \to R^H_{\mu\nu}$ where;

$$R^H_{\mu\nu} \equiv R_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)}\,, \quad \xi^\alpha \equiv g^{\mu\nu}\left(\Gamma^\alpha{}_{\mu\nu} - \bar{\Gamma}^\alpha{}_{\mu\nu}\right)\,.$$

- Γ is our usual Levi-Civita connection of g
- $\bar{\Gamma}$ is the 'reference connection' - any fixed smooth connection which we are free to choose
- For convenience, take $\bar{\Gamma}$ to be connection of a reference metric $\bar{g}_{\mu\nu}$.

Note ξ is a globally defined vector field.
Now the character of the equations is definite and given by the signature of the metric $g_{\mu\nu}$;

$$\delta R^H_{\mu\nu} =_P -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}$$

Relation to generalized harmonic coordinates

When $\xi^\mu = 0$ then $R^H_{\mu\nu} = R_{\mu\nu}$ and,

$$\nabla^2_S x^\alpha = H^\alpha \equiv -g^{\mu\nu} \bar{\Gamma}^\alpha{}_{\mu\nu}$$

ie. analog of **generalised harmonic coordinates**.

Harmonic Einstein equation and dynamics

Consider solving dynamic vacuum equations $R_{\mu\nu} = 0$.

For general Lorentzian $g_{\mu\nu}$ then $R^H_{\mu\nu} = 0$ is hyperbolic.

- Bianchi identity: $\nabla^2 \xi_\mu + R_\mu{}^\nu \xi_\nu = 0$
- Choose initial data and reference metric so $\xi^\mu = 0$ at $t = 0$: eg. $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\partial_t g_{\mu\nu} = \partial_t \bar{g}_{\mu\nu}$ at $t = 0$.
- Also require that $\partial_t \xi^\alpha = 0$. *Equivalent to Ham and Mom constraints.*
- Then Bianchi implies $\xi^\mu = 0$ for all $t > 0$.

Solve the hyperbolic $R^H_{\mu\nu} = 0$ to obtain a solution to $R_{\mu\nu} = 0$.

The static problem

Consider a static vacuum black hole solution.

Simplest way to manifest ellipticity:

- For a static black hole we may perform a Euclidean continuation $t = i\tau$.
- With a Riemannian metric then $R^H_{\mu\nu}$ has elliptic character.

Static vacuum black holes as a boundary value problem

Consider a general (non-extremal) static black hole solution;

$$ds^2 = -N(x)^2 dt^2 + h_{ij}(x) dx^i dx^j$$

where $\partial / \partial t$ is the static timelike Killing vector.

$N = 0$ is the horizon. The zeroth law implies surface gravity $\kappa = \partial_n N|_{N=0}$ constant (where n is unit normal vector to horizon)

Any such static black hole may be analytically continued to imaginary time $\tau = it$ with $\tau \sim \tau + 2\pi/\kappa$;

$$ds^2 = +N(x)^2 d\tau^2 + h_{ij}(x) dx^i dx^j$$

Surprise: The horizon is smooth with no boundary there.

Static vacuum black holes as a boundary value problem

Take coordinates $x^i = \{r, x^a\}$ where the horizon is located at $r = 0$. Then near the horizon;

$$ds^2 \sim \left(\kappa^2 r^2 d\tau^2 + dr^2\right) + \tilde{h}_{ab}(r, x) dx^a dx^b$$

- The Euclidean time circle forms angle of polar coordinates in $\mathbb{R}^2$, with r the radial coordinate.
- Polar coordinates break down at the origin $r = 0$, may take 'Cartesian' coordinates $X = r \cos \kappa\tau$ and $Y = r \sin \kappa\tau$ to cover the horizon.

So a static black hole can be written as a smooth Euclidean geometry; τ is periodic, $\partial/\partial\tau$ generates $U(1)$ isometry.

Static vacuum black holes as a boundary value problem

We see the static problem has an elegant formulation as a boundary value problem;

- Only asymptotic boundary, where the size of the periodic time circle is fixed (ie. the temperature)
- Formally there is no boundary at the horizon

Now certainly a solution $R_{\mu\nu} = 0$ in a gauge $\xi^\mu = 0$ implies $R^H_{\mu\nu} = 0$.

BUT $R^H_{\mu\nu} = 0$ only naively implies $R_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$ - a **Ricci soliton**. This is the **key difference** with the dynamic situation.

Ricci Solitons

Since $R^H_{\mu\nu} = 0$ is elliptic then a solution must be locally unique.

- Hence generically one can distinguish a soliton from a Ricci flat solution.

However the existence of solitons is very constrained. Example: Bourguignon ('79) proved none exist on a compact manifold.

May simply prove for appropriate choice of reference connection no asymptotically flat solitons exist. **[Figueras, Lucietti, TW '11]**

Ricci Solitons

Take metric and reference metric to be asymptotically flat:

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= d\tau^2 + \delta_{ij}dx^i dx^j + O(r^{-p}) \\ \partial_i g_{\mu\nu} &= O(r^{-p-1})\,, \quad \partial_i\partial_j g_{\mu\nu} = O(r^{-p-2}) \end{aligned}$$

Then;

$$\xi^\tau = O(r^{-p-1}) \qquad \xi^i = O(r^{-p-1})\,.$$

Contracting the Bianchi identity and using $R^H_{\mu\nu} = 0$;

$$\nabla^2\xi_\mu + R_\mu{}^\nu\xi_\mu = 0 \quad \to \quad \nabla^2\phi + \xi^\mu\partial_\mu\phi = 2(\nabla_\mu\xi_\nu)(\nabla^\mu\xi^\nu)$$

where $\phi = \xi^\mu\xi_\mu$. Asymptotically we have $\phi \sim O(r^{-2p-2}) \to 0$
Note $\phi \geq 0$ and $(\nabla_\mu\xi_\nu)(\nabla^\mu\xi^\nu) \geq 0$ for a Riemannian geometry.

Ricci Solitons

Hence a necessary condition for a soliton is that there exists a non-trivial function $f \neq 0$ satisfying;

$$\nabla^2 f + \xi^\mu \partial_\mu f \geq 0$$

and asymptotically $f \to 0$.
BUT maximum principle implies this is impossible. Hence no asymptotically flat solitons.

Can extend analysis to negative cosmological constant, and also Kaluza-Klein asymptotics. Also may extend to extremal horizons.

AdS/CFT example: No solitons may exist!

Solving the elliptic system I:

Local relaxation (eg. Jacobi) = Diffusion

Consider the Laplace equation, $\nabla^2 \psi = 0$ using finite difference and jacobi.
Use coordinates $(x_1, x_2, \ldots, x_D)$ on $\mathbb{R}^D$ and rectangular lattice of points, spacing Δ.
Point positions $(m_1 \Delta, m_2 \Delta, \ldots, m_D \Delta)$ for $m_1, m_2 \ldots, m_D \in \mathbb{Z}$.
Denote the set of points $\{p_i\}$ in lattice in domain with
$i = 1, \ldots, N$.
Real space finite difference; function ψ represented by values $\{\psi_i\}$ where $\psi_i = \psi(p_i)$.

Solving the elliptic system I:

Represent Laplace equation by second order finite difference. Denote $2D$ nearest neighbour points in rectangular lattice as $p_{i->j}$, where $j = 1, 2, \ldots, 2D$. Then;

$$\nabla^2 \psi|_{p_i} \simeq \frac{1}{\Delta^2} \left(-(2D)\psi_i + \sum_{j=1}^{2D} \psi_{i->j} \right) .$$

Fix boundary values, and use Jacobi method; consider a sequence of guesses, $\{\psi_i^{(A)}\}$ for integer $A = 0, 1, 2, \ldots$.
Jacobi method; given a guess $\{\psi_i^{(A)}\}$ improve it by;

$$\psi_i^{(A+1)} = \frac{1}{2D} \left(\sum_{j=1}^{2D} \psi_{i->j}^{(A)} \right)$$

Then iterate until have solution to required accuracy. Solution is a fixed point.

Solving the elliptic system I:

We may rearrange as;

$$\frac{2D}{\Delta^2}\left(\psi_i^{(A+1)} - \psi_i^{(A)}\right) = \frac{1}{\Delta^2}\left(\sum_{j=1}^{2D}\psi_{i->j}^{(A)} - (2D)\psi_i^{(A)}\right)$$

but this is a finite differencing of the diffusion equation; now ψ is a function of flow time λ;

$$\frac{\partial\psi(\lambda, x)}{\partial\lambda} = \nabla^2\psi(\lambda, x)$$

so that $\psi_i^{(A)} = \psi(A\,\delta)|_{p_i}$ with $\delta = \frac{\Delta^2}{2D}$.
The left-hand side is forward Euler differencing,

$$\left.\frac{\partial\psi}{\partial\lambda}\right|_{\lambda=A\delta} \simeq \frac{1}{\delta}\left(\psi_i^{(A+1)} - \psi_i^{(A)}\right) .$$

Local relaxation is diffusion on scales larger than lattice scale Δ.

Solving the elliptic system I:

Recall that $R^H_{\mu\nu} \sim -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} \ldots$

Hence local relaxation for $R^H_{\mu\nu}$ on large scales is diffusion for the metric;

$$\frac{\partial g_{\mu\nu}(\lambda)}{\partial \lambda} = -2 R^H_{\mu\nu} = -2 R_{\mu\nu} + 2 \nabla_{(\mu} \xi_{\nu)}$$

Begin with an initial smooth guess metric, and flow until (hopefully) reach a fixed point.

Solving the elliptic system I:

In fact this is diffeomorphic to the famous Ricci flow;

$$\frac{\partial g_{\mu\nu}(\lambda)}{\partial \lambda} = -2R_{\mu\nu}$$

since $\nabla_{(\mu}\xi_{\nu)}$ infinitessimal diffeo.

Important consequence: diffusion gives a geometric flow. No dependence on $\bar{\Gamma}$ for;

- trajectory of flow
- basin of attraction of fixed point

Solving the elliptic system I:

Consider a fixed point $g^{(0)}_{\mu\nu}$, so $R_{\mu\nu}[g^{(0)}] = 0$.

Perturb it $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. Then,

$$\frac{\partial h_{\mu\nu}}{\partial \lambda} = -2 \triangle_L h_{\mu\nu}$$

in suitable gauge ($v^\mu = 0$).

- Fixed point is **stable iff** $\triangle_L$ is a **positive** operator.
- Gross, Perry, Yaffe showed that for Schwarzschild $\triangle_L$ has a single negative eigenmode.

In such a case may still use relaxation but if N negative modes, must 'tune' N parameter set of initial data.

Solving the elliptic system II:

Discretize system, to obtain finite set of coupled non-linear equations. Solve $R^H_{\mu\nu} = 0$ using Newton method.

Linearize;

$$R^H[g+h]_{\mu\nu} = R^H[g]_{\mu\nu} + \mathcal{O}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} + O(h^2)$$

Then given a guess $g^{(i)}_{\mu\nu}$, construct a new guess $g^{(i+1)}_{\mu\nu}$;

$$g^{(i+1)}_{\mu\nu} = g^{(i)}_{\mu\nu} - b_{\mu\nu} \; , \quad \mathcal{O}_{\mu\nu}{}^{\alpha\beta} b_{\alpha\beta} = R^H[g^{(i)}]_{\mu\nu}$$

Hence must invert $\mathcal{O}$ so **non-local** update.

Solving the elliptic system II:

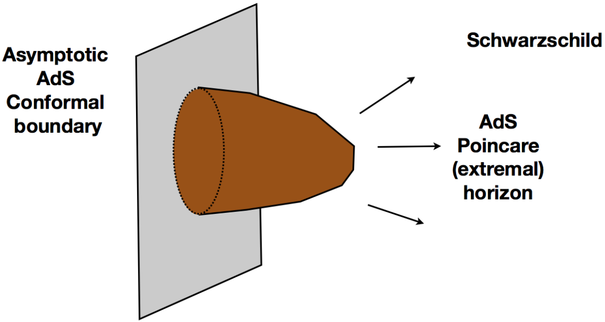
In continuum, provided $g_{\mu\nu}^{(i)}$ smooth, the update will be smooth.

Advantage: No problem with negative modes of $\triangle_L$

Disadvantages:

- More complicated than Ricci flow to implement
- Not geometric. Trajectory and basin of attraction of fixed point depends on choice of $\bar{\Gamma}$.

AdS-CFT example



- The solution enjoys the static isometry generated by $\partial/\partial\tau$, and axisymmetry.
- If the metric $g_{\mu\nu}$ and reference $\bar{g}_{\mu\nu}$ is symmetric under isometries then so is $R^H_{\mu\nu}$
- To exploit these isometrics choose adapted charts, treat fixed points as boundaries.

Isometries and 'fictitious' boundaries

Consider a function f that has spherical symmetry in Euclidean space $\mathbb{R}^m$.

- In Cartesian coordinates $ds^2 = \delta_{ij}dx^i dx^j$ the origin of the symmetry $x^i = 0$ is not a special point; $f(x)$ is smooth (C^∞) there.
- In adapted coordinates $ds^2 = dr^2 + r^2 d\Omega^2$, the function only depends r. But coordinates break down at $r = 0$.
- Treat $r = 0$ as a 'fictitious boundary' and deduce boundary conditions.
- Use $r^2 = \delta_{ij}x^i x^j$ and $f(x)$ smooth in x and f only depends on r; then f must be a smooth function in r^2, i.e. $f = f(r^2)$ is smooth.

Isometries and 'fictitious' boundaries

Consider Euclidean static metric with horizon;

The metric in polar coordinates is,

$$ds^2 = Adr^2 + r^2 Bd\tau^2 + rC_a dr dx^a + h_{ab} dx^a dx^b$$

where A, B, C_a, h_{ab} depend on radial and x coordinates, but not τ (recall $\tau \sim \tau + 2\pi/\kappa$).

Take Cartesian coordinates X and Y; $X = r\cos\kappa\tau$, $Y = r\sin\kappa\tau$

If metric smooth in X and Y implies;

- A, B, C_a, h_{ab} smooth functions of $r^2 = X^2 + Y^2$
- $\kappa^2 A = B$ at $r = 0$, for a constant κ (surface gravity)

Find $\partial_r \phi|_{r=0} = 0$ so there can be no maximum at horizon (as not really a boundary).

Isometries and 'fictitious' boundaries

Similarly for rotational axisymmetry;

$$ds^2 = Adr^2 + r^2 Bd\Omega^2 + rC_a drdx^a + h_{ab}dx^a dx^b$$

where $d\Omega^2$ is line element on unit $(n-1)$-sphere.

Transforming to Cartesian coordinates; find A, B, C_a, h_{ab} smooth functions of r^2 and $A = B$ at $r = 0$.

Static and axisymmetric metrics: coordinates for the problem

Consider AdS_5 in Poincare coordinates

$$g_{AdS_5} = \frac{\ell^2}{z^2}\left(dz^2 + dR^2 + R^2 d\Omega_{(2)}^2 - dt^2\right)$$

Take coordinates (r, x);

$$z = \frac{1-x^2}{1-r^2}\,, \qquad R = \frac{x\sqrt{2-x^2}}{1-r^2}$$

with $0 \le x < 1$ and $0 \le r < 1$.

Static and axisymmetric metrics: coordinates for the problem

Then metric in these coordinates is

$$g_{AdS_5} = \frac{\ell^2}{(1-x^2)^2}\left(-f(r)^2 dt^2 + \frac{4\,r^2}{f(r)^2}\,dr^2 + \frac{4}{g(x)}\,dx^2 + x^2 g(x)\,d\Omega_{(2)}^2\right)$$

$$f(r) \;=\; 1-r^2\,, \qquad g(x) = 2 - x^2$$

Conformal boundary is $x \to 1$, and $x=0$ is $SO(3)$ symmetry axis.

Coordinates are adapted to Poincare horizon which is at $r=1$ and $x<1$.

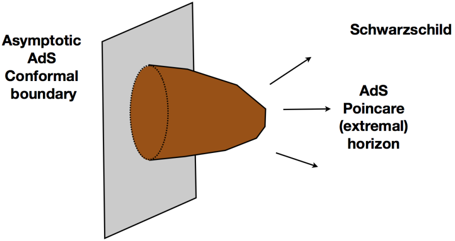
Static and axisymmetric metrics: coordinates for the problem

The geometry induced on Poincare horizon is;

$$\gamma_{ab}dx^a dx^b = \frac{\ell^2}{(1-x^2)^2}\left(\frac{4}{g(x)}\,dx^2 + x^2 g(x)\,d\Omega^2_{(2)}\right)$$

where $0 \leq x < 1$; locally conformal to round S^3 (by changing coordinate to $y = 1 - x^2$).

Metric ansatz



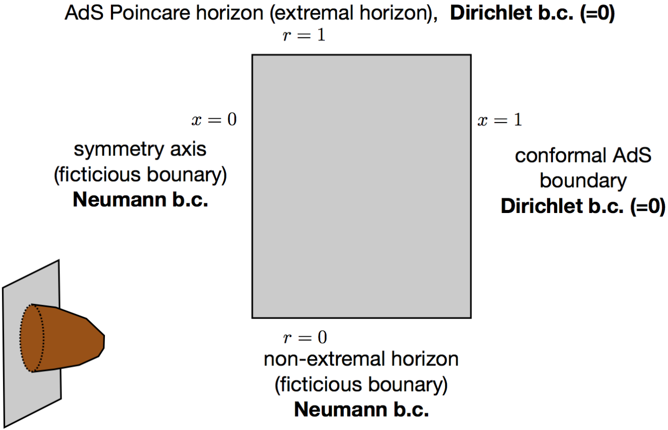
Take general metric in coordinates adapted to isometries;

$$
\begin{aligned}
ds^2 &= \frac{1}{(1-x^2)^2}\Big(4r^2 f^2 e^T d\tau^2 + x^2 g\, e^S d\Omega^2_{(2)} + \frac{4}{f^2} e^{T+r^2 A} dr^2 \\
&\qquad + \frac{4}{g} e^{S+x^2 B} dx^2 + \frac{2rx}{f} F dr dx\Big)
\end{aligned}
$$

where T, A, B, F, S are unknown functions of r, x and $g = 2 - x^2$ $f = 1 - r^2$.

Metric ansatz

- Horizon and axis regularity implies they are smooth in r^2 and x^2.
- At $r = 1$ (Poincare horizon) and $x = 0$ (conformal boundary) $T, A, B, F, S = 0$.



Metric ansatz

As $x \to 1$;

$$ds^2 \to \frac{1}{(1-x)^2}\left(dx^2 + f^2\left(2r^2 d\tau^2 + 2\frac{1}{f^4}dr^2 + \frac{1}{f^2}d\Omega^2\right)\right)$$

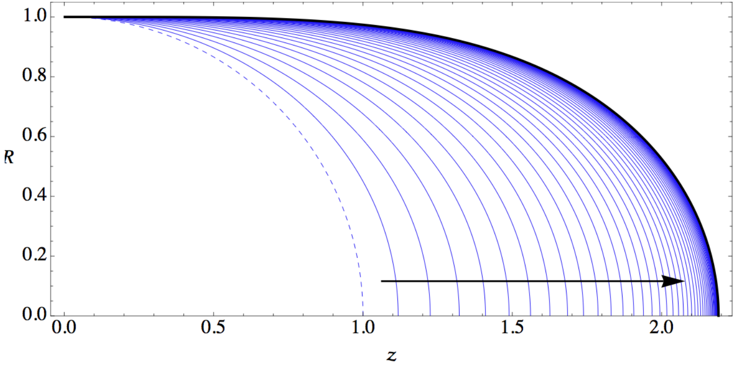
where,

$$2r^2 d\tau^2 + 2\frac{1}{f^4}dr^2 + \frac{1}{f^2}d\Omega^2 = \\ 2\left(1-\frac{1}{\rho}\right)d\tau^2 + \left(1-\frac{1}{\rho}\right)^{-1}d\rho^2 + \rho^2 d\Omega^2$$

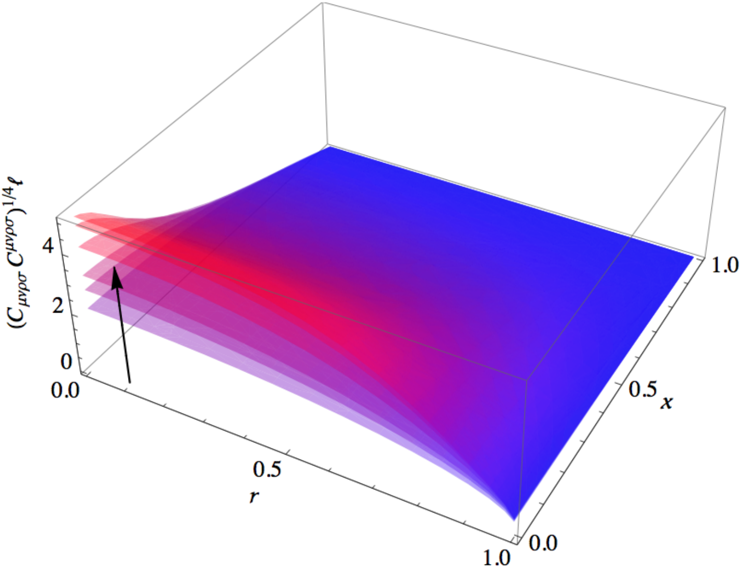
where $\rho = 1/\left(\sqrt{2}(1-r^2)\right)$.

So the conformal boundary metric is conformal to Schwarzschild.

Horizon embedding



Weyl curvature



Static solutions from a Lorentzian perspective

Consider static chart;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N(x)^2 dt^2 + h_{ij}(x) dx^i dx^j \tag{1}$$

so that $N^2 > 0$. Choose reference metric also static w.r.t. $\partial/\partial t$;

$$\bar{ds}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\bar{N}(x)^2 dt^2 + \bar{h}_{ij}(x) dx^i dx^j \tag{2}$$

Take both to have constant surface gravity, with same value.
Could continue both to smooth Euclidean metrics on same manifold.
$R^H_{\mu\nu}$ shares the static symmetry. PDEs for components of g are invariant under $t \to \tau = it$.
Hence Harmonic Einstein equation restricted to Lorentzian *static* metrics (and reference metrics) is elliptic.
Behaviour of local relaxation/Ricci flow, and Newton method same in either signature.

Static solutions from a Lorentzian perspective

In Lorentzian signature we have no option but to think of the horizon as a boundary.

But Harmonic Einstein equations are independent of signature, so boundary conditions must be too.

Consider Lorentizan coordinates;

$$ds^2 = -r^2 V dt^2 + U dr^2 + r\, U_a dr dx^a + h_{ab} dx^a dx^b \tag{3}$$

where metric functions are functions of r and x^a.

Coordinates break down at horizon.

Static solutions from a Lorentzian perspective

Change coordinates;

$$a = r \cosh \kappa t \,, \quad b = r \sinh \kappa t \tag{4}$$

to cover the static horizon.
Metric components are smooth provided;

- V, U, U_a, h_{ab} are smooth (C^∞) functions of r^2 and x^a
- $V = \kappa^2 U$ at $r = 0$, with constant κ surface gravity

Same conditions apply to reference metric.
Of course these are same conditions as in Euclidean case.

Static solutions from a Lorentzian perspective

Now $R^H_{\mu\nu}$ shares the same regularity properties as the metric;
In the coordinates (a, b, x^a) the metric and reference metric components are smooth, so are those of $R^H_{\mu\nu}$.
Transform back to the static coordinates (t, r, x^a) giving;

$$R^H = -r^2 f dt^2 + g dr^2 + r\, g_a dr dx^a + r_{ab} dx^a dx^b \tag{5}$$

where f, g, g_a and r_{ab} are smooth in r^2, x^a, and $f = \kappa^2 g$.
Thus in the Lorentzian picture Ricci flow and Newton method preserve regularity of the horizon boundary, and preserve its surface gravity.
View horizon as boundary and impose physical data there, namely surface gravity.

Stationary case

Now no Euclidean continuation; must treat directly in Lorentzian signature.
Consider globally timelike stationary Killing vector $\partial/\partial t$. This cannot be a black hole. Then,

$$ds^2 = -N(x)\left(dt + A_i(x)dx^i\right)^2 + h_{ij}(x)dx^i dx^j \tag{6}$$

with $N > 0$ and h_{ij} Riemannian. Then;

$$R^H_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \ldots = -\frac{1}{2}h^{ij}\partial_i\partial_j g_{\mu\nu} + \ldots \tag{7}$$

This is still an elliptic problem as h_{ij} is Riemannian.
Take same form for reference, the $R^H_{\mu\nu}$ is Ricci flow and Newton methods truncate into this stationary class of metrics.

Stationary case

- If one considers black holes, then $\partial/\partial t$ will not be timelike inside ergosurfaces or horizons. In this case one must treat stationary horizons as boundaries with appropriate boundary conditions (analogous to the Lorenztian treatment of static horizons). In addition to obtain an elliptic problem one should use the rigidity property of stationary black holes.
- For information about this stationary case, the reader is referred to the review; `arXiv:1107.5513`
- For a more recent method that solves the black hole in the interior of the black hole, and can treat non-Killing horizons, please see the paper `arXiv:1212.4498`.