

Black holes on supercomputers: Numerical relativity applications to astrophysics and high-energy physics

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1st July 2013

Overview

- Introduction, motivation
- Foundations of numerical relativity
 - Formulations of Einstein's eqs.: 3+1, BSSN, GHG, characteristic
 - Beyond 4D: Reducing dimensionality
 - Initial data, Gauge, Boundaries
 - Technical ingredients: Discretization, mesh refinement,...
 - Diagnostics: Horizons, Momenta, GWs
- Applications and Results of NR
 - Astrophysics
 - Gravitational wave physics
 - High-energy physics
 - Fundamental properties of BHs

1. Introduction, motivation

The Schwarzschild solution

- Einstein 1915

General relativity: geometric theory of gravity

- Schwarzschild 1916

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

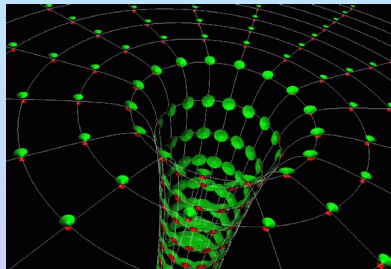
- Singularities:

$r = 0$: physical

$r = 2M$: coordinate

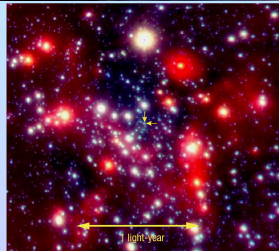
- Newtonian escape velocity

$$v = \sqrt{\frac{2M}{r}}$$



Evidence for astrophysical black holes

- X-ray binaries
 - e. g. Cygnus X-1 (1964)
 - MS star + compact star
 - ⇒ Stellar Mass BHs
 - ~ 5 ... 50 M_{\odot}
- Stellar dynamics
 - near galactic centers,
 - iron emission line profiles
 - ⇒ Supermassive BHs
 - ~ $10^6 \dots 10^9 M_{\odot}$
 - AGN engines

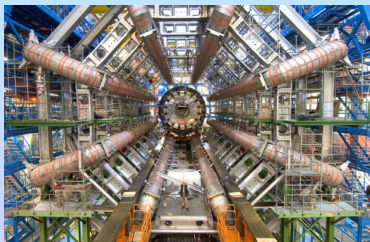
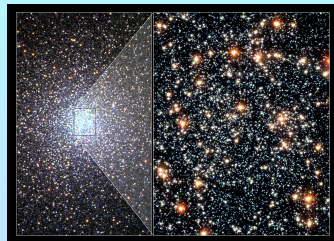


The Centre of the Milky Way
(VLT YEPUN + NACO)

ESO PR Photo 29a/02 (9 October 2002) ©European Southern Observatory

Conjectured BHs

- Intermediate mass BHs
 $\sim 10^2 \dots 10^5 M_{\odot}$
- Primordial BHs
 $\leq M_{Earth}$
- Mini BHs, LHC
 $\sim TeV$

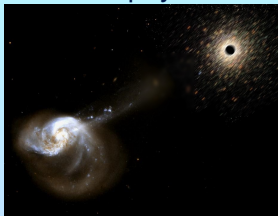


Note: BH solution is scale invariant!

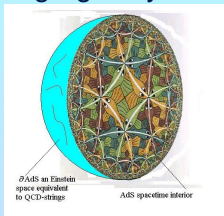


Research areas: Black holes have come a long way!

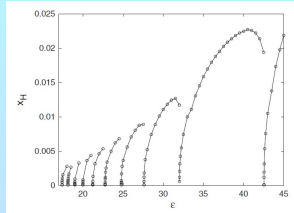
Astrophysics



Gauge-gravity duality



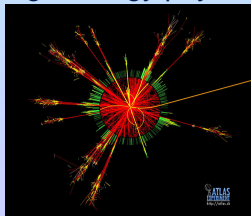
Fundamental studies



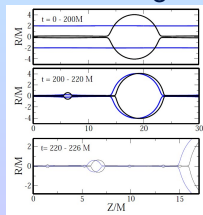
GW physics



High-energy physics



Fluid analogies



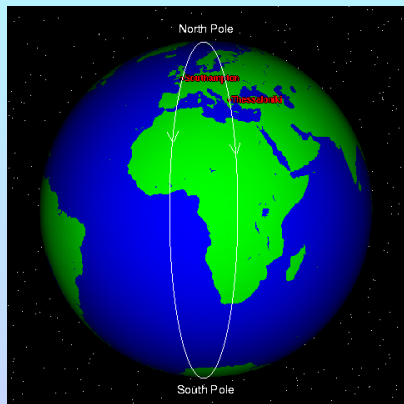
General Relativity: Curvature

- Curvature generates acceleration
“geodesic deviation”
No “force”!!
- Description of geometry

Metric $g_{\alpha\beta}$

Connection $\Gamma_{\beta\gamma}^{\alpha}$

Riemann Tensor $R^{\alpha}{}_{\beta\gamma\delta}$



The metric defines everything

- Christoffel connection

$$\Gamma_{\beta\gamma}^{\alpha} = \frac{1}{2}g^{\alpha\mu} (\partial_{\beta}g_{\gamma\mu} + \partial_{\gamma}g_{\mu\beta} - \partial_{\mu}g_{\beta\gamma})$$

- Covariant derivative

$$\nabla_{\alpha}T^{\beta}_{\gamma} = \partial_{\alpha}T^{\beta}_{\gamma} + \Gamma_{\mu\alpha}^{\beta}T^{\mu}_{\gamma} - \Gamma_{\gamma\alpha}^{\mu}T^{\beta}_{\mu}$$

- Riemann Tensor

$$R^{\alpha}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma_{\beta\delta}^{\alpha} - \partial_{\delta}\Gamma_{\beta\gamma}^{\alpha} + \Gamma_{\mu\gamma}^{\alpha}\Gamma_{\beta\delta}^{\mu} - \Gamma_{\mu\delta}^{\alpha}\Gamma_{\beta\gamma}^{\mu}$$

- \Rightarrow Geodesic deviation,
Parallel transport,
...

How to get the metric?



Train cemetery
Uyuni, Bolivia

- Solve for the metric $g_{\alpha\beta}$

How to get the metric?

- The metric must obey the Einstein Equations
- Ricci-Tensor, Einstein Tensor, Matter Tensor

$$R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta}$$

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^{\mu}{}_{\mu} \quad \text{“Trace reversed” Ricci}$$

$$T_{\alpha\beta} \quad \text{“Matter”}$$

- Einstein Equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

- Solutions: Easy! \Rightarrow Calculate $G_{\alpha\beta}$

\Rightarrow Use that as matter tensor

- Physically meaningful solutions: Difficult!

Solving Einstein's equations: Different methods

- Analytic solutions
 - Symmetry assumptions
 - Schwarzschild, Kerr, FLRW, Myers-Perry, Emparan-Reall,...
- Perturbation theory
 - Assume solution is close to known solution $g_{\alpha\beta}$
 - Expand $\hat{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h_{\alpha\beta}^{(1)} + \epsilon^2 h_{\alpha\beta}^{(2)} + \dots \Rightarrow$ linear system
 - Regge-Wheeler-Zerilli-Moncrief, Teukolsky, QNMs, EOB,...
- Post-Newtonian Theory
 - Assume small velocities \Rightarrow expansion in $\frac{v}{c}$
 - N^{th} order expressions for GWs, momenta, orbits,...
 - Blanchet, Buonanno, Damour, Kidder, Will,...
- Numerical Relativity

2. Foundations of numerical relativity

A list of tasks

- Target: Predict time evolution of BBH in GR
- Einstein equations:
 - 1) Cast as evolution system
 - 2) Choose specific formulation
 - 3) Discretize for computer
- Choose coordinate conditions: Gauge
- Fix technical aspects:
 - 1) Mesh refinement / spectral domains
 - 2) Singularity handling / excision
 - 3) Parallelization
- Construct realistic initial data
- Start evolution and waaaaiiiit...
- Extract physics from the data

2.1 Formulations of Einstein's equations

The Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$\Leftrightarrow R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{D-2} T g_{\mu\nu} \right) + \frac{2}{D-2} \Lambda g_{\mu\nu}$$

- In this form no well-defined mathematical character
hyperbolic, elliptic, parabolic?
- Coordinate x^α on equal footing; time only through signature of $g_{\alpha\beta}$
- Well-posedness of the equations? Suitable for numerics?
- Several ways to identify character and coordinates
→ Formulations

2.1.1 ADM like $D - 1 + 1$ formulations

3+1 Decomposition

- NR: ADM 3+1 split Arnowitt, Deser & Misner '62
York '79, Choquet-Bruhat & York '80

- Spacetime = Manifold (\mathcal{M}, g)

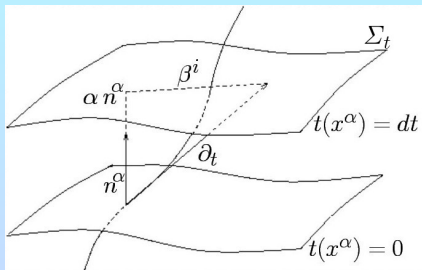
- Hypersurfaces

Scalar field $t : \mathcal{M} \rightarrow \mathbb{R}$

such that $t = \text{const}$ defines Σ_t

\rightarrow 1 form \mathbf{dt} , vector ∂_t

$$\langle \mathbf{dt}, \partial_t \rangle = 1$$



- Def.:** Timelike unit vector: $n_\mu \equiv -\alpha(\mathbf{dt})_\mu$

Lapse: $\alpha = 1/\|\mathbf{dt}\|$

Shift: $\beta^\mu = (\partial_t)^\mu - \alpha n^\mu$

Adapted coordinate basis: $\partial_t = \alpha n + \beta$, $\partial_i = \frac{\partial}{\partial x^i}$

3+1 Decomposition

Def.: A vector v^α is tangent to Σ_t $\Leftrightarrow \langle \mathbf{dt}, v \rangle = (\mathbf{dt})_\mu v^\mu = 0$

Projector: $\perp^\alpha{}_\mu = \delta^\alpha{}_\mu + n^\alpha n_\mu$

For a vector tangent to Σ_t one easily shows

- $n_\mu v^\mu = 0$
- $\perp^\mu{}_\alpha v^\mu = v^\alpha$

Projection of the metric

- $\gamma_{\alpha\beta} := \perp^\mu{}_\alpha \perp^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta} + n_\alpha n_\beta \Rightarrow \gamma_{\alpha\beta} = \perp_{\alpha\beta}$
- For v^α tangent to Σ_t : $g_{\mu\nu} v^\mu v^\nu = \gamma_{\mu\nu} v^\mu v^\nu$

Adapted coordinates: $x^\alpha = (t, x^i)$

\Rightarrow we can ignore t components for tensors tangential to Σ_t

$\Rightarrow \gamma_{ij}$ is the metric on Σ_t First fundamental form

3+1 decomposition of the metric

In adapted coordinates, we write the spacetime metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array} \right)$$

$$\Leftrightarrow g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2} \beta^j \\ \hline \alpha^{-2} \beta^i & \gamma^{ij} - \alpha^{-2} \beta^i \beta^j \end{array} \right)$$

$$\Leftrightarrow ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

Gauge variables: **Lapse** α , **Shift vector** β^i

For any tensor tangent in all components to Σ_t we raise and lower indices with γ_{ij} :

$$S^i_j{}_k = \gamma^{jm} S^i_{mk} \text{ etc.}$$

Projections and spatial covariant derivative

- For an arbitrary tensor S of type $\binom{p}{q}$, its projection is

$$(\perp S)^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = \perp^{\alpha_1}_{\mu_1} \dots \perp^{\alpha_p}_{\mu_p} \perp^{\nu_1}_{\beta_1} \dots \perp^{\nu_q}_{\beta_q} S^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$$

“Project every free index”

- For a tensor S on Σ_t , its covariant derivative is $DS := \perp(\nabla S)$

$$D_\rho S^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = \perp^{\alpha_1}_{\mu_1} \dots \perp^{\alpha_p}_{\mu_p} \perp^{\nu_1}_{\beta_1} \dots \perp^{\nu_q}_{\beta_q} \perp^\sigma_\rho \nabla_\sigma S^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$$

- One can show that

- $D = \perp \nabla$ is torsion free on Σ_t if ∇ is on \mathcal{M}
- $(\perp \nabla \gamma)_{ijk} = 0$ metric compatible
- $\perp \nabla$ is unique in satisfying these properties

Extrinsic curvature

Def.: $K_{\alpha\beta} = -\perp \nabla_{\beta} n_{\alpha}$

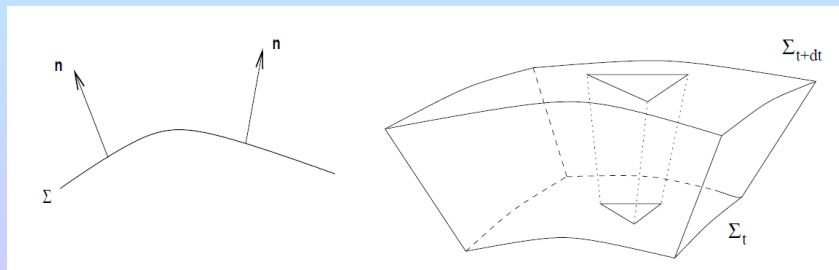
- $\nabla_{\beta} n_{\alpha}$ is not symmetric, but $\perp \nabla_{\beta} n_{\alpha}$ and, thus, $K_{\alpha\beta}$ is!

- One can show that

$$\mathcal{L}_n \gamma_{\alpha\beta} = n^{\mu} \nabla_{\mu} \gamma_{\alpha\beta} + \gamma_{\mu\beta} \nabla_{\alpha} n^{\mu} + \gamma_{\alpha\mu} \nabla_{\beta} n^{\mu} = -2K_{\alpha\beta}$$

$$K_{\alpha\beta} = -\frac{1}{2} \mathcal{L}_n \gamma_{\alpha\beta}$$

- Two interpretations of $K_{\alpha\beta} \rightarrow$ embedding of Σ_t in \mathcal{M}



The projections of the Riemann tensor

$$\perp^\mu_\alpha \perp^\nu_\beta \perp^\gamma_\rho \perp^\sigma_\delta R^\rho_{\sigma\mu\nu} = \mathcal{R}^\gamma_{\delta\alpha\beta} + K^\gamma_\alpha K_{\delta\beta} - K^\gamma_\beta K_{\delta\alpha} \quad \text{Gauss Eq.}$$

$$\perp^\mu_\alpha \perp^\nu_\beta R_{\mu\nu} + \perp_{\mu\alpha} \perp^\nu_\beta n^\rho n^\sigma R^\mu_{\rho\nu\sigma} = \mathcal{R}_{\alpha\beta} + KK_{\alpha\beta} - K^\mu_\beta K_{\alpha\mu} \quad \text{contracted}$$

$$R + 2 R_{\mu\nu} n^\mu n^\nu = \mathcal{R} + K^2 - K^{\mu\nu} K_{\mu\nu} \quad \text{scalar Gauss eq.}$$

$$\perp^\gamma_\rho n^\sigma \perp^\mu_\alpha \perp^\nu_\beta R^\rho_{\sigma\mu\nu} = D_\beta K^\gamma_\alpha - D_\alpha K^\gamma_\beta \quad \text{Codazzi eq.}$$

$$n^\sigma \perp^\nu_\beta R_{\sigma\nu} = D_\beta K - D_\mu K^\mu_\beta \quad \text{contracted}$$

$$\perp_{\alpha\mu} \perp^\nu_\beta n^\sigma n^\rho R^\mu_{\rho\nu\sigma} = \frac{1}{\alpha} \mathcal{L}_m K_{\alpha\beta} + K_{\alpha\mu} K^\mu_\beta + \frac{1}{\alpha} D_\alpha D_\beta \alpha$$

$$\perp^\mu_\alpha \perp^\nu_\beta R_{\mu\nu} = -\frac{1}{\alpha} \mathcal{L}_m K_{\alpha\beta} - 2K_{\alpha\mu} K^\mu_\beta - \frac{1}{\alpha} D_\alpha D_\beta \alpha + \mathcal{R}_{\alpha\beta} + KK_{\alpha\beta}$$

$$R = -\frac{2}{\alpha} \mathcal{L}_m K - \frac{2}{\alpha} \gamma^{\mu\nu} D_\mu D_\nu \alpha + \mathcal{R} + K^2 + K^{\mu\nu} K_{\mu\nu}$$

- Here \mathcal{L} is the Lie derivative and $m^\mu = \alpha n^\mu = (\partial_t)^\mu + \beta^\mu$
- Summation of spatial tensors: ignore time indices;
 $\mu, \nu, \dots \rightarrow m, n, \dots$

Decomposition of the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$\Leftrightarrow R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} T \right) + \frac{2}{D-2} \Lambda g_{\alpha\beta}$$

Energy momentum tensor

- $\rho = T_{\mu\nu} n^\mu n^\nu$ energy density

$$j_\alpha = -T_{\mu\nu} n^\mu \perp^\nu{}_\alpha \quad \text{momentum density}$$

$$S_{\alpha\beta} = \perp^\mu{}_\alpha \perp^\nu{}_\beta T_{\mu\nu}, \quad S = \gamma^{\mu\nu} S_{\mu\nu} \quad \text{stress tensor}$$

- $T_{\alpha\beta} = S_{\alpha\beta} + n_\alpha j_\beta + n_\beta j_\alpha + \rho n_\alpha n_\beta, \quad T = S - \rho$

Lie derivative $\mathcal{L}_m = \mathcal{L}_{(\partial_t - \beta)}$

- $\mathcal{L}_m K_{ij} = \partial_t K_{ij} - \beta^m \partial_m K_{ij} - K_{mj} \partial_i \beta^m - K_{im} \partial_j \beta^m$

$$\mathcal{L}_m \gamma_{ij} = \partial_t \gamma_{ij} - \beta^m \partial_m \gamma_{ij} - \gamma_{mj} \partial_i \beta^m - \gamma_{im} \partial_j \beta^m$$

Decomposition of the Einstein equations

Definition:

$$\mathcal{L}_m \gamma_{ij} = -2\alpha K_{ij}$$

$\perp^\mu_\alpha \perp^\nu_\beta$ projection:

$$\mathcal{L}_m K_{ij} = -D_i D_j \alpha + \alpha (\mathcal{R}_{ij} + K K_{ij} - 2K_{im} K^m_j) + 8\pi\alpha \left[\frac{S_{-\rho}}{D-2} \gamma_{ij} - S_{ij} \right] - \frac{2}{D-2} \Lambda \gamma_{ij}$$

Evolution equations

$n^\mu n^\nu$ projection

$$\mathcal{R} + K^2 - K^{mn} K_{mn} = 2\Lambda + 16\pi\rho \quad \text{Hamiltonian constraint}$$

$\perp^\mu_\alpha n^\nu$ projection

$$D_i K - D_m K^m_i = -8\pi j_i \quad \text{Momentum constraint}$$

Well-posedness

- Consider a field ϕ evolved with a first-order system of PDEs
- The system has a **well posed** initial value formulation
 - ⇔ There exists some norm and a smooth function $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\|\phi(t)\| \leq F(\|\phi(0)\|, t) \|\phi(0)\|$
- Well-posed systems have **unique solutions** for given initial data
- There can still be fast growth, e.g. **exponential**
- **Strong hyperbolicity** is necessary for well-posedness
- The general ADM equations are only **weakly hyperbolic**
- Details depend on: **gauge, constraints, discretization**

Sarbach & Tiglio, Living Reviews Relativity **15** (2012) 9; Gundlach & Martín-García, PRD **74** (2006) 024016; Reula, gr-qc/0403007

The BSSN system

- Goal: modify ADM to get a strongly hyperbolic system

Baumgarte & Shapiro, PRD **59** (1998) 024007, Shibata & Nakamura, PRD **52** (1995) 5428

- Conformal decomposition, trace split, auxiliary variable

$$\phi = \frac{1}{4(D-1)} \ln \gamma, \quad K = \gamma^{ij} K_{ij}$$

$$\tilde{\gamma}_{ij} = e^{-4\phi} \Leftrightarrow \tilde{\gamma}^{ij} = e^{4\phi} \gamma^{ij}$$

$$\tilde{A}_{ij} = e^{-4\phi} \left(K_{ij} - \frac{1}{D-1} \gamma_{ij} K \right) \Leftrightarrow K_{ij} = e^{4\phi} \left(\tilde{A}_{ij} + \frac{1}{D-1} \tilde{\gamma}_{ij} K \right)$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{mn} \tilde{\Gamma}_{mn}^i$$

- Auxiliary constraints

$$\tilde{\gamma} = \det \tilde{\gamma}_{ij} = 1, \quad \tilde{\gamma}^{mn} \tilde{A}_{mn} = 0$$

The BSSN equations

$$\partial_t \phi = \beta^m \partial_m \phi - \frac{1}{2(D-1)} (\partial_m \beta^m - \alpha K)$$

$$\partial_t \tilde{\gamma} = \beta^m \partial_m \tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i} \partial_{i)} \beta^m - \frac{2}{D-1} \tilde{\gamma}_{ij} \partial_m \beta^m - 2\alpha \tilde{\mathbf{A}}_{ij}$$

$$\begin{aligned} \partial_t K &= \beta^m \partial_m K - e^{4\phi} \tilde{\gamma}^{mn} D_m D_n \alpha + \alpha \tilde{\mathbf{A}}^{mn} \tilde{\mathbf{A}}_{mn} + \frac{1}{D-1} \alpha K^2 \\ &\quad + \frac{8\pi}{D-2} \alpha [\mathbf{S} + (D-3)\rho] - \frac{2}{D-2} \alpha \Lambda \end{aligned}$$

$$\begin{aligned} \partial_t \tilde{\mathbf{A}}_{ij} &= \beta^m \partial_m \tilde{\mathbf{A}}_{ij} + 2\tilde{\mathbf{A}}_{m(i} \partial_{i)} \beta^m - \frac{2}{D-1} \tilde{\mathbf{A}}_{ij} \partial_m \beta^m + \alpha K \tilde{\mathbf{A}}_{ij} - 2\alpha \tilde{\mathbf{A}}_{im} \tilde{\mathbf{A}}^m_j \\ &\quad + e^{-4\phi} (\alpha \mathcal{R}_{ij} - D_i D_j \alpha - 8\pi \alpha \mathbf{S}_{ij})^{\text{TF}} \end{aligned}$$

$$\begin{aligned} \partial_t \tilde{\Gamma}^i &= \beta^m \partial_m \tilde{\Gamma}^i + \frac{2}{D-1} \tilde{\Gamma}^i \partial_m \beta^m + \tilde{\gamma}^{mn} \partial_m \partial_n \beta^i + \frac{D-3}{D-1} \tilde{\gamma}^{im} \partial_m \partial_n \beta^n \\ &\quad + 2\tilde{\mathbf{A}}^{im} [2(D-1)\alpha \partial_m \phi - \partial_m \alpha] + 2\alpha \tilde{\Gamma}^i_{mn} \tilde{\mathbf{A}}^{mn} - 2\frac{D-2}{D-1} \alpha \tilde{\gamma}^{im} \partial_m K - 16\pi \alpha j^i \end{aligned}$$

Note: There are alternative versions using $\chi = e^{-4\phi}$ or $W = e^{-2\phi}$

The BSSN equations

In the BSSN equations we use

$$\Gamma_{jk}^i = \tilde{\Gamma}_{jk}^i + 2(\delta^i_k \partial_j \phi + \delta^i_j \partial_k \phi - \tilde{\gamma}_{jk} \tilde{\gamma}^{im} \partial_m \phi)$$

$$\mathcal{R}_{ij} = \tilde{\mathcal{R}}_{ij} + \mathcal{R}_{ij}^\phi$$

$$\mathcal{R}_{ij}^\phi = 2(3-D)\tilde{D}_i \tilde{D}_j \phi - 2\tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \tilde{D}_m \tilde{D}_n \phi + 4(D-3)(\partial_i \phi \partial_j \phi - \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \phi \partial_n \phi)$$

$$\tilde{\mathcal{R}}_{ij} = -\frac{1}{2}\tilde{\gamma}^{mn} \partial_m \partial_n \tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i} \partial_j) \tilde{\Gamma}^m + \tilde{\Gamma}^m \tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn} [2\tilde{\Gamma}_{m(i}^k \tilde{\Gamma}_{j)kn} + \tilde{\Gamma}_{im}^k \tilde{\Gamma}_{kjn}]$$

$$D_i D_j \alpha = \tilde{D}_i \tilde{D}_j \alpha - 2(\partial_i \phi \partial_j \alpha + \partial_j \phi \partial_i \alpha) + 2\tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \phi \partial_n \alpha$$

The constraints are

$$\mathcal{H} = \mathcal{R} + \frac{D-2}{D-1} K^2 - \tilde{A}^{mn} \tilde{A}_{mn} - 16\pi\rho - 2\Lambda = 0$$

$$\mathcal{M}_i = \tilde{D}_m \tilde{A}^m_i - \frac{D-2}{D-1} \partial_i K + 2(D-1)\tilde{A}^m_i \partial_m \phi - 8\pi j_i = 0$$

2.1.2 Generalized Harmonic formulation

The Generalized Harmonic (GH) formulation

- **Harmonic gauge**: choose coordinates such that

$$\square x^\alpha = \nabla_\mu \nabla^\mu x^\alpha = -g^{\mu\nu} \Gamma_{\mu\nu}^\alpha = 0$$

- **4-dim.** version of Einstein equations

$$R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + \dots$$

Principal part of wave equation \Rightarrow **Manifestly hyperbolic**

- **Problem**: Start with spatial hypersurface $t = \text{const.}$

Does t remain timelike?

- **Solution**: **Generalize** harmonic gauge

Garfinkle, APS Meeting (2002) 12004, Pretorius, CQG **22** (2005) 425,
Lindblom et al, CQG **23** (2006) S447

\rightarrow **Source functions** $H^\alpha = \nabla_\mu \nabla^\mu x^\alpha = -g^{\mu\nu} \Gamma_{\mu\nu}^\alpha$

The Generalized harmonic formulation

- Any spacetime in any coordinates can be formulated in GH form!

Problem: find the corresponding H^α

- Promote H^α to evolution variables
- Einstein field equations in GH form:

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = -\partial_\nu g_{\mu(\alpha}\partial_{\beta)}g^{\mu\nu} - \partial_{(\alpha}H_{\beta)} + H_\mu\Gamma_{\alpha\beta}^\mu \\ - \Gamma_{\nu\alpha}^\mu\Gamma_{\mu\beta}^\nu - \frac{2}{D-2}\Lambda g_{\alpha\beta} - 8\pi\left(T_{\mu\nu} - \frac{1}{D-2}Tg_{\alpha\beta}\right)$$

with constraints

$$\mathcal{C}^\alpha = H^\alpha - \square x^\alpha = 0$$

- Still principal part of wave equation !!! Manifestly hyperbolic

Friedrich, Comm.Math.Phys. **100** (1985) 525, Garfinkle, PRD **65** (2002) 044029, Pretorius, CQG **22** (2005) 425

Constraint damping in the GH system

- One can show that

$$\mathcal{C}^\alpha|_{t=0} = 0, \quad \partial_t \mathcal{C}^\alpha|_{t=0} = 0 \Leftrightarrow \text{The ADM } \mathcal{H} = 0, \quad \mathcal{M}_i = 0$$

- Bianchi identities imply evolution of \mathcal{C}^α :

$$\square \mathcal{C}_\alpha = -\mathcal{C}^\mu \nabla_{(\mu} \mathcal{C}_{\alpha)} - \mathcal{C}^\mu \left[8\pi \left(T_{\mu\alpha} - \frac{1}{D-2} T g_{\mu\alpha} \right) + \frac{2}{D-2} \Lambda g_{\mu\alpha} \right]$$

- In practice: numerical violations of $\mathcal{C}^\mu = 0 \Rightarrow$ unstable modes

- Solution: add constraint damping

$$\begin{aligned} \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} &= -\partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma_{\alpha\beta}^\mu - \Gamma_{\nu\alpha}^\mu \Gamma_{\mu\beta}^\nu \\ &\quad - \frac{2}{D-2} \Lambda g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{D-2} T g_{\alpha\beta} \right) - \kappa \left[2n_{(\alpha} \mathcal{C}_{\beta)} - \lambda g_{\alpha\beta} n^\mu \mathcal{C}_\mu \right] \end{aligned}$$

Gundlach et al, CQG **22** (2005) 3767

- E.g. Pretorius, PRL **95** (2005) 121101: $\kappa = 1.25/m$, $\lambda = 1$

Summary GH formulation

- Specify initial data $g_{\alpha\beta}$, $\partial_t g_{\alpha\beta}$ at $t = 0$
which satisfy the constraints $\mathcal{C}^\mu = \partial_t \mathcal{C}^\mu = 0$
- Constraints **preserved** due to Bianchi identities
- Alternative first-order version of GH formulation

Lindblom et al, CQG **23** (2006) S447

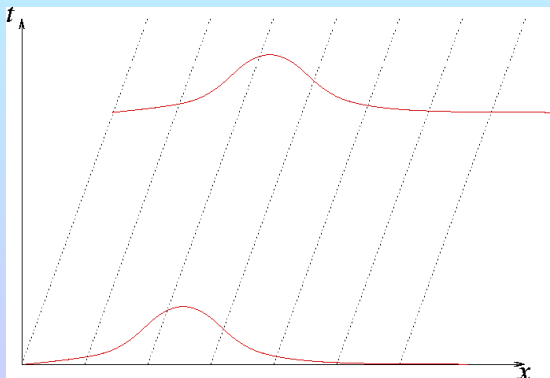
- Auxiliary variables \rightarrow First-order system
- Symmetric hyperbolic system
 \rightarrow **constraint-preserving boundary conditions**
- Used for spectral BH code **SpEC**

Caltech, Cornell, CITA

2.1.3 Characteristic formulation

Characteristic coordinates

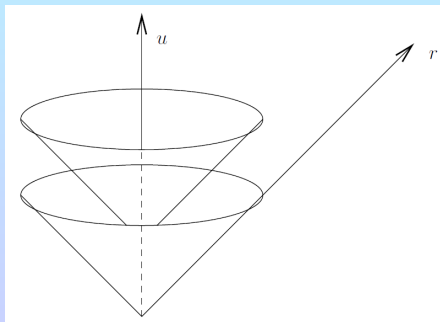
- Consider advection equation $\partial_t f + a \partial_x f = 0$
 - Characteristics: curves $\mathcal{C} : x \rightarrow at + x_0 \Leftrightarrow \frac{dx}{dt} = a$
- $$\frac{df}{dt}|_{\mathcal{C}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt}|_{\mathcal{C}} = \frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} = 0 \Rightarrow f \text{ constant along } \mathcal{C}$$



Characteristic “Bondi-Sachs” formulation

Here: $D = 4$, $\Lambda = 0$

- Foliate spacetime using characteristic surfaces; light cones
Bondi, Proc.Roy.Soc.A **269** (1962), 21; Sachs, Proc.Roy.Soc.A **270** (1962), 103
- “ $u = t - r$, $v = t + r$ ” → double null, ingoing or outgoing



outgoing null timelike foliation

Characteristic “Bondi-Sachs” formulation

- Write metric as

$$ds^2 = V \frac{e^{2\beta}}{r} du^2 - 2e^{2\beta} du dr + r^2 h_{AB} (dx^A - U^A du)(dx^B - U^B du)$$
$$2h_{AB} dx^A dx^B = (e^{2\gamma} + e^{2\delta}) d\theta^2 + 4 \sin \theta \sinh(\gamma - \delta) d\theta d\phi$$
$$+ \sin^2 \theta (e^{-2\gamma} + e^{-2\delta}) d\phi^2$$

- Introduce tetrad k , ℓ , m , \bar{m} such that

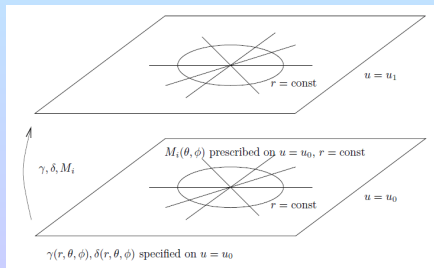
$$g(k, \ell) = 1, \quad g(m, \bar{m}) = 1 \text{ and all other products vanish}$$

- The Einstein equations become

- 4 hypersurface eqs.: $R_{\mu\nu} k^\mu k^\nu = R_{\mu\nu} k^\mu m^\nu = R_{\mu\nu} m^\mu \bar{m}^\nu = 0$
- 2 evolution eqs.: $R_{\mu\nu} m^\mu m^\nu = 0$
- 1 trivial eq.: $R_{\mu\nu} k^\mu \ell^\nu = 0$
- 3 supplementary eqs.: $R_{\mu\nu} \ell^\mu m^\nu = R_{\mu\nu} \ell^\mu \ell^\nu = 0$

Integration of the characteristic equations

- Provide initial data for γ , δ on hypersurface $u = \text{const}$
- Integrate hypersurface eqs. along $r \rightarrow \beta$, V , U^A at u
 \rightarrow 3 “constants” of integration $M_i(\theta, \phi)$
- Evolve γ , δ using evolution eqs.
 \rightarrow 2 “constants” of integration \rightarrow complex news $\partial_u c(u, \theta, \phi)$
- Evolve the M_i through the supplementary eqs.



Summary characteristic formulation

- Naturally adapted to the causal structure of GR
- Clear hierarchy of equations \rightarrow isolated degrees of freedom
- Problem: **caustics** \rightarrow breakdown of coordinates
- Well suited for symmetric spacetimes, planar BHs
- Solution for binary problem?

Recent investigation: Babiuc, Kreiss & Winicour, arXiv:1305.7179 [gr-qc]

- Application to characteristic **GW extraction**

Babiuc, Winicour & Zlochower, CQG **28** (2011) 134006

Reisswig et al, CQG **27** (2010) 075014

Direct methods

- Use symmetry to write line element, e.g.

$$ds^2 = -a^2(\mu, t)dt^2 + b^2(\mu, t)d\mu^2 - R^2(\mu, t)d\Omega^2$$

May & White, PR **141** (1966) 1232

- Energy momentum tensor

$$T^0_0 = -\rho(1 + \epsilon), \quad T^1_1 = T^2_2 = T^3_3 = 0 \quad \text{Lagrangian coords.}$$

- GRTENSOR, MATHEMATICA,...

⇒ Field equations:

$$a' = \dots$$

$$b' = \dots$$

$$\ddot{R} = \dots$$

Further reading

- 3+1 formalism

Gourgoulhon, gr-qc/0703035

- Characteristic formalism

Winicour, Liv. Rev. Rel. **15** 2012 2

- Numerical relativity in general

Alcubierre, "*Introduction to 3+1 Numerical Relativity*", Oxford University Press

Baumgarte & Shapiro, "*Numerical Relativity*", Cambridge University Press

- Well-posedness, Einstein eqs. as an Initial-Boundary-Value problem

Sarbach & Tiglio, Liv. Rev. Rel. **15** (2012) 9

2.2. NR beyond 4D

A list of tasks

- NR in 3+1 dimensions: $\mathcal{O}(100)$ cores, Gb of memory
- Each extra dimension can introduce a factor of $\mathcal{O}(100)$
 \Rightarrow reduce D to 3 + 1 dimensions; **symmetries**
- Three approaches:
 - **Dimensional reduction** to 3 + 1 GR + quasi matter
 - **CARTOON** type methods
 - Simplify **line element** using symmetry
- Outer boundary conditions: **regularization**, **background subtraction**

2.2.1 Dimensional reduction

Conventions

- Reduce D dimensions to d ; typically $d = 3 + 1 = 4$
- Indices
 - $A, B, C, \dots = 0 \dots D - 1$: D dimensional spacetime
 - $\alpha, \beta, \gamma, \dots = 0 \dots d - 1$: d dimensional base spacetime
 - $a, b, c, \dots = d \dots D - 1$: $D - d$ dimensional fibre
- Symmetry: $SO(D - d + 1)$
 \Rightarrow rotations in $D - d + 1$ space dimensions \Leftrightarrow on S^{D-d} sphere
Typically: $SO(D - 3)$ symmetry, S^{D-4} sphere

General formalism

Cho, Phys. Lett. B **186** (1987) 38

Cho & Kim, J. Math. Phys. **30** (1987) 1570

Zilhão, arXiv:1301.1509 [gr-qc]

The general D metric can be written

$$\begin{aligned} ds^2 &= g_{AB} dx^A dx^B \\ &= (g_{\mu\nu} + e^2 \kappa^2 g_{ab} B^a{}_{\mu} B^b{}_{\nu}) dx^{\mu} dx^{\nu} + 2e\kappa B^a{}_{\mu} g_{ab} dx^{\mu} dx^b + g_{ab} dx^a dx^b \end{aligned}$$

Comments

- e , κ are **coupling** and **scale** parameters; they'll eventually drop out
- This metric is **completely general!**
- We used a special case of this: ADM 3+1 decomposition!

General formalism

- Assumption: g_{AB} admits m Killing vectors $\xi^{(i)} = \xi^{(i)a} \partial_a$
 $\Rightarrow \mathcal{L}_{\xi^{(i)}} g_{AB} = 0$
- Def.: dual form $\xi^{(j)} = \xi_a^{(j)} dx^a$ such that $\xi_a^{(j)} \xi^{(i)a} = \delta_j^i$
- Def.: $F^a{}_{bc} \equiv -\xi_b^{(i)} \partial_c \xi^{(i)a}$
- Then $\Rightarrow \mathcal{L}_{\xi^{(i)}} g_{AB} = 0$ implies
$$\partial_a g_{bc} = F^d{}_{ab} g_{dc} + F^d{}_{ac} g_{db}$$
$$\partial_a B_\mu^b = -F^b{}_{ad} B_\mu^d$$
$$\partial_a g_{\mu\nu} = 0$$

General formalism

$$\text{Def.: } D_\mu \equiv \partial_\mu - e\kappa B_\mu^d \partial_d$$

$$\mathcal{F}^a{}_{\mu b} \equiv e\kappa \partial_b B_\mu^a = -e\kappa F^a{}_{bc} B_\mu^c$$

$$\mathcal{F}^a{}_{\mu\nu} \equiv -e\kappa [\partial_\mu B_\nu^a - \partial_\nu B_\mu^a + e\kappa (F^a{}_{bc} - F^a{}_{cb})]$$

The covariant derivatives are defined as

$$\nabla_\sigma T^{a\mu}{}_{b\nu} \equiv D_\sigma T^{a\mu}{}_{b\nu} + \mathcal{F}^a{}_{\sigma c} T^{c\mu}{}_{b\nu} - \mathcal{F}^c{}_{\sigma b} T^{a\mu}{}_{c\nu} + \Gamma_{\lambda\sigma}^\mu T^{a\lambda}{}_{b\nu} - \Gamma_{\nu\sigma}^\lambda T^{a\mu}{}_{b\lambda}$$

$$\nabla_c T^{a\mu}{}_{b\nu} \equiv \partial_c T^{a\mu}{}_{b\nu} + \Gamma_{dc}^a T^{d\mu}{}_{b\nu} - \Gamma_{bc}^d T^{a\mu}{}_{d\nu}$$

Here, $\Gamma_{\lambda\sigma}^\mu$ and Γ_{dc}^a are the connections associated with $g_{\mu\nu}$ and g_{ab} .

Note: $\nabla_\sigma g_{\mu\nu} = \nabla_c g_{ab} = 0$, but $\nabla_\sigma g_{ab} \neq 0!$

General formalism

A tedious but straightforward calculation gives us the Ricci tensor as

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4}g^{cd}\nabla_{\mu}g_{cd}\nabla^{\mu}g_{ab} + \frac{1}{2}g^{cd}\nabla_{\mu}g_{ac}\nabla^{\mu}g_{bd} \\ + \frac{1}{4}g^{\mu\nu}g^{\rho\sigma}g_{bc}g_{ad}\mathcal{F}^c_{\rho\nu}\mathcal{F}^d_{\sigma\mu} - \frac{1}{2}\nabla^{\mu}\nabla_{\mu}g_{ab}$$

$$R_{\mu a} = e\kappa R_{ac}B^c_{\mu} + \frac{1}{2}g^{\rho\sigma}\nabla_{\rho}(g_{ac}\mathcal{F}^c_{\sigma\mu}) + \frac{1}{4}g^{cd}\nabla_{\rho}g_{cd}g^{\rho\sigma}\mathcal{F}^e_{\sigma\mu}g_{ae} \\ + \frac{1}{2}\nabla_c(g^{cd}\nabla_{\mu}g_{da})$$

$$R_{\mu\nu} = \mathcal{R}_{\mu\nu} + 2e\kappa B^c_{(\mu}R_{\nu)c} - e^2\kappa^2 R_{cd}B^c_{\mu}B^d_{\nu} - \frac{1}{2}g^{\rho\sigma}g_{cd}\mathcal{F}^c_{\sigma\mu}\mathcal{F}^d_{\rho\nu} \\ - \frac{1}{2}\nabla_{\nu}(g^{cd}\nabla_{\mu}g_{cd}) - \frac{1}{4}g^{cd}g^{ab}\nabla_{\mu}g_{ca}\nabla_{\nu}g_{db} - \frac{1}{2}\nabla_c\mathcal{F}^c_{\mu\nu}$$

$$R = \mathcal{R}(g_{\mu\nu}) + \mathcal{R}(g_{ab}) - \frac{1}{4}g_{cd}g^{\rho\sigma}g^{\mu\nu}\mathcal{F}^d_{\sigma\mu}\mathcal{F}^c_{\rho\nu} - \nabla^{\mu}(g^{cd}\nabla_{\mu}g_{cd}) \\ - \frac{1}{4}g^{ca}g^{bd}\nabla^{\mu}g_{cd}\nabla_{\mu}g_{ab} - \frac{1}{4}g^{cd}g^{ab}\nabla^{\mu}g_{cd}\nabla_{\mu}g_{ab}$$

Case: $SO(D - d + 1)$ symmetry, $\Leftrightarrow S^{D-d}$ sphere

In practice: $SO(D - 3)$, S^{D-4} sphere (e.g. $D = 6 \Rightarrow SO(3)$, S^2)

- S^n sphere: $(n + 1)n/2$ Killing vectors $\xi_{(i)}$
- E.g. S^2 sphere: ∂_ϕ , $\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi$, $\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi$
Rotations around x , y , z axes
- Killing's equation $\mathcal{L}_{\xi_{(i)}} g_{AB} = 0$ implies
 $\mathcal{L}_{\xi_{(i)}} g_{ab} = 0$, $\mathcal{L}_{\xi_{(i)}} B_\mu^a = 0$, $\mathcal{L}_{\xi_{(i)}} g_{\mu\nu} = 0$
- Consequences:
 - $g_{ab} = e^{2\psi(x^\mu)} h_{ab}$ with $h_{ab} =$ metric on S^{D-d} with unit radius
 - $g_{\mu\nu} = g_{\mu\nu}(x^\mu)$ in adapted coordinates
 - $[\xi_{(i)}, B_\mu] = 0$ for $n \geq 2$ (only vector field commuting with all KVs: 0)
 $\Rightarrow B_\mu^a = 0$

Case: $SO(D - d + 1)$ symmetry, $\Leftrightarrow S^{D-d}$ sphere

With these consequences we get

- $R_{ab} = \{(D - d - 1) - e^{2\psi} [(D - d)\partial^\mu\psi \partial_\mu\psi + \nabla^\mu\partial_\mu\psi]\} h_{ab}$

$$R_{\mu a} = 0$$

$$R_{\mu\nu} = \mathcal{R}_{\mu\nu} - (D - d)(\nabla_\nu\partial_\mu\psi - \partial_\mu\psi \partial_\nu\psi)$$

$$R = \mathcal{R} + (D - d) [(D - d - 1)e^{-2\psi} - 2\nabla^\mu\partial_\nu\psi - (D - d + 1)\partial^\mu\psi \partial_\mu\psi]$$

- The D dimensional vacuum Einstein equations $R_{\mu\nu}$ thus become

$$e^{2\psi} [(D - d)\partial^\mu\psi \partial_\mu\psi + \nabla^\mu\partial_\mu\psi] = (D - d - 1)$$

$$\mathcal{R}_{\mu\nu} = (D - d)(\nabla_\nu\partial_\mu\psi - \partial_\mu\psi \partial_\nu\psi)$$

i.e. the d dimensional Einstein equations plus quasi-matter terms

Regularity of the variables

- Note: One of the $d - 1$ spatial coordinates x, y, z, \dots is a **radius**
Without loss of generality we choose y
 \Rightarrow Computational domain: $x, z, \dots \in \mathbb{R}, y \geq 0$
- Analysing our equations for some analytically known data, e.g. **Brill-Lindquist**, shows that $e^{2\psi} = 0$ at $y = 0$
- Solution: Use instead $\zeta = \frac{e^{-4\phi}}{y^2} e^{2\psi}$
where $e^{-4\phi}$ is the BSSN conformal factor.
- With that we get the BSSN equations with matter terms...

BSSN source terms for d=4

Note: We set $d = 4$, $i, j, \dots = 1, 2, 3$ and use $\chi \equiv e^{-4\phi}$

$$\begin{aligned} \frac{4\pi(\rho+S)}{D-4} &= (D-5) \frac{\chi}{\zeta} \frac{\tilde{\gamma}^{yy}\zeta-1}{y^2} - \frac{2D-7}{4\zeta} \tilde{\gamma}^{mn} \partial_m \eta \partial_n \chi - \chi \frac{\tilde{\Gamma}^y}{y} + \frac{D-6}{4} \frac{\chi}{\zeta^2} \tilde{\gamma}^{mn} \partial_m \zeta \partial_n \zeta \\ &+ \frac{1}{2\zeta} \tilde{\gamma}^{mn} (\chi \tilde{D}_m \partial_n \zeta - \zeta \tilde{D}_m \partial_n \chi) + (D-4) \frac{\tilde{\gamma}^{ym}}{y} \left(\frac{\chi}{\zeta} \partial_m \zeta - \partial_m \chi \right) \\ &- \frac{K K_\zeta}{\zeta} - \frac{K^2}{3} - \frac{1}{2} \frac{\tilde{\gamma}^{ym}}{y} \partial_m \chi + \frac{D-1}{4} \tilde{\gamma}^{mn} \frac{\partial_m \chi}{\chi} \frac{\partial_n \chi}{\chi} - (D-5) \left(\frac{K_\zeta}{\zeta} + \frac{K}{3} \right)^2 \end{aligned}$$

$$\begin{aligned} \frac{8\pi\chi S_{ij}^{\text{TF}}}{D-4} &= \frac{1}{2} \left[\frac{2\chi}{y\zeta} (\delta^y_{(j} \partial_{i)} \zeta - \zeta \tilde{\Gamma}^y_{ij}) + \frac{1}{2\chi} \partial_i \chi \partial_j \chi - \tilde{D}_i \partial_j \chi + \frac{\chi}{\zeta} \tilde{D}_i \partial_j \zeta \right. \\ &+ \left. \frac{1}{2\chi} \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_n \chi \left(\partial_m \chi - \frac{\chi}{\zeta} \partial_m \zeta \right) - \tilde{\gamma}_{ij} \frac{\tilde{\gamma}^{ym}}{y} \partial_m \chi - \frac{\chi}{2\zeta^2} \partial_i \zeta \partial_j \zeta \right]^{\text{TF}} \\ &- \left(\frac{K_\zeta}{\zeta} + \frac{K}{3} \right) \tilde{A}_{ij} \end{aligned}$$

BSSN source terms for d=4

$$\frac{16\pi j_i}{D-4} = \frac{2}{y} \left(\delta^y_i \frac{K_\zeta}{\zeta} - \tilde{\gamma}^{ym} \tilde{A}_{mi} \right) + \frac{2}{\zeta} \partial_i K_\zeta - \frac{K_\zeta}{\zeta} \left(\frac{1}{\chi} \partial_i \chi + \frac{1}{\zeta} \partial_i \zeta \right) + \frac{2}{3} \partial_i K$$

$$- \tilde{\gamma}^{nm} \tilde{A}_{mi} \left(\frac{1}{\zeta} \partial_n \zeta - \frac{1}{\chi} \partial_n \chi \right)$$

The matter evolution is given by

$$\partial_t \zeta = \beta^m \partial_m \zeta - 2\alpha K_\zeta - \frac{2}{3} \zeta \partial_m \beta^m + 2\zeta \frac{\beta^y}{y}$$

$$\begin{aligned} \partial_t K_\zeta = & \beta^m \partial_m K_\zeta - \frac{2}{3} K_\zeta \partial_m \beta^m + 2 \frac{\beta^y}{y} K_\zeta - \frac{1}{3} \zeta (\partial_t - \mathcal{L}_\beta) K - \chi \zeta \frac{\tilde{\gamma}^{ym}}{y} \partial_m \alpha \\ & - \frac{1}{2} \tilde{\gamma}^{mn} \partial_m \alpha (\chi \partial_n \zeta - \zeta \partial_n \chi) + \alpha \left[(5 - D) \chi \frac{\zeta \tilde{\gamma}^{yy} - 1}{y^2} + (4 - D) \chi \frac{\tilde{\gamma}^{ym}}{y} \partial_m \zeta \right. \\ & + \frac{2D-7}{2} \zeta \frac{\tilde{\gamma}^{ym}}{y} \partial_m \chi + \frac{6-D}{4} \frac{\chi}{\zeta} \tilde{\gamma}^{mn} \partial_m \zeta \partial_n \zeta + \frac{2D-7}{4} \tilde{\gamma}^{mn} \partial_m \zeta \partial_n \chi \\ & + \frac{1-D}{4} \frac{\zeta}{\chi} \tilde{\gamma}^{mn} \partial_m \chi \partial_n \chi + (D-6) \frac{K_\zeta^2}{\zeta} + \frac{2D-5}{3} K K_\zeta + \frac{D-1}{9} \zeta K^2 \\ & \left. + \frac{1}{2} \tilde{\gamma}^{mn} (\zeta \tilde{D}_m \partial_n \chi - \chi \tilde{D}_m \partial_n \zeta) + \chi \zeta \frac{\tilde{\Gamma}^y}{y} \right] \end{aligned}$$

Regularization at $y = 0$

- Note that the previous equations contain divisions by y ,
E.g. : $\frac{\beta^y}{y}, \frac{\tilde{\gamma}^{ym}}{y} \partial_m \zeta, \dots$

- These can all be **regularized!**

- E.g. : Symmetry of a vector across $y = 0$ implies

$$\beta^y(-y) = -\beta^y(y)$$

We can therefore Taylor expand β^y around $y = 0$ as

$$\beta^y(y) = b_1 y + \mathcal{O}(y^2)$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{\beta^y}{y} = b_1 = \partial_y \beta^y$$

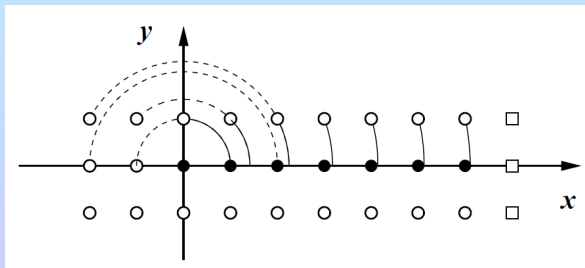
- Similar tricks work for all such terms

see Zilhão et al, PRD **81** (2010) 084052

2.2.2 CARTOON methods

Dimensional reduction through CARTOON

- Originally developed for **axisymmetry** around z in 3+1 dimensions
Alcubierre et al, IJMPD **10** (2001) 273
- Coordinates $(z, x, y) \leftrightarrow (z, \rho, \theta)$ where $x = \rho \cos \phi$, $y = \rho \sin \phi$
- Killing vector $\partial_\phi = x\partial_y - y\partial_x$
- Extend 2D grid by ghostzones for derivatives; **rotate**, **interpolate**



Dimensional reduction through CARTOON

Yoshino & Shibata. PRD **80** (2009) 084025

- **Scalar:** $\psi(z, x, y) = \psi(z, \rho, 0)$
- **Vector:** Express $v^x(z, x, y)$, $v^y(z, x, y)$ through $v^\rho(z, \rho)$, $v^\phi(z, \rho)$ and replace those through the relation on the xy plane

$$v^x(z, \rho, 0) = v^\rho(z, \rho), \quad v^y(z, \rho, 0) = \rho v^\phi(z, \rho)$$

$$\Rightarrow v^x(z, x, y) = \frac{x}{\rho} v^x(z, \rho, 0) - \frac{y}{\rho} v^y(z, \rho, 0)$$

$$v^y(z, x, y) = \frac{y}{\rho} v^x(z, \rho, 0) + \frac{x}{\rho} v^y(z, \rho, 0)$$

- **Likewise for tensors:** T_{zz} like scalar, T_{zx} , T_{zy} like vector

$$T_{xx}(z, x, y) = \left(\frac{x}{\rho}\right)^2 T_{xx}(z, \rho, 0) + \left(\frac{y}{\rho}\right)^2 T_{yy}(z, \rho, 0) - \frac{2xy}{\rho^2} T_{xy}(z, \rho, 0)$$

$$T_{yy}(z, x, y) = \left(\frac{y}{\rho}\right)^2 T_{xx}(z, \rho, 0) + \left(\frac{x}{\rho}\right)^2 T_{yy}(z, \rho, 0) + \frac{2xy}{\rho^2} T_{xy}(z, \rho, 0)$$

$$T_{xy}(z, x, y) = \frac{xy}{\rho^2} [T_{xx}(z, \rho, 0) - T_{yy}(z, \rho, 0)] + \frac{x^2 - y^2}{\rho^2} T_{xy}(z, \rho, 0)$$

CARTOON in D=5 with $SO(3)$ symmetry

- Cartesian coordinates: (w, x, y, z)
each hypersurface $w = \text{const}$ is spher. symmetric
- 3 Killing vectors
 $\xi_1 = y\partial_z - z\partial_y$, $\xi_2 = z\partial_x - x\partial_z$, $\xi_3 = x\partial_y - y\partial_x$
- Use data in xw plane, set $r = \sqrt{x^2 + y^2 + z^2}$
- Scalar: $\psi(w, x, y, z) = \psi(w, r, 0, 0)$
- Vector, Tensor fields: ...
cf. Yoshino & Shibata, PRD **80** (2009) 084025
- \Rightarrow effective 2+1 Cartesian code

A modified CARTOON method

Shibata & Yoshino, PRD **81** (2010) 104035

Yoshino & Shibata, PTPS **189** (2011) 269

- For larger D , CARTOON ghostzones require considerable memory
- Solution: Trade derivatives

- Coordinates: (x, y, z, w_i) , $i = 1 \dots D - 4$, $\rho = \sqrt{z^2 + \sum_i w_i^2}$

- Symmetry: $SO(D - 3)$, i.e. Rotations in w_i

- Scalar: $\psi(x, y, z, w_i) = \psi(x, y, \rho, 0)$

$$\Rightarrow \partial_{w_i} \psi = \partial_{(x,y)} \partial_{w_i} \psi = \partial_z \partial_{w_i} = 0, \quad \partial_{w_i} \partial_{w_j} \psi = \frac{\partial_z \psi}{z} \delta_{ij}$$

where (x, y) stands for either x or y

A modified CARTOON method

- **Vector:** $\beta^z(x, y, z, w_i) = \frac{z}{\rho} \beta^z(x, y, \rho, 0)$

$$\beta^{w_i}(x, y, z, w_i) = \frac{w_i}{\rho} \beta^z(x, y, \rho, 0)$$

$$\begin{aligned} \Rightarrow \partial_{w_i} \beta^z &= \partial_{(x,y)} \beta^{w_i} = \partial_z \beta^{w_i} = \partial_{(x,y)} \partial_{w_i} \beta^z = \partial_z \partial_{w_i} \beta^z \\ &= \partial_{(x,y)} \partial_{(x,y)} \beta^{w_i} = \partial_{(x,y)} \partial_z \beta^{w_i} = \partial_{w_j} \partial_{w_k} \beta^{w_i} = 0 \end{aligned}$$

$$\partial_{w_j} \beta^{w_i} = \frac{\beta^z}{z} \delta_{ij}, \quad \partial_{(x,y)} \partial_{w_j} \beta^{w_i} = \frac{\partial_{(x,y)} \beta^z}{z} \delta_{ij},$$

$$\partial_{w_i} \partial_{w_j} \beta^z = \partial_z \partial_{w_j} \beta^{w_i} = \left(\frac{\partial_z \beta^z}{z} - \frac{\beta^z}{z^2} \right) \delta_{ij}$$

A modified CARTOON method

- Tensors:

$$T_{zz}(x, y, z, w_i) = \frac{z^2}{\rho^2} T_{zz}(x, y, \rho, 0) + \left(1 - \frac{z^2}{\rho^2}\right) T_{ww}(x, y, \rho, 0)$$

$$T_{w_i w_i}(x, y, z, w_i) = \frac{w_i^2}{\rho^2} T_{zz}(z, y, \rho, 0) + \left(1 - \frac{w_i^2}{\rho^2}\right) T_{ww}(x, y, \rho, 0)$$

$$T_{zw_i}(x, y, z, w_i) = \frac{zw_i}{\rho^2} [T_{zz}(x, y, \rho, 0) - T_{ww}(x, y, \rho, 0)]$$

$$T_{w_i w_j}(x, y, z, w_i) = \frac{w_i w_j}{\rho^2} [T_{zz}(x, y, \rho, 0) - T_{ww}(x, y, \rho, 0)]$$

where $T_{ww} \equiv T_{w_1 w_1} = T_{w_2 w_2} = \dots$ which are all equal

$$\begin{aligned} \Rightarrow \partial_{w_i} T_{zz} &= \partial_{w_j} T_{w_i w_i} = \partial_{(x,y)} T_{zw_i} = \partial_z T_{zw_i} = \partial_{(x,y)} \partial_{w_i} T_{zz} \\ &= \partial_z \partial_{w_i} T_{zz} = \partial_{(x,y)} \partial_{w_j} T_{w_i w_i} = \partial_z \partial_{w_j} = \partial_{(x,y)} \partial_{(x,y)} T_{zw_i} \\ &= \partial_{(x,y)} \partial_z T_{zw_i} = \partial_z \partial_{w_i} T_{zz} = 0 \end{aligned}$$

A modified CARTOON method

- Tensors continued: for $i \neq j$ we also have

$$\begin{aligned}\partial_{(x,y)} T_{w_i w_j} &= \partial_z T_{w_i w_j} = \partial_{w_k} T_{w_i w_j} = \partial_{(x,y)} \partial_{(x,y)} T_{w_i w_j} = \partial_{(x,y)} \partial_z T_{w_i w_j} \\ &= \partial_z \partial_z T_{w_i w_j} = \partial_{(x,y)} \partial_{w_k} T_{w_i w_j} = \partial_z \partial_{w_k} T_{w_i w_j} = 0\end{aligned}$$

- The non-zero derivatives appearing in the BSSN eqs. are

$$\partial_{w_j} T_{zw_i} = \frac{T_{zz} - T_{ww}}{z} \delta_{ij}, \quad \partial_{(x,y)} \partial_{w_j} T_{zw_i} = \frac{\partial_{(x,y)} T_{zz} - \partial_{(x,y)} T_{ww}}{z} \delta_{ij},$$

$$\partial_{w_i} \partial_{w_j} \frac{1}{z} \left[\partial_z T_{zz} + \frac{2(T_{ww} - T_{zz})}{z} \right] \delta_{ij},$$

$$\partial_{w_j w_k} T_{w_i w_i} = \frac{2(T_{zz} - T_{ww})}{z^2} \delta_{ik} \delta_{ij} + \frac{\partial_z T_{ww}}{z} \delta_{jk},$$

$$\partial_z \partial_{w_j} T_{zw_i} = \frac{1}{z} \left[\partial_z T_{zz} - \partial_z T_{ww} - \frac{T_{zz} - T_{ww}}{z} \right] \delta_{ij},$$

$$\partial_{w_k} \partial_{w_l} T_{w_i w_j} = \frac{T_{zz} - T_{ww}}{z^2} (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \quad \text{for } i \neq j$$

A modified CARTOON method

- Plugging these into the D dim. equations enables us to work on a genuine d dim. hypersurface with no ghost zones
- Note: There are additional fields, e.g. T_{zw_j} , but these are only required on the (xyz) hyper plane!
- Note: There are divisions by z
⇒ regularization at $z = 0$ required!
cf. $y = 0$ in the dimensional reduction

Further reading

- Dimensional reduction

Zilhão, arXiv:1301.1509

- Modified Cartoon

Yoshino & Shibata, PTPS **189** (2011) 269

2.3. Initial data, Gauge, Boundaries

2.3.1. Initial data

Analytic initial data

- Schwarzschild, Kerr, Tangherlini, Myers Perry,...

e.g. Schwarzschild in isotropic coordinates:

$$ds^2 = -\frac{M-2r}{M+2r} dt^2 + \left(1 + \frac{M}{2r}\right) [r^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

- Time symmetric N BH initial data: Brill-Lindquist, Misner 1960s
- Problem: Finding initial data for dynamic systems
- Goals
 - 1) Solve constraints
 - 2) Realistic snapshot of physical system
- This is mostly done using the **ADM 3+1** split

The York-Lichnerowicz split

- We work in $D = 4$
- Conformal metric: $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$
Lichnerowicz, J.Math.Pures Appl. **23** (1944) 37
York, PRL **26** (1971) 1656, PRL **28** (1972) 1082
- Note: in contrast to BSSN we do not set $\bar{\gamma} = 1$
- Conformal traceless split of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K$$

$$A^{ij} = \psi^{-10}\bar{A}_{ij} \Leftrightarrow A_{ij} = \psi^{-2}\bar{A}_{ij}$$

Bowen-York data

- By further splitting \bar{A}_{ij} into a longitudinal and a transverse traceless part, the momentum constraint simplifies significantly
Cook, Living Review Relativity (2000) 05
- Further assumptions: **vacuum**, $K = 0$, $\bar{\gamma}_{ij} = f_{ij}$, $\psi|_{\infty} = 1$
where f_{ij} is the flat metric in arbitrary coordinates.
Conformal flatness, asymptotic flatness, traceless
- Then there exists an analytic solution to the momentum constraint

$$\bar{A}_{ij} = \frac{3}{2r^2} [P_i n_j + P_j n_i - (f_{ij} - n_i n_j) P^k n_k] \\ + \frac{3}{r^3} (\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i)$$

where r is a coordinate radius and $n^i = \frac{x^i}{r}$

Bowen & York, PRD **21** (1980) 2047

Properties of the Bowen York solution

- The momentum in an asymptotically flat hypersurface associated with the asymptotic translational and rotational Killing vectors $\xi_{(a)}^i$ is

$$\Pi^i = \frac{1}{8\pi} \oint_{\infty} (K^j_i - \delta^j_i K) \xi_{(a)}^i d^2 A_j$$

$\Rightarrow \dots \Rightarrow P^i$ and S^i are the physical **linear** and **angular momentum** of the spacetime

- The momentum constraint is **linear**
 \Rightarrow we can **superpose** Bowen-York data.

The momenta then simply add up

- Bowen-York data generalizes (analytically!) to higher D
Yoshino, Shiromizu & Shibata, PRD **74** (2006) 124022

Puncture data

Brandt & Brügmann, PRL **78** (1997) 3606

- The Hamiltonian constraint is now given by

$$\bar{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \bar{A}_{mn} \bar{A}^{mn} = 0$$

- Ansatz for conformal factor: $\psi = \psi_{\text{BL}} + u$,
where $\psi_{\text{BL}} = \sum_{i=1}^N \frac{m_i}{2|\vec{r} - \vec{r}_i|}$ is the Brill-Lindquist conformal factor,
i.e. the solution for $\bar{A}_{ij} = 0$.

- There then exist unique C^2 solutions u to the Hamiltonian constraints
- The Hamiltonian constraint in this form is further suitable for numerical solution

e.g. Ansorg, Brügmann & Tichy, PRD **70** (2004) 064011

Properties of the puncture solutions

- m_i and \vec{r}_i are bare mass and position of the i^{th} BH.
- In the limit of vanishing Bowen York parameters $P^i = S^i = 0$, the puncture solution reduces to Brill Lindquist data

$$\gamma_{ij} dx^i dx^j = \left(1 + \sum_i \frac{m_i}{2|\vec{r} - \vec{r}_i|}\right)^4 (dx^2 + dy^2 + dz^2)$$

- The numerical solution of the Hamiltonian constraint generalizes rather straightforwardly to higher D

Yoshino, Shiromizu & Shibata, PRD **74** (2006) 124022

Zilhão et al, PRD **84** (2011) 084039

- Punctures generalize to asymptotically de-Sitter BHs

Zilhão et al, PRD **85** (2012) 104039

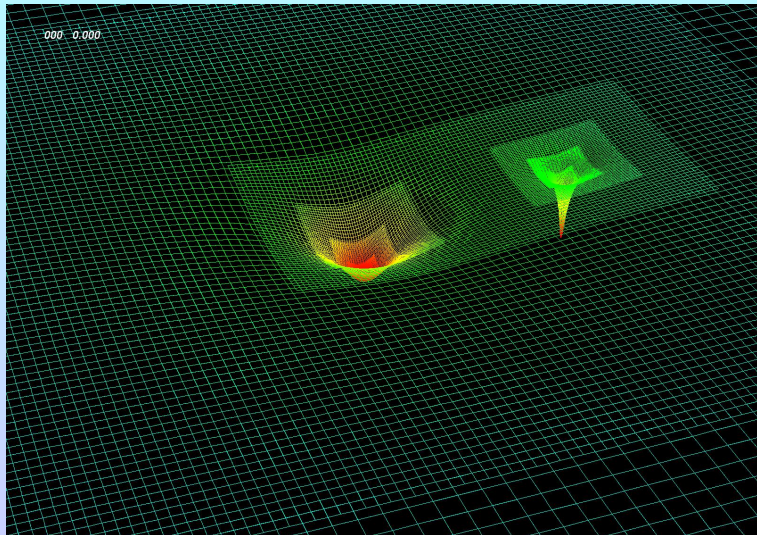
using McVittie coordinates McVittie, MNRAS **93** (1933) 325

2.3.2. Gauge

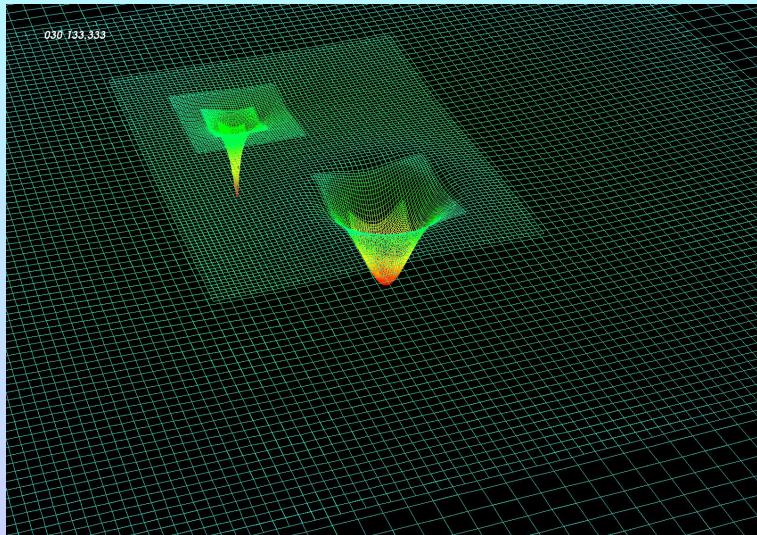
The gauge freedom

- Remember: Einstein equations say nothing about α , β^i
- Any choice of lapse and shift gives a solution
- This represents the coordinate freedom of GR
- Physics do not depend on α , β^i
So why bother?
- The performance of the numerics DO depend strongly on the gauge!

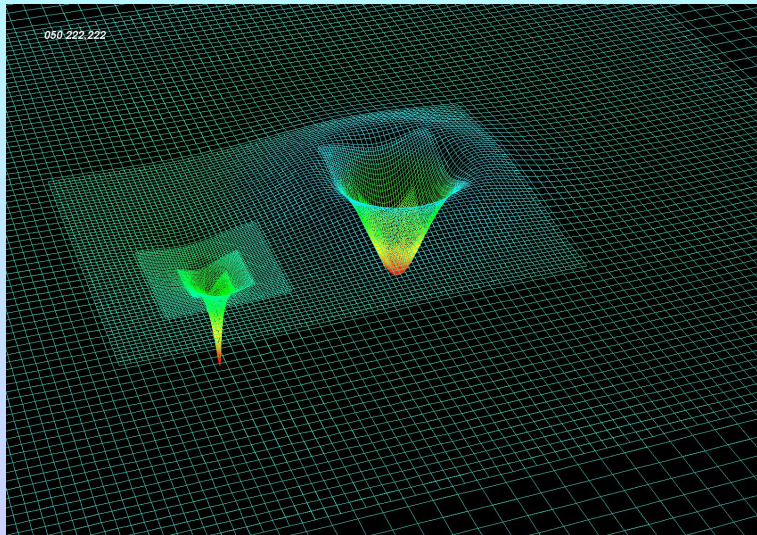
What goes wrong with bad gauge?



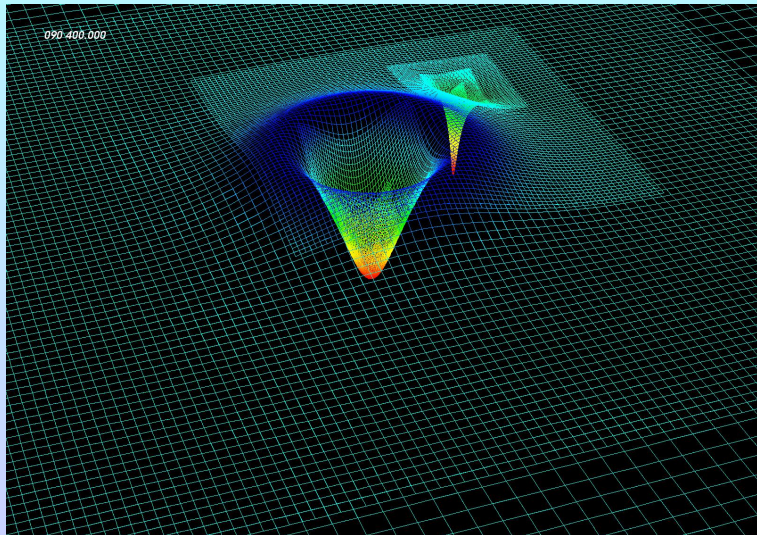
What goes wrong with bad gauge?



What goes wrong with bad gauge?



What goes wrong with bad gauge?



Ingredients for good gauge

- Singularity avoidance
- Avoid slice stretching
- Aim at stationarity in comoving frame
- Well posedness of system
- Generalize “good” gauge, e .g. harmonic
- Lots of good luck!

Bona et al, PRL **75** (1995) 600,

Alcubierre *et al.*, PRD **67** (2003) 084023,

Alcubierre, CQG **20** (2003) 607,

Garfinkle, PRD **65** (2001) 044029

Moving puncture gauge

- Gauge was a key ingredient in the **Moving puncture** breakthroughs

Campanelli et al, PRL **96** (2006) 111101

Baker et al, PRL **96** (2006) 111102

- Variant of 1 + log slicing and Γ -driver shift

Alcubierre et al, PRD **67** (2003) 084023

- Now in use as

$$\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha K$$

and

$$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} B^i$$

$$\partial_t B^i = \beta^m \partial_m B^i + \partial_t \tilde{\Gamma}^i - \beta^m \partial_m \tilde{\Gamma}^i - \eta B^i$$

or

$$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} \tilde{\Gamma}^i - \eta \beta^i$$

Moving puncture gauge continued

- Some people drop the advection derivatives $\beta^m \partial_m \dots$
- η is a damping parameter or position-dependent function

Alic et al, CQG **27** (2010) 245023, Schnetter, CQG **27** (2010) 167001, Müller et al, PRD **82** (2010) 064004

- Modifications in higher D :
 - Dimensional reduction Zilhão et al, PRD **81** (2010) 084052

$$\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha(\eta_K K + \eta_{K_\zeta} K_\zeta)$$

- CARTOON Yoshino & Shibata, PTPS **189** (2011) 269

$$\partial_t \beta^i = \frac{D-1}{2(D-2)} v_{\text{long}}^2 B^i$$

$$\partial_t B^i = \partial_t \tilde{\Gamma}^i - \eta B^i$$

- Here $\eta_K, \eta_{K_\zeta}, v_{\text{long}}$ are parameters

Gauge conditions in the GH formulation

- How to choose H_μ ? \rightarrow some experimentation...

- Pretorius' breakthrough

$$\square H_t = -\xi_1 \frac{\alpha-1}{\alpha^\eta} + \xi_2 n^\mu \partial_\mu H_t \text{ with}$$

$$\xi_1 = 19/m, \quad \xi_2 = 2.5/m, \quad \eta = 5 \quad \text{where } m = \text{mass of 1 BH}$$

- Caltech-Cornell-CITA spectral code:

Initialize H_α to minimize time derivatives of metric,

adjust H_α to harmonic and damped harmonic gauge condition

Lindblom & Szilágyi, PRD **80** (2009) 084019, with Scheel, PRD **80** (2009) 124010

- The H_α are related to lapse and shift: $n^\mu H_\mu = -K - n^\mu \partial_\mu \ln \alpha$

$$\gamma^{\mu i} H_\mu = -\gamma^{mn} \Gamma_{mn}^i + \gamma^{im} \partial_m (\ln \alpha) + \frac{1}{\alpha} n^\mu \partial_\mu \beta^i$$

2.3.3. Boundaries

Inner boundary: Singularity treatment

- Cosmic censorship \Rightarrow horizon protects outside

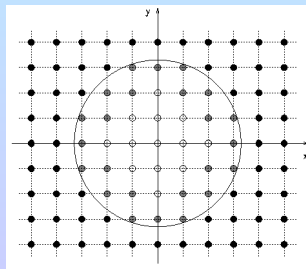
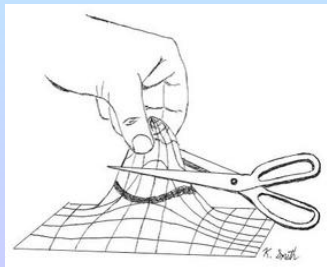
- We get away with it...

Moving Punctures

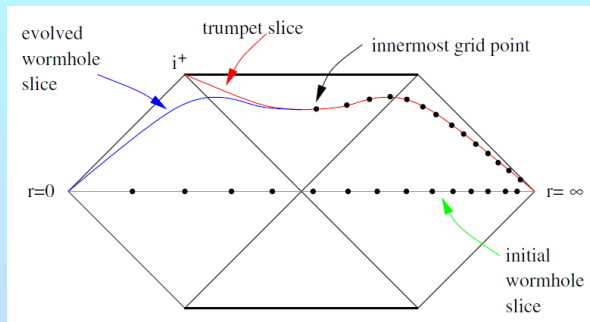
UTB, NASA Goddard '05

- Excision: Cut out region around singularity

Caltech-Cornell, Pretorius



Moving puncture slices: Schwarzschild



- Wormhole \rightarrow Trumpet slice = stationary $1+\log$ slice

Hannam et al, PRL **99** (2007) 241102, PRD **78** (2008) 064020

Brown, PRD **77** (2008) 044018, CQG **25** (2008) 205004

- Gauge might propagate at $> c$, no pathologies

Natural excision Brown, PRD **80** (2009) 084042

Outer boundary: Asymptotically flat case

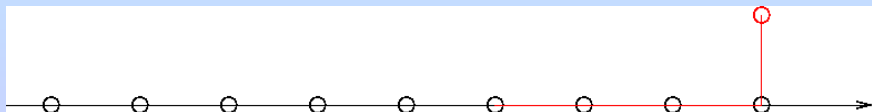
- Computational domains often don't extend to ∞
- Outgoing Sommerfeld conditions

Assume $f = f_0 + \frac{u(t-r)}{r^n}$ where $f_0 =$ asymptotic value

$$\partial_t u + \partial_r u = 0$$

$$\partial_t f + n \frac{f-f_0}{r} + \frac{x^i}{r} \partial_i f = 0$$

- Use upwinding, i.e. one-sided, derivatives!



Non-asymptotically flat case: de Sitter

- In McVittie coordinates:

$$r \rightarrow \infty \Rightarrow ds^2 = -dt^2 + a(t)^2(r^2 + r^2 d\Omega_2^2)$$

$$\text{where } a(t) = e^{Ht}, \quad H = \sqrt{\Lambda/3}$$

- Radial null geodesics: $dt = \pm adr$

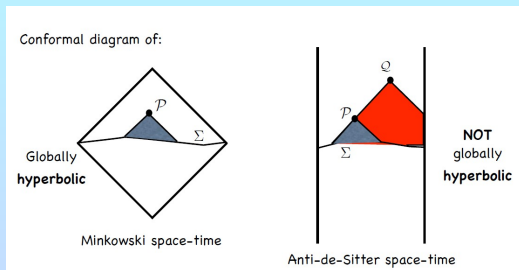
$$\text{We expect: } f = f_0 + \frac{a u(t-ar)}{r^n}$$

$$\Rightarrow \partial_t f - \partial_t f_0 + \frac{1}{a(t)} \partial_r f + n \frac{f-f_0}{r a(t)} - H(f-f_0) = 0$$

Zilhão et al, PRD **85** (2012) 104039

Anti de Sitter

Much more complicated!



- Time-like outer boundary \Rightarrow affects interior
- AdS metric **diverges** at outer boundary

Anti de Sitter metric

- Maximally symmetric solution to Einstein eqs. with $\Lambda < 0$

- Hyperboloid $X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2$

embedded in $D + 1$ dimensional flat spacetime of signature
- - + ... +

- Global AdS

$$X_0 = L \frac{\cos \tau}{\cos \rho}, \quad X_d = L \frac{\sin \tau}{\cos \rho}$$

$X_i = L \tan \rho \Omega_i$, for $i = 1 \dots D - 1$, Ω_i hyperspherical coords.

$$\Rightarrow ds^2 = \frac{L^2}{\cos^2 \rho} (-d\tau^2 + d\rho^2 + \sin^2 \rho d\Omega_{D-2}^2)$$

where $0 \leq \rho < \pi/2$, $-\pi < \tau \leq \pi$

- Outer boundary at $\rho = \pi/2$

Anti de Sitter metric continued

- Poincaré coordinates

$$X_0 = \frac{1}{2z} \left[z^2 + L^2 + \sum_{i=1}^{D-2} (x^i)^2 - t^2 \right]$$

$$X_i = \frac{Lx^i}{z} \text{ for } i = 1 \dots D-2$$

$$X_{D-1} = \frac{1}{2z} \left[z^2 - L^2 + \sum_{i=1}^{D-2} (x^i)^2 - t^2 \right]$$

$$X_d = \frac{Lt}{z}$$

$$\Rightarrow ds^2 = \frac{L^2}{z^2} \left[-dt^2 + dz^2 + \sum_{i=1}^{D-2} (dx^i)^2 \right]$$

where $z > 0$, $t \in \mathbb{R}$

- Outer boundary at $z = 0$

e.g. Ballón Bayona & Braga, hep-th/0512182

AdS spacetimes: Outer boundary

- AdS boundary: $\rho \rightarrow \pi/2$ (global)
 $z \rightarrow 0$ (Poincaré)
- AdS metric becomes **singular**
 \Rightarrow induced metric determined up to **conformal rescaling** only
- Global: $ds_{\text{gl}}^2 \sim -d\tau^2 + d\Omega_{D-2}$
Poincaré: $ds_{\text{P}}^2 \sim -dt^2 + \sum_{i=1}^{D-2} d(x^i)^2$
 \Rightarrow Different **topology**: $\mathbb{R} \times S_{D-2}$ and \mathbb{R}^{D-1}
- The dual theories live on spacetimes of different topology

Regularization methods

- Decompose metric into AdS part plus deviation
Bantilan & Pretorius, PRD **85** (2012) 084038
- Factor out appropriate factors of the bulk coordinate
Chesler & Yaffe, PRL **106** (2011) 021601
Heller, Janik & Witaszczyk, PRD **85** (2012) 126002
- Factor out singular term of the metric
Bizoń & Rostworowski, PRL **107** (2011) 031102
- Regularity of the outer boundary may constrain the gauge freedom
Bantilan & Pretorius, PRD **85** (2012) 084038

Further reading

- Initial data construction

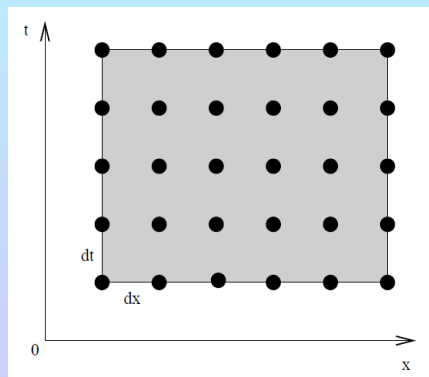
Cook, Liv. Rev. Rel. **3** (2000) 5

Pfeiffer, gr-qc/0510016

2.4 Discretization of the equations

Finite differencing

- Consider one spatial, one time dimension t, x
- Replace computational domain by discrete points
 $x_i = x_0 + i dx, t_n = t_0 + n dt$
- Function values $f(t_n, x_i) \sim f_{n,i}$



Derivatives and finite derivatives

- Goal: represent $\frac{\partial^m f}{\partial x^m}$ in terms of $f_{n,i}$

- Fix index n ; Taylor expansion:

$$f_{i-1} = f_i - f'_i dx + \frac{1}{2} f''_i dx^2 + \mathcal{O}(dx^3)$$

$$f_i = f_i$$

$$f_{i+1} = f_i + f'_i dx + \frac{1}{2} f''_i dx^2 + \mathcal{O}(dx^3)$$

- Write f'_i as linear combination: $f'_i = Af_{i-1} + Bf_i + Cf_{i+1}$
- Insert Taylor expressions and compare coefficients on both sides

$$\Rightarrow 0 = A + B + C, \quad 1 = (-A + B)dx, \quad 0 = \frac{1}{2}A dx^2 + \frac{1}{2}C dx^2$$

$$\Rightarrow A = -\frac{1}{2dx}, \quad B = 0, \quad C = \frac{1}{2dx}$$

$$\Rightarrow f'_i = \frac{f_{i+1} - f_{i-1}}{2dx} + \mathcal{O}(dx^2)$$

- Higher order accuracy \rightarrow more points; works same in time

Mesh refinement

3 Length scales :	BH	$\sim 1 M$
	Wavelength	$\sim 10 \dots 100 M$
	Wave zone	$\sim 100 \dots 1000 M$

- Critical phenomena

Choptuik '93

- First used for BBHs

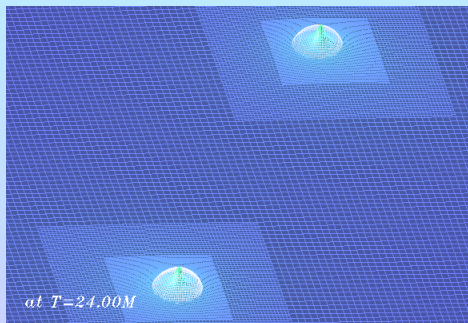
Brügmann '96

- Available Packages:

Paramesh MacNeice *et al.* '00

Carpet Schnetter *et al.* '03

SAMRAI MacNeice *et al.* '00



Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for 1 + 1 dimensions

0) data at t

$t+dt$

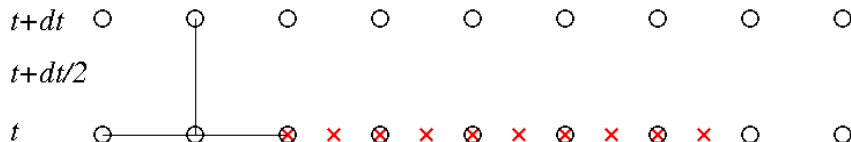
$t+dt/2$

t ○ ○ ⊗ × ⊗ × ⊗ × ⊗ × ⊗ × ⊗ × ⊗ × ○ ○

Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for 1 + 1 dimensions

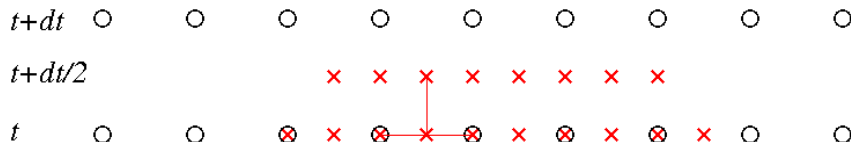
1) update coarse grid



Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for 1 + 1 dimensions

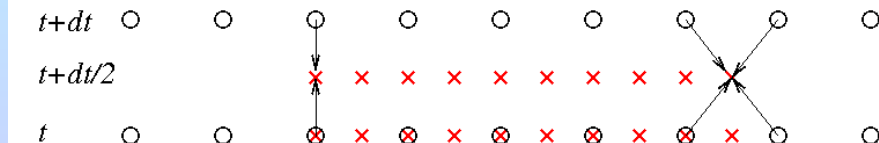
2) first update on fine grid



Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for 1 + 1 dimensions

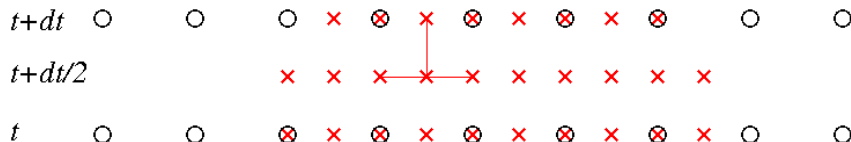
3) prolongation



Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for 1 + 1 dimensions

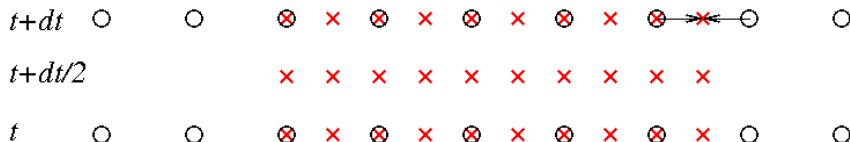
4) second update on fine grid



Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for 1 + 1 dimensions

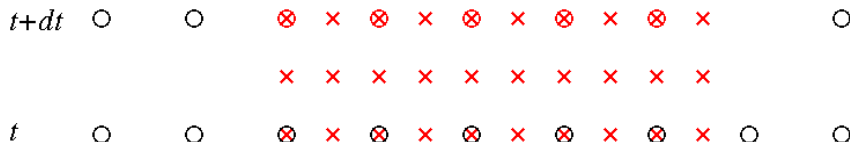
5) prolongation



Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for 1 + 1 dimensions

6) restriction



Alternative discretization schemes

- Spectral methods: high accuracy, efficiency, complexity

Caltech-Cornell-CITA code SpEC

<http://www.black-holes.org/SpEC.html>

Applications to moving punctures still in construction

e.g. Tichy, PRD **80** (2009) 104034

Also used in symmetric asymptotically AdS spacetimes

e.g. Chesler & Yaffe, PRL **106** (2011) 021601

- Finite Volume methods
- Finite Element methods

D. N. Arnold, A. Mukherjee & L. Pouly, gr-qc/9709038

C. F. Sopuerta, P. Sun & J. Xu, CQG **23** (2006) 251

C. F. Sopuerta & P. Laguna, PRD **73** (2006) 044028

Further reading

- Numerical methods

Press et al, “*Numerical Recipes*”, Cambridge University Press

2.5 Diagnostics

The subtleties of diagnostics in GR

- Successful numerical simulation \Rightarrow Numbers for grid functions
- Typically: Spacetime metric $g_{\alpha\beta}$ and time derivative or ADM variables $\gamma_{ij}, K_{ij}, \alpha, \beta^i$
- Challenges
 - Coordinate dependence of numbers \Rightarrow Gauge invariants
 - Global quantities at ∞ , computational domain finite \Rightarrow Extrapolation
 - Complexity of variables, e.g. GWs \Rightarrow Spherical harmonics
 - Local quantities: meaningful? \Rightarrow Horizons
- AdS/CFT correspondence: Dictionary

Newton's gravitational constant

- **Note:** We wrote the Einstein equations for $\Lambda = 0$ as

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi GT_{\alpha\beta}$$

- The (areal) **horizon radius** of a static BH in D dimensions then is

$$r_s^{D-3} = \frac{16\pi GM}{(D-2)\Omega_{D-2}},$$

where $\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$ is the area of the $D - 2$ hypersphere

- The **Hawking entropy formula** is $S = \frac{A_{AH}}{4G}$

- But **Newton's force law** picks up geometrical factors:

$$\mathbf{F} = \frac{(D-3)8\pi G}{(D-2)\Omega_{D-2}} \frac{Mm}{r^{D-2}} \hat{\mathbf{r}}$$

See e.g. Emparan & Reall, Liv. Rev. Rel. **6** (2008)

Global quantities

- Assumptions
 - Asymptotically, the metric is flat and time independent
 - The expressions also refer to Cartesian coordinates

- ADM mass = Total mass-energy of spacetime

$$M_{ADM} = \frac{1}{4\Omega_{D-2}G} \lim_{r \rightarrow \infty} \int_{S_r} \sqrt{\gamma} \gamma^{mn} \gamma^{kl} (\partial_n \gamma_{mk} - \partial_k \gamma_{mn}) dS_l$$

- Linear momentum of spacetime

$$P_i = \frac{1}{8\pi G} \lim_{r \rightarrow \infty} \int_{S_r} \sqrt{\gamma} (K^m_i - \delta^m_i K) dS_m$$

- Angular momentum in $D = 4$

$$J_i = \frac{1}{8\pi} \epsilon_{il}{}^m \lim_{r \rightarrow \infty} \int_{S_r} \sqrt{\gamma} x^l (K^n_m - \delta^n_m K) dS_n$$

- By construction, these are time independent!

Apparent horizons

- By Cosmic censorship, existence of an **apparent horizon** implies an **event horizon**
- Consider **outgoing null geodesics** with tangent vector k^μ
- Def.: **Expansion** $\Theta = \nabla_\mu k^\mu$
- Def.: **Apparent horizon** = outermost surface where $\Theta = 0$
- On a hypersurface Σ_t , the condition for $\Theta = 0$ becomes
$$\hat{D}_m s^m - K + K_{mn} s^m s^n,$$
where s^i = unit normal to the $(D - 2)$ dimensional AH surface
e.g. Thornburg, PRD **54** (1996) 4899

Apparent horizons continued

- Parametrize the horizon by $r = f(\varphi^i)$,
where r is the radial and φ^i are angular coordinates
- Rewrite the condition $\Theta = 0$ in terms $f(\varphi^i)$
 \Rightarrow Elliptic equation for $f(\varphi^i)$
- This can be solved e.g. with **Flow**, **Newton** methods
Thornburg, PRD **54** (1996) 4899, Gundlach, PRD **57** (1998) 863
Alcubierre et al, CQG **17** (2000) 2159, Schnetter, CQG **20** (2003) 4719

- Irreducible mass $M_{irr} = \sqrt{\frac{A_{AH}}{16\pi G^2}}$

- BH mass in $D = 4$: $M^2 = M_{irr}^2 + \frac{S^2}{4M_{irr}^2} (+P^2)$,

where S is the spin of the BH, Christodoulou, PRL **25** (1970) 1596

Gravitational waves in $D = 4$: Newman Penrose

- Construct a Tetrad

- $n^\alpha =$ Timelike unit normal field

- Spatial triad u, v, w through Gram-Schmidt orthogonalization

E.g. starting with $u^i = [x, y, z]$, $v^i = [xz, yz, -x^2 - y^2]$,

$$w^i = \epsilon^i{}_{mn} v^m w^n$$

- $\ell^\alpha = \frac{1}{\sqrt{2}}(n^\alpha + u^\alpha)$, $k^\alpha = \frac{1}{\sqrt{2}}(n^\alpha - u^\alpha)$, $m^\alpha = \frac{1}{\sqrt{2}}(v^\alpha + iw^\alpha)$

$\Rightarrow -\ell \cdot k = 1 = m \cdot \bar{m}$, all other products vanish

- Newman-Penrose scalar $\Psi_4 = C_{\alpha\beta\gamma\delta} k^\alpha \bar{m}^\beta k^\gamma \bar{m}^\delta$

- In vacuum, $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$

- For more details, see e.g.

Nerozzi, PRD **72** (2005) 024014, Brüggmann et al, PRD **77** (2008) 024027

Analysis of Ψ_4

- Multipolar decomposition: $\Psi_4 = \sum_{\ell,m} \psi_{\ell m}(t, r) Y_{\ell m}^{-2}(\theta, \phi)$,

where $\psi_{\ell m} = \int_0^{2\pi} \int_0^\pi \Psi_4 \overline{Y_{\ell m}^{-2}} \sin \theta d\theta d\phi$

- Radiated energy: $\frac{dE}{dt} = \lim_{r \rightarrow \infty} \left[\frac{r^2}{16\pi} \int_{\Omega} \left| \int_{-\infty}^t \Psi_4 d\tilde{t} \right|^2 d\Omega \right]$

- Momentum: $\frac{dP_i}{dt} = - \lim_{r \rightarrow \infty} \left[\frac{r^2}{16\pi} \int_{\Omega} \ell_i \left| \int_{-\infty}^t \Psi_4 d\tilde{t} \right|^2 d\Omega \right]$,

where $\ell_i = [-\sin \theta \cos \phi, -\sin \theta \sin \phi, -\cos \theta]$

- Angular mom.: $\frac{dJ_z}{dt} =$

$$- \lim_{r \rightarrow \infty} \left\{ \frac{r^2}{16\pi} \operatorname{Re} \left[\int_{\Omega} \left(\partial_\phi \int_{-\infty}^t \Psi_4 d\tilde{t} \right) \left(\int_{-\infty}^t \int_{-\infty}^{\hat{t}} \overline{\Psi_4} d\tilde{t} d\hat{t} \right) d\Omega \right] \right\}$$

see e.g. Ruiz et al, GRG **40** (2008) 2467

Wave extraction in $D \geq 4$

- Newman Penrose formalism: Algebraically special not as powerful

- Landau-Lifshitz pseudo tensor \Rightarrow radiated energy

Yoshino & Shibata, PRD **80** (2009) 084025, PTPS **189** (2011) 269

- Regge-Wheeler-Zerilli-Moncrief formalism in $D = 4$

Regge & Wheeler, PR **108** (1957) 1063, Zerilli, PRL **24** (1970) 737,
Moncrief, Ann.Phys. **88** (1974) 323

For applications in NR see e.g.

Reisswig et al, PRD **83** (2011) 064008, Sperhake et al, PRD **71** (2005) 124042
Rezzolla, gr-qc/0302025

- Generalization to $D > 4 \Rightarrow$ KI formalism

Kodama & Ishibashi, PTP **110** (2003) 701

The Kodama-Ishibashi formalism

- We follow notation from Witek et al, PRD **82** (2010) 104014
- Assumption: Metric \approx **spherically symmetric** far from GW sources
- Coordinates adapted to rotational symmetry on S^{D-2} :

$$(t, r, \theta, \bar{\theta}, \phi^1, \dots, \phi^{D-4})$$

- **Tangherlini background metric** Tangherlini, Nuovo Cim. **27** (1963) 636

$$ds_{(0)}^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + r^2[d\bar{\theta}^2 + \sin^2 \bar{\theta}(d\theta^2 + \sin^2 \theta d\Omega_{D-4})]$$

$$\text{where } A(r) = \left(1 - \frac{r_s^{D-3}}{r^{D-3}}\right)^{-1}$$

- **Perturbation:** $ds_{(1)}^2 = h_{ab}dx^a dx^b + h_{a\bar{\theta}}dx^a d\bar{\theta} + h_{\bar{\theta}\bar{\theta}}d\bar{\theta}^2 + h_{\theta\theta}d\Omega_{D-3}$

$$\text{where } x^a = (t, r)$$

The Kodama-Ishibashi formalism: Axisymmetry

- We now consider the case of $SO(D - 2)$ symmetry

$$\Rightarrow h_{a\theta} = h_{\bar{\theta}\theta} = 0,$$

perturbative expansion contains only **Scalar harmonics**

- Scalar harmonics: $\mathcal{S}(\phi^{\bar{i}})$ which satisfy

$$\square \mathcal{S} = -k^2 \mathcal{S}, \quad k = \ell(\ell + D - 3), \quad \ell = 0, 1, 2, \dots$$

where \square refers to the background metric $\gamma_{\bar{i}\bar{j}}$ induced onto $(\bar{\theta}, \theta, \phi^1, \dots, \phi^{D-4})$

- Def.: $\mathcal{S}_{\bar{i}} = -\frac{1}{k} \partial_{\bar{i}} \mathcal{S}, \quad \mathcal{S}_{\bar{i}\bar{j}} = \frac{1}{k^2} \bar{D}_{\bar{i}} \partial_{\bar{j}} \mathcal{S} + \frac{1}{D-2} \gamma_{\bar{i}\bar{j}} \mathcal{S}$

The Kodama-Ishibashi formalism: Axisymmetry

- The metric perturbations can be written as

$$h_{ab} = f_{ab}S, \quad h_{a\bar{i}} = rf_aS_{\bar{i}}, \quad h_{\bar{i}\bar{j}} = 2r^2(H_L\gamma_{\bar{i}\bar{j}}S + H_T S_{\bar{i}\bar{j}}),$$

where f_{ab} , f_a , H_L , H_T are functions of (t, r) .

They are obtained from **projecting** metric components onto spherical harmonics.

- Note: We are **suppressing** indices ℓ, m here!

We mean $h_{ab}^{\ell m} = f_{ab}^{\ell m} S_{\ell m}$, etc.

- Gauge invariant functions: $F = H_L + \frac{1}{D-2}H_T + \frac{1}{r}X_a\hat{D}^a r$

$$F_{ab} = f_{ab} + \hat{D}_b X_a + \hat{D}_a X_b$$

where $X_a = \frac{r}{k} \left(f_a + \frac{r}{k} \hat{D}_a H_T \right)$

and $\hat{D}_a =$ cov.deriv. of the (t, r) subsector of the backgr. metric

The Kodama-Ishibashi formalism: Master function

- Master function:

$$\partial_t \Phi = (D-2)r^{(D-4)/2} \frac{-F^r_t + 2r\partial_t F}{k^2 - D + 2 + \frac{(D-2)(D-1)}{2} \frac{r_s^{D-3}}{r^{D-3}}}$$

- Energy flux; we restore the index ℓ , ($m = 0$ in axisymmetry)

$$\frac{dE_\ell}{dt} = \frac{1}{32\pi} \frac{D-3}{D-2} k^2 (k^2 - D + 2) (\partial_t \Phi^\ell)^2$$

- Total radiated energy:

$$E = \sum_{\ell=2}^{\infty} \int_{-\infty}^{\infty} \frac{dE_\ell}{dt} dt$$

The AdS/CFT dictionary: Fefferman-Graham coords.

- AdS/CFT correspondence

⇒ Vacuum expectation values $\langle T_{ij} \rangle$ of the field theory given by quasi-local Brown-York stress-energy tensor

Brown & York, PRD **47** (1993) 1407

- Consider asymptotically AdS metric in Fefferman-Graham coordinates

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{r^2} (dr^2 + \gamma_{ij} dx^i dx^j),$$

where

$$\gamma_{ij}(r, x^i) = \gamma_{(0)ij} + r^2 \gamma_{(2)ij} + \dots + r^D \gamma_{(D)ij} + h_{(D)ij} r^D \log r^2 + \mathcal{O}(r^{D+1}),$$

- Note: This asymptotes to Poincaré coordinates as $r \rightarrow 0$

The AdS/CFT dictionary: Fefferman-Graham coords.

- Here, the $\gamma_{(a)ij}$, $h_{(D)ij}$ are functions of x^i , logarithmic terms only appear for even D , powers of r are exclusively even up to order $D - 1$
- Vacuum expectation values of CFT momentum tensor for $D = 4$ is

$$\langle T_{ij} \rangle = \frac{4L^3}{16\pi G} \left\{ \gamma_{(4)ij} - \frac{1}{8} \gamma_{(0)ij} [\gamma_{(2)}^2 - \gamma_{(0)}^{km} \gamma_{(0)}^{ln} \gamma_{(2)kl} \gamma_{(2)mn}] - \frac{1}{2} \gamma_{(2)i}{}^m \gamma_{(2)jm} + \frac{1}{4} \gamma_{(2)ij} \gamma_{(2)} \right\}$$

where $\gamma_{(n)} \equiv \text{Tr}(\gamma_{(n)ij}) = \gamma_{(0)}^{ij} \gamma_{(n)ij}$

de Haro et al, Commun.Math.Phys. **217** (2001) 595; also for other D

- Note: $\gamma_{(2)ij}$ is determined by $\gamma_{(0)ij} \Rightarrow$ CFT freedom given by $\gamma_{(4)ij}$

AdS/CFT: Renormalized stress-tensor

- Again: Brown-York stress-tensor \rightarrow as the VEVs of the field theory

- Divergencies in $\langle T^{ab} \rangle = \frac{\delta S_{\text{eff}}}{\delta \gamma_{ab}}$

Regularize by adding boundary curvature invariants to S_{eff}

Balasubramanian & Kraus, Commun.Math.Phys. **208** (1999) 413

- Foliate D dimensional spacetime into timelike hypersurfaces Σ_r homoeomorphic to the boundary

$$\Rightarrow ds^2 = \alpha^2 dr^2 + \gamma_{ab}(dx^a + \beta^a dr)(dx^b + \beta^b dr) \quad (\text{like ADM})$$

- $\hat{n}^\mu =$ outward pointing normal vector to the boundary

$$\Theta^{\mu\nu} = -\frac{1}{2}(\nabla^\mu \hat{n}^\nu + \nabla^\nu \hat{n}^\mu) \quad \text{Extrinsic curvature}$$

AdS/CFT: Renormalized stress-tensor

- Including counter terms, for ADS_5 :

$$T^{\mu\nu} = \frac{1}{8\pi G} \left[\Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} - \frac{3}{L} \gamma^{\mu\nu} - \frac{L}{2} \mathcal{G}^{\mu\nu} \right]$$

where $\mathcal{G}_{\mu\nu}$ is the Einstein tensor of the induced metric $\gamma_{\mu\nu}$

- Note: Applying this to the global ADS_5 metric gives $T^{\mu\nu} \neq 0$
 \Rightarrow Casimir energy of quantum field theory on $S^3 \times \mathbb{R}$
- Other D : cf. Balasubramanian & Kraus, Commun.Math.Phys. **208** (1999) 413
- AdS/CFT Dictionary for additional fields, see e.g.
Skenderis, CQG **19** (2002) 5849
de Haro et al, Commun.Math.Phys. **217** (2001) 595

Further reading

- Isolated and dynamical horizons
Ashtekar & Krishnan, Liv. Rev. Rel. **7** (2004) 10

3 Results from BH evolutions

3.1 BHs in GW physics

Gravitational waves

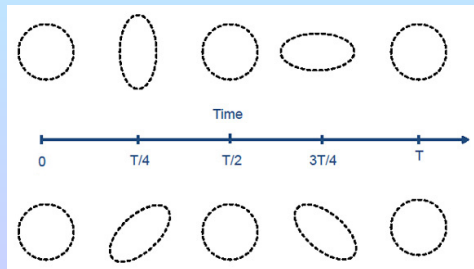
- Weak field limit: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$
- Trace reversed perturbation $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta}$
 \Rightarrow Vacuum field eqs.: $\square\bar{h}_{\alpha\beta} = 0$

- Appropriate gauge \Rightarrow

$$\bar{h}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\sigma x^\sigma}$$

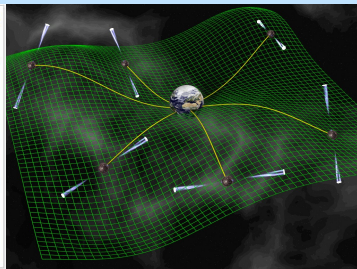
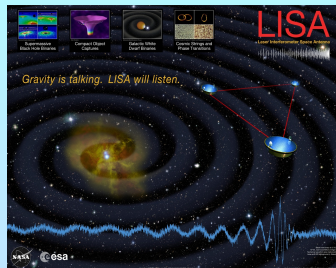
where $k^\sigma = \text{null vector}$

- GWs displace particles

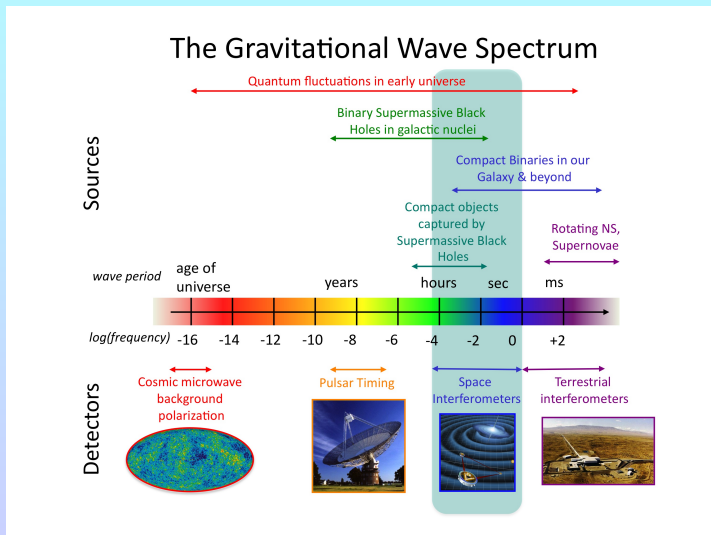


Gravitational wave detectors

- Accelerated masses \Rightarrow GWs
- Weak interaction!
- Laser interferometric detectors

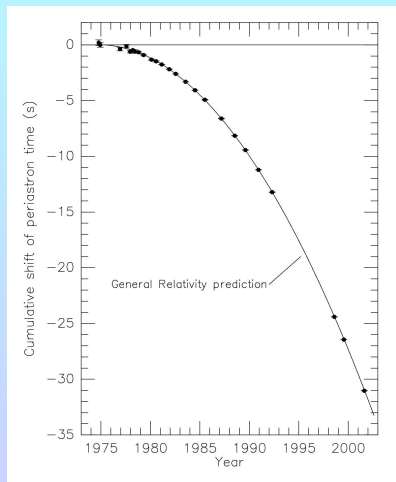


The gravitational wave spectrum



Some targets of GW physics

- Confirmation of GR
 - Hulse & Taylor 1993 Nobel Prize
- Parameter determination of BHs: M , \vec{S}
- Optical counter parts
 - Standard sirens (candles)
 - Mass of graviton
- Test Kerr Nature of BHs
- Cosmological sources
- Neutron stars: EOS



Free parameters of BH binaries

- Total mass M

Relevant for GW detection: Frequencies scale with M

Not relevant for source modeling: trivial rescaling

- Mass ratio $q \equiv \frac{M_1}{M_2}$, $\eta \equiv \frac{M_1 M_2}{(M_1 + M_2)^2}$

- Spin: \vec{S}_1, \vec{S}_2 (6 parameters)

- Initial parameters

Binding energy E_b

Separation

Orbital ang. momentum L

Eccentricity

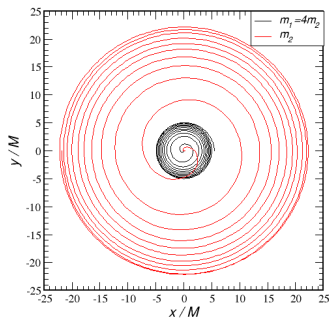
Alternatively: frequency, eccentricity

BBH trajectory and waveform

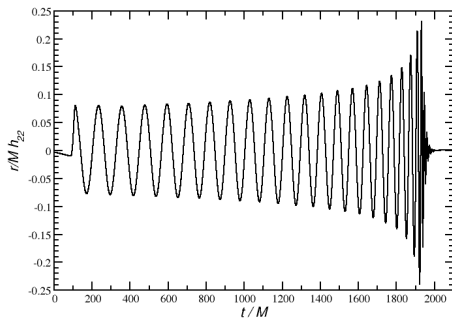
- $q = 4$, non-spinning binary; ~ 11 orbits

US, Brügmann, Müller & Sopena '11

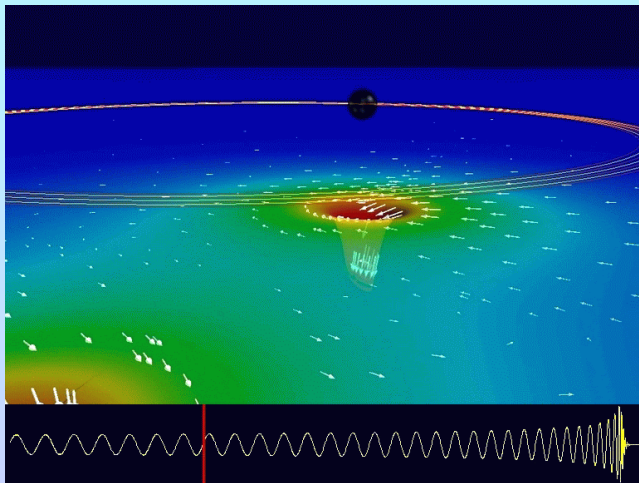
Trajectory



Quadrupole mode





Morphology of a BBH inspiral

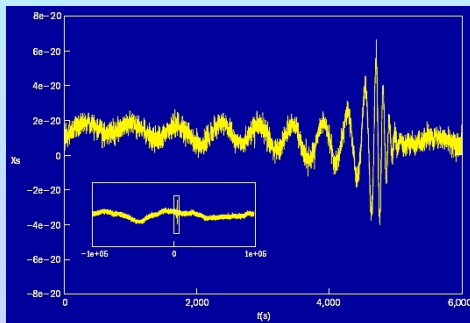


Thanks to Caltech, Cornell, CITA

Matched filtering

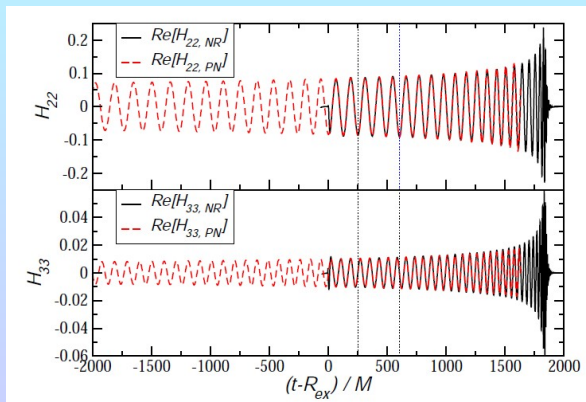
- BH binaries have 7 parameters: 1 mass ratio, 2×3 for spins
- Sample parameter space, generate waveform for each point

- NR + PN
- Effective one body
- Ninja, NRAR Projects
-  GEO 600 noise
-  chirp signal



Template construction

- Stitch together PN and NR waveforms
- EOB or phenomenological templates for ≥ 7 -dim. par. space



Template construction

- Phenomenological waveform models

- Model phase, amplitude with simple functions → Model parameters
- Create **map** between physical and model parameters
- Time or frequency domain

Ajith et al, CQG **24** (2007) S689, PRD **77** (2008) 104017, CQG **25** (2008) 114033, PRL **106** (2011) 241101; Santamaria et al, PRD **82** (2010) 064016, Sturani et al, arXiv:1012.5172 [gr-qc]

- Effective-one-body (EOB) models

- Particle in effective metric, PN, ringdown model
Buonanno & Damour PRD **59** (1999) 084006, PRD **62** (2000) 064015
- Resum PN, calibrate **pseudo PN** parameters using NR
Buonanno et al, PRD **77** (2008) 026004, Pan et al, PRD **81** (2010) 084041, PRD **84** (2012) 124052; Damour et al, PRD **77** (2008) 084017, PRD **78** (2008) 044039, PRD **83** (2011) 024006

The Ninja project

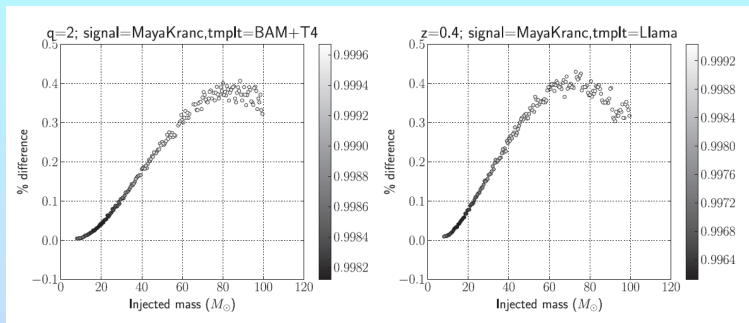
<https://www.ninja-project.org/>

Aylott et al, CQG **26** (2009) 165008, CQG **26** (2009) 114008

Ajith et al, CQG **29** (2012) 124001

- Use PN/NR hybrid waveforms in GW data analysis
- Ninja2: 56 hybrid waveforms from 8 NR groups
- Details on hybridization procedures
- Overlap and mass bias study:
 - Take one waveform as signal, fixing M_{tot}
 - Search with other waveform (same config.) varying t_0 , ϕ_0 , M_{tot}

The Ninja project



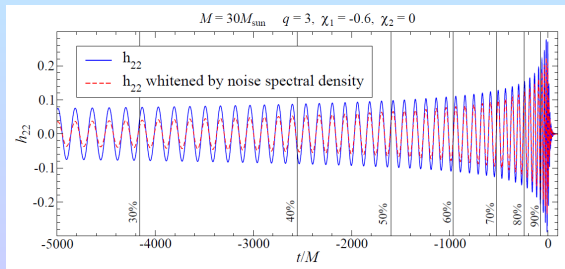
- Left: $q = 2$, non-spinning waveforms, MAYAKRANC, BAM + T4
- Right: $q = 1$, $\chi_1 = \chi_2 = 0.4$ waveform, MAYAKRANC, LLAMA + T4
- Mass bias $< 0.5\%$

The NRAR project

<https://www.ninja-project.org/doku.php?id=nrar:home>

Hinder, Buonanno et al, in preparation

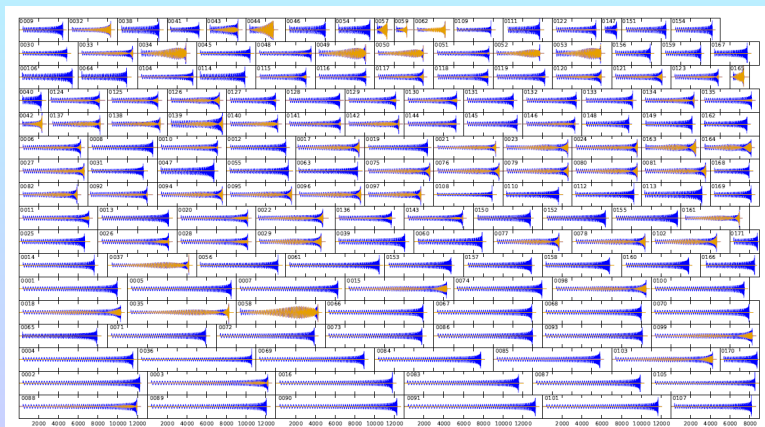
- Pool efforts from 9 NR groups
- 11M core hours on XSEDE Kraken
- 22 waveforms, including precessing runs
- Common, automatized analysis, uncertainty measures



Tools of mass production

- SpEC catalog: 171 waveforms: $q \leq 8$, 90 preprocessing, ≤ 34 orbits

Mroué et al, arXiv:1304.6077[gr-qc]



Strategies in parameter space

- SpEC: 16 orbits in 40 hours
- Still, 7-dimensional parameter space $\rightarrow N \sim 10^7$ waveforms?
- Probably too many...
- Reduce # of parameters describing dominant spin effects
Ajith et al, PRL **106** (2011) 241101, PRD **84** (2011) 084037,
Pürrer et al, arXiv:1306.2320 [gr-qc]
- Spin-orbit resonances \Rightarrow preferred regions in parameter space?
Gerosa et al, arXiv:1302.4442 [gr-qc]
- **Trade-off:** Quantity or quality of waveforms?
Both affects parameter estimation!

Limits in the parameter space

- Mass ratio $q = 100$

Lousto & Zlochower, PRL **106** (2011) 041101

Head-on case: Sperhake et al, PRD **84** (2011) 084038

- Spin magnitude $\chi = 0.97$

Superposed Kerr-Schild data (non-conformally flat)

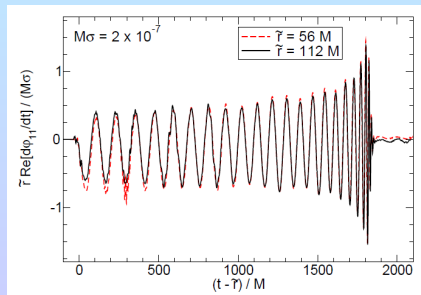
Lovelace et al, CQG **29** (2012) 045003

- Separations $D = 100 M$; few orbits

Lousto & Zlochower, arXiv:1304.3937 [gr-qc]

Going beyond GR: Scalar-tensor theory of gravity

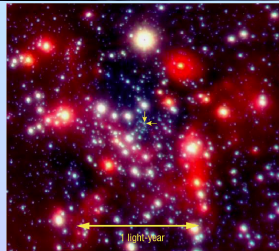
- Brans-Dicke theory: 1 parameter ω_{BD} ; well constrained
- Bergmann-Wagoner theories: **Generalize** $\omega = \omega(\phi)$
- **No-hair theorem**: BHs solutions same as in GR
e.g. Hawking, Comm.Math.Phys. **25** (1972) 167
Sotiriou & Faraoni, PRL **108** (2012) 081103
- **Circumvent no-hair theorem**:
Scalar bubble
Healey et al, arXiv:1112.3928 [gr-qc]
- **Circumvent no-hair theorem**:
Scalar gradient
Berti et al, arXiv:1304.2836 [gr-qc]



3.2 BHs in Astrophysics

Evidence for astrophysical black holes

- X-ray binaries
 - e. g. Cygnus X-1 (1964)
 - MS star + compact star
 - ⇒ Stellar Mass BHs
 - ~ 5 ... 50 M_{\odot}
- Stellar dynamics
 - near galactic centers,
 - iron emission line profiles
 - ⇒ Supermassive BHs
 - ~ $10^6 \dots 10^9 M_{\odot}$
 - AGN engines

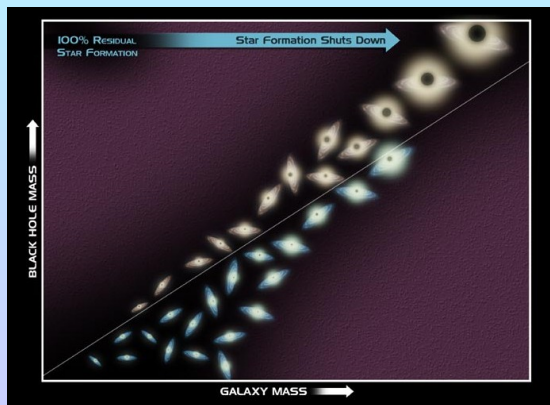


The Centre of the Milky Way
(VLT YEPUN + NACO)

ESO PR Photo 29a/02 (9 October 2002) ©European Southern Observatory

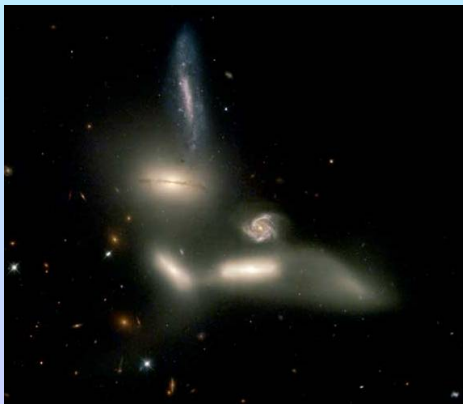
Correlation of BH and host galaxy properties

- Galaxies ubiquitously harbor BHs
 - BH properties correlated with bulge properties
- e. g. J. Magorrian *et al.*, AJ 115, 2285 (1998)



SMBH formation

- Most widely accepted scenario for galaxy formation: hierarchical growth; “bottom-up”
- Galaxies undergo frequent mergers \Rightarrow BH merger



Gravitational recoil

- Anisotropic GW emission \Rightarrow recoil of remnant BH

Bonnor & Rotenburg '61, Peres '62, Bekenstein '73

- Escape velocities: Globular clusters 30 km/s
 dSph 20 – 100 km/s
 dE 100 – 300 km/s
 Giant galaxies \sim 1000 km/s

Ejection / displacement of BH \Rightarrow

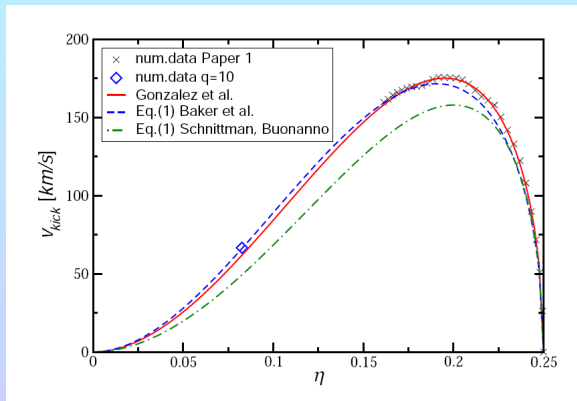
- Growth history of SMBHs
- BH populations, IMBHs
- Structure of galaxies



Kicks from non-spinning BHs

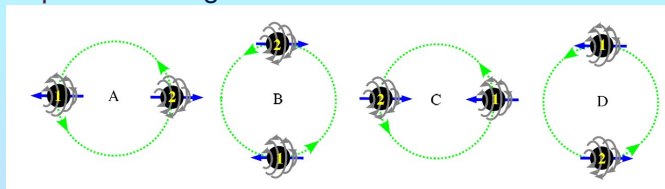
- Max. kick: ~ 180 km/s, harmless!

González et al., PRL 98, 091101 (2009)



Spinning BHs: Superkicks

- Superkick configuration:

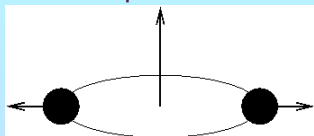


- Kicks up to $v_{\max} \approx 4\,000$ km/s
Campanelli *et al.*, PRL **98** (2007) 231102
González *et al.* PRL **98** (2007) 231101
- Suppression via **spin alignment** and **Resonance** effects in inspiral
Schnittman, PRD **70** (2004) 124020
Bogdanovicz *et al.*, ApJ **661** (2007) L147
Kesden *et al.*, PRD **81** (2010) 084054, ApJ **715** (2010) 1006

Even larger kicks: superkick and hang-up

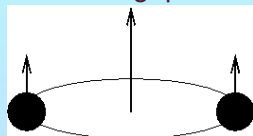
Lousto & Zlochower, arXiv:1108.2009 [gr-qc]

Superkicks

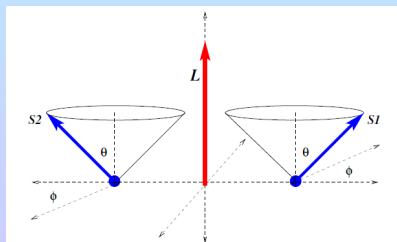


- Moderate GW generation
- Large kicks

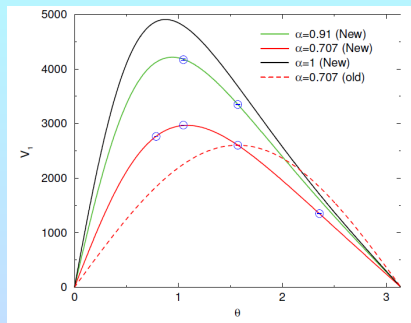
Hangup



- Strong GW generation
- No kicks



Superkicks and orbital hang-up



- Maximum kick about 25 % larger: $v_{\max} \approx 5000$ km/s
- Distribution asymmetric in θ ; v_{\max} for partial alignment
- Suppression through resonances still works

Berti et al, PRD **85** (2012) 124049

Spin precession and flip

- X-shaped radio sources

Merrit & Ekers, Science **297**
(2002) 1310

- Jet along spin axis

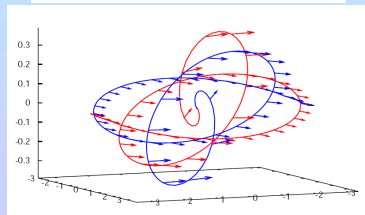
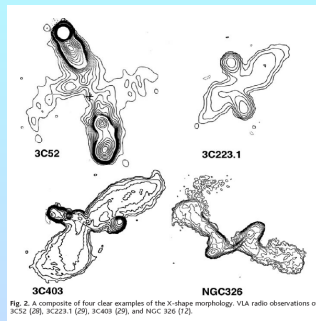
- Spin re-alignment

⇒ new + old jet

- Spin precession 98°

Spin flip 71°

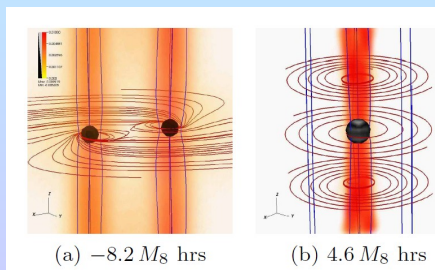
Campanelli et al, PRD **75** (2006)
064030



Jets generated by binary BHs

Palenzuela et al, PRL **103** (2009) 081101, Science **329** (2010) 927

- Non-spinning BH binary
- Einstein-Maxwell equations with “force free” plasma
- Electromagnetic field extracts energy from $\mathbf{L} \Rightarrow$ jets
- Optical signature: double jets



3.3. High-energy collisions of BHs

The Hierarchy Problem of Physics

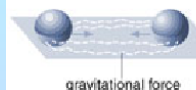
- Gravity $\approx 10^{-39} \times$ other forces
- Higgs field $\approx \mu_{obs} \approx 250 \text{ GeV} = \sqrt{\mu^2 - \Lambda^2}$
where $\Lambda \approx 10^{16} \text{ GeV}$ is the grand unification energy
- Requires enormous finetuning!!!
- Finetuning exist: $\frac{987654321}{123456789} = 8.0000000729$
- Or E_{Planck} much lower? Gravity strong at small r ?
 \Rightarrow BH formation in high-energy collisions at LHC
- Gravity not measured below 0.16 mm ! Diluted due to...
 - Large extra dimensions Arkani-Hamed, Dimopoulos & Dvali '98
 - Extra dimension with warp factor Randall & Sundrum '99

Stages of BH formation

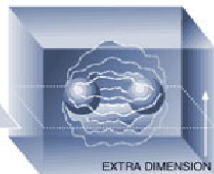
Black Holes on Demand

Scientists are exploring the possibility of producing miniature black holes on demand by smashing particles together. Their plans hinge on the theory that the universe contains more than the three dimensions of everyday life. Here's the idea:

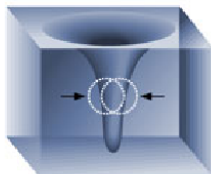
Particles collide in three dimensional space, shown below as a flat plane.



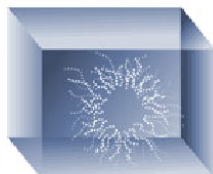
As the particles approach in a particle accelerator, their gravitational attraction increases steadily.



When the particles are extremely close, they may enter space with more dimensions, shown above as a cube.



The extra dimensions would allow gravity to increase more rapidly so a black hole can form.



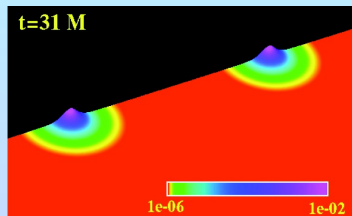
Such a black hole would immediately evaporate, sending out a unique pattern of radiation.

- Matter does not matter at energies well above the Planck scale
⇒ Model particle collisions by black-hole collisions

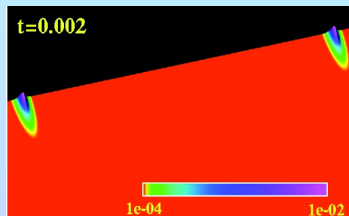
Banks & Fischler, gr-qc/9906038; Giddings & Thomas, PRD **65** (2002) 056010

Does matter “matter”?

- Hoop conjecture \Rightarrow kinetic energy triggers BH formation
- Einstein plus minimally coupled, massive, complex scalar field
“Boson stars” Pretorius & Choptuik, PRL **104** (2010) 111101



$$\gamma = 1$$

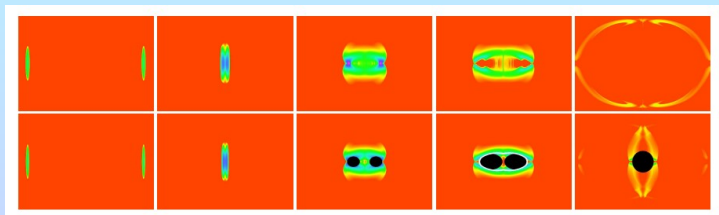


$$\gamma = 4$$

- BH formation threshold: $\gamma_{\text{thr}} = 2.9 \pm 10 \% \sim 1/3 \gamma_{\text{hoop}}$
- Model particle collisions by BH collisions

Does matter “matter”?

- Perfect fluid “stars” model
- $\gamma = 8 \dots 12$; BH formation below Hoop prediction
East & Pretorius, PRL **110** (2013) 101101
- Gravitational focussing \Rightarrow Formation of individual horizons



- Type-I critical behaviour
- Extrapolation by 60 orders would imply no BH formation at LHC

Rezzolla & Tanaki, CQG **30** (2013) 012001

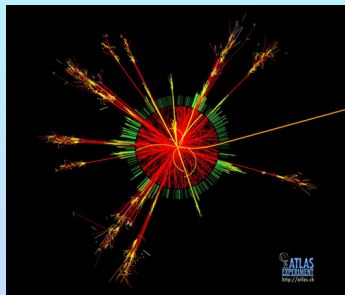
Experimental signature at the LHC

Black hole formation at the LHC could be detected by the properties of the jets resulting from Hawking radiation. BlackMax, Charybdis

- Multiplicity of partons: Number of jets and leptons
- Large transverse energy
- Black-hole mass and spin are important for this!

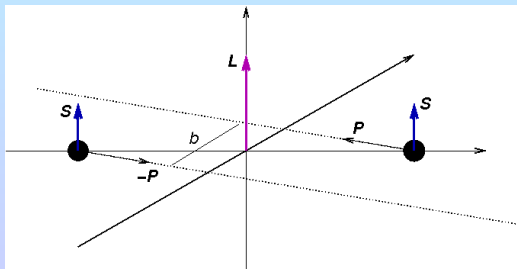
ToDo:

- Exact cross section for BH formation
- Determine loss of energy in gravitational waves
- Determine spin of merged black hole



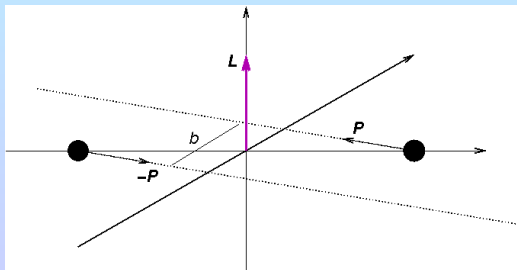
$D = 4$: Initial setup: 1) Aligned spins

- Orbital hang-up Campanelli et al, PRD **74** (2006) 041501
- 2 BHs: Total rest mass: $M_0 = M_{A,0} + M_{B,0}$
Boost: $\gamma = 1/\sqrt{1-v^2}$, $M = \gamma M_0$
- Impact parameter: $b \equiv \frac{L}{P}$



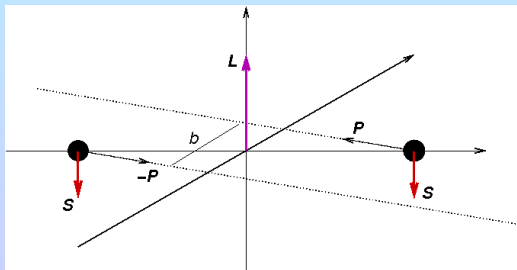
$D = 4$: Initial setup: 2) No spins

- Orbital hang-up Campanelli et al, PRD **74** (2006) 041501
- 2 BHs: Total rest mass: $M_0 = M_{A,0} + M_{B,0}$
Boost: $\gamma = 1/\sqrt{1-v^2}$, $M = \gamma M_0$
- Impact parameter: $b \equiv \frac{L}{P}$



$D = 4$: Initial setup: 3) Anti-aligned spins

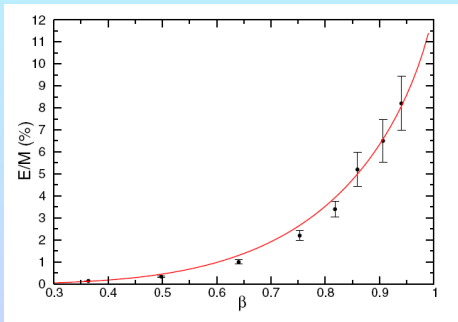
- Orbital hang-up Campanelli et al, PRD **74** (2006) 041501
- 2 BHs: Total rest mass: $M_0 = M_{A,0} + M_{B,0}$
Boost: $\gamma = 1/\sqrt{1-v^2}$, $M = \gamma M_0$
- Impact parameter: $b \equiv \frac{L}{P}$



$D = 4$: Head-on: $b = 0$, $\vec{S} = 0$

- Total radiated energy: 14 ± 3 % for $v \rightarrow 1$
US et al, PRL **101** (2008) 161101

About half of Penrose '74



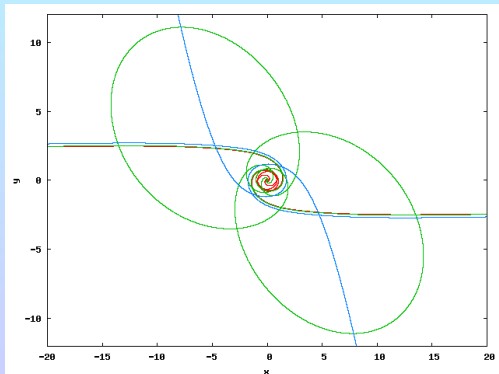
- Agreement with approximative methods

Flat spectrum, GW multipoles Berti et al, PRD **83** (2011) 084018

$D = 4$: Grazing: $b \neq 0$, $\vec{S} = 0$, $\gamma = 1.52$

- Radiated energy up to at least 35 % M
- Immediate vs. Delayed vs. No merger

US et al, PRL **103** (2009) 131102



$D = 4$: Scattering threshold b_{scat} for $\vec{S} = 0$

● $b < b_{\text{scat}} \Rightarrow$ Merger

$b > b_{\text{scat}} \Rightarrow$ Scattering

● Numerical study: $b_{\text{scat}} = \frac{2.5 \pm 0.05}{v} M$

Shibata et al, PRD **78** (2008) 101501(R)

● Independent study US et al, PRL **103** (2009) 131102, arXiv:1211.6114

$\gamma = 1.23 \dots 2.93$:

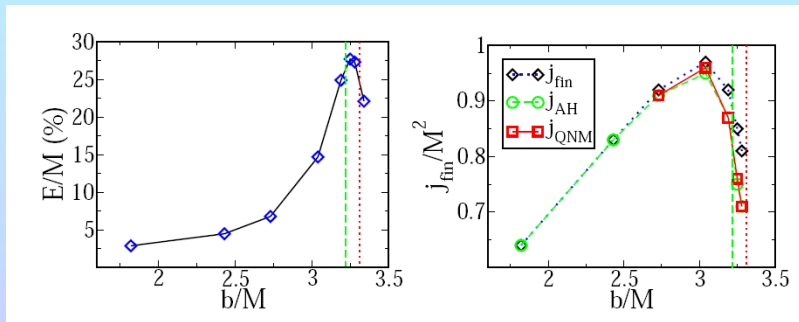
$\chi = -0.6, 0, +0.6$ (anti-aligned, nonspinning, aligned)

● Limit from Penrose construction: $b_{\text{crit}} = 1.685 M$

Yoshino & Rychkov, PRD **74** (2006) 124022

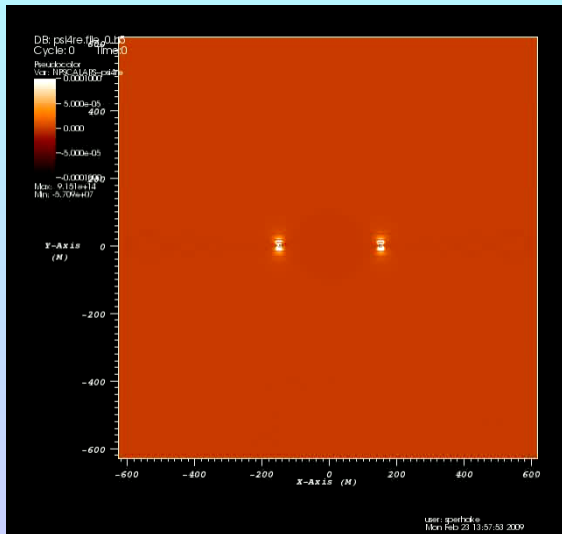
$D = 4$: Radiated quantities $b \neq 0$, $\vec{S} = 0$

- b -sequence with $\gamma = 1.52$
- Threshold of immediate merger Pretorius & Khurana, CQG **24** (2007) S83
- $E_{\text{rad}} \sim 35\%$ for $\gamma = 2.93$; about 10% of Dyson luminosity

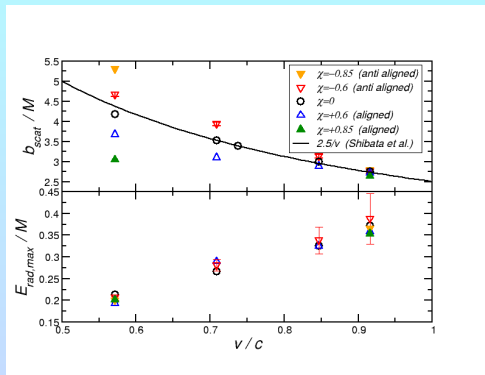


US *et al.*, PRL **103** (2009) 131102

$D = 4$: Gravitational radiation: Delayed merger



$D = 4$: Scattering threshold and radiated energy $\vec{S} \neq 0$

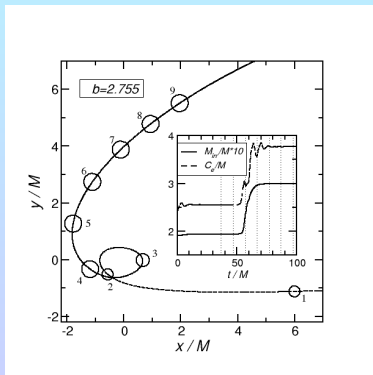


US et al, arXiv:1211.6114

- At speeds $v \gtrsim 0.9$ spin effects washed out
- E_{rad} always below $\lesssim 50\% M$

$D = 4$: Absorption

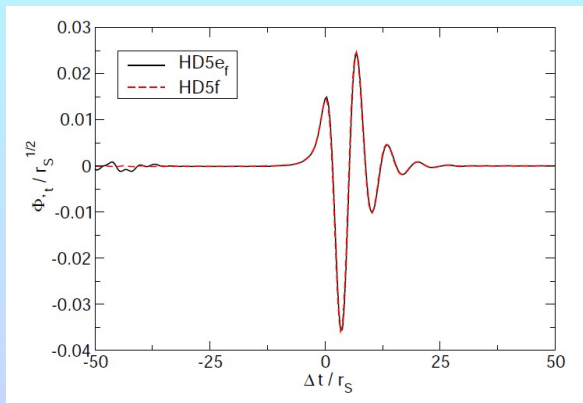
- For large γ : $E_{kin} \approx M$
- If E_{kin} is not radiated, where does it go?
- Answer: $\sim 50\%$ into E_{rad} , $\sim 50\%$ is absorbed



US et al, arXiv:1211.6114

$D = 5$: GWs from head-on

Wave extraction based on Kodama & Ishibashi, PTP **110** (2003) 701

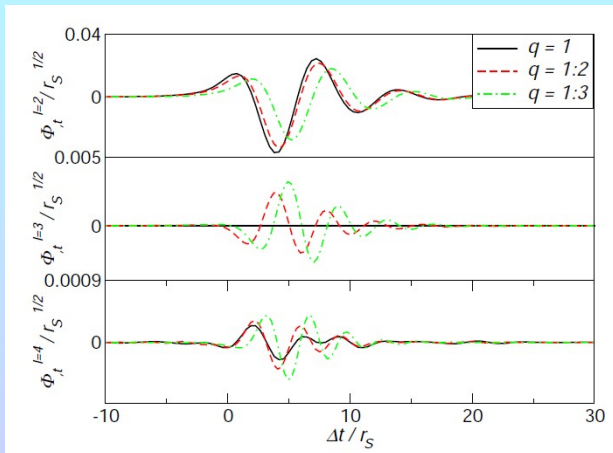


$E_{\text{rad}} = 0.089 \% M$ cf. $0.055 \% M$ in $D = 4$

Witek *et al.*, PRD **82** (2010) 104014

$D = 5$: Unequal-mass head-on

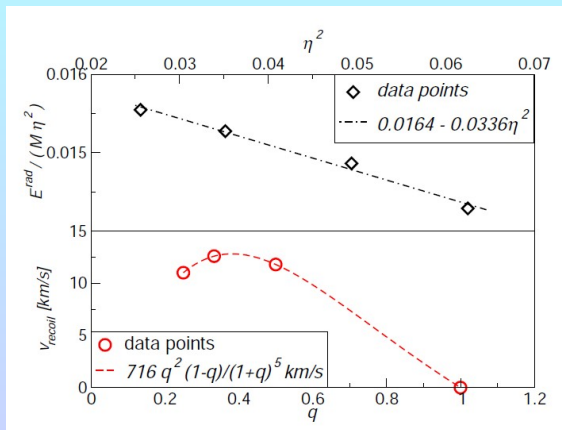
Kodama-Ishibashi multipoles



Witek *et al.*, PRD **83** (2011) 044017

$D = 5$: Unequal-mass head-on

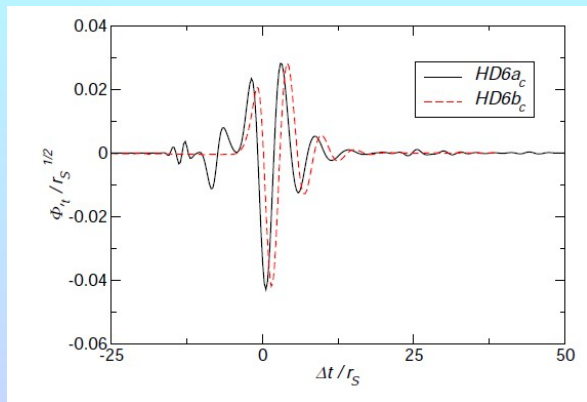
Radiated energy and momentum



Agreement within $< 5\%$ with extrapolated point particle calculations

$D = 6$: First black-hole collisions

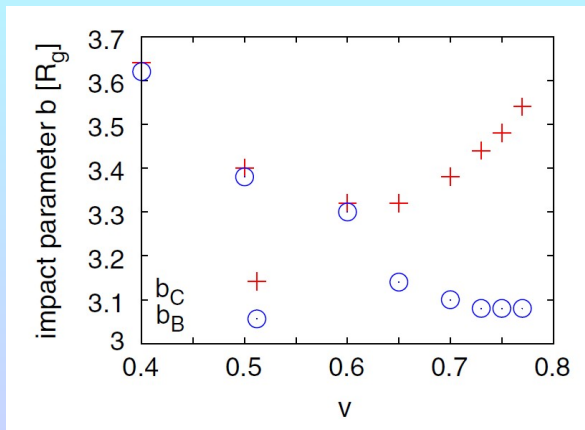
Witek *et al.* '10



- Adjust shift parameters
- Use LaSh system Witek *et al.*, PRD **83** (2011) 104041

$D = 5$: Scattering threshold

Okawa et al, PRD **83** (2011) 121501



Numerical stability still an issue...

3.4. BH Holography

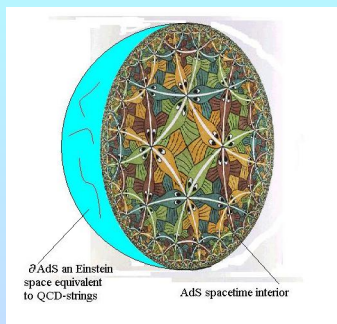
Large N and holography

- Holography

- BH entropy $\propto A_{Hor}$
- For a Local Field Theory entropy $\propto V$
- Gravity in D dims
 \Leftrightarrow local FT in $D - 1$ dims

- Large N limit

- Perturbative expansion of gauge theory in $g^2 N$
 \sim loop expansion in string theory
- N : # of “colors”
 $g^2 N$: t’Hooft coupling



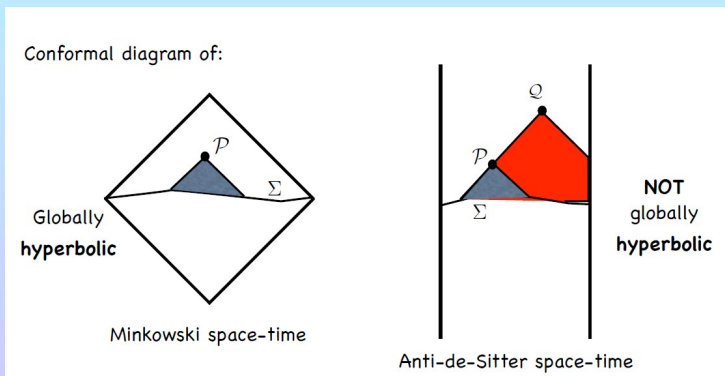
The AdS/CFT conjecture

Maldacena, Adv.Theor.Math.Phys. **2** (1998) 231

- “strong form”: Type IIb string theory on $AdS_5 \times S^5$
 $\Leftrightarrow \mathcal{N} = 4$ super Yang-Mills in $D = 4$
Hard to prove; non-perturbative Type IIb String Theory?
- “weak form”: low-energy limit of string-theory side
 \Rightarrow Type IIb Supergravity on $AdS_5 \times S^5$
- Some assumptions, factor out S^5
 \Rightarrow General Relativity on AdS_5
- Corresponds to limit of large N , $g^2 N$ in the field theory
- E. g. Stationary AdS BH \Leftrightarrow Thermal Equil. with T_{Haw} in dual FT
Witten, Adv.Theor.Math.Phys. **2** (1998) 253

The boundary in AdS

- Dictionary between metric properties and vacuum expectation values of CFT operators.
E. g. $T_{\alpha\beta}$ operator of CFT \leftrightarrow transverse metric on *AdS* boundary.
- The boundary plays an active role in *AdS*! Metric singular!



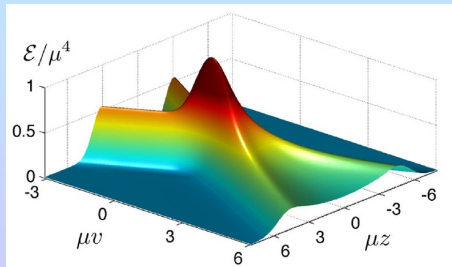
Collision of planar shockwaves in $\mathcal{N} = 4$ SYM

- Dual to colliding gravitational shock waves in AADS
- Characteristic study with translational invariance

Chesler & Yaffe PRL **102** (2009) 211601, PRD **82** (2010) 026006, PRL **106** (2011) 021601

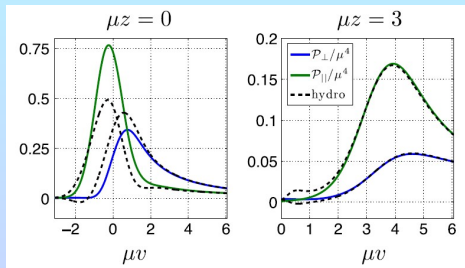
- Initial data: 2 superposed shockwaves

$$ds^2 = r^2[-dx_+ dx_- + d\mathbf{x}_\perp] + \frac{1}{r^2}[dr^2 + h(x_\pm) dx_\pm^2]$$



Collision of planar shockwaves in $\mathcal{N} = 4$ SYM

- Initially system far from equilibrium
- Isotropization after $\Delta v \sim 4/\mu \sim 0.35 \text{ fm}/c$
- Confirms hydro sims. of QGP $\sim 1 \text{ fm}/c$ Heinz, nucl-th/0407067



- Non-linear vs. linear Einstein Eqs. agree within $\sim 20 \%$
Heller et al, PRL **108** (2012) 191601
- Thermalization in ADM formulation Heller et al, PRD **85** (2012) 126002

Cauchy (“4+1”) evolutions in asymptotically AdS

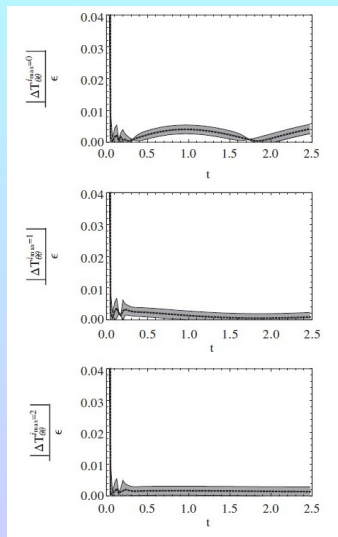
- Characteristic coordinates successful numerical tool in AdS/CFT
- But: restricted to symmetries, caustics problem...
- Cauchy evolution needed for general scenarios? Cf. BBH inspiral!!
- Cauchy scheme based on generalized harmonic formulation

Bantilan & Pretorius, PRD **85** (2012) 084038

- $SO(3)$ symmetry
- Compactify “bulk radius”
- Asymptotic symmetry of AdS_5 : $SO(4, 2)$
- Decompose metric into AdS_5 piece and deviation
- Gauge must preserve asymptotic fall-off

Cauchy (“4+1”) evolutions in asymptotically AdS

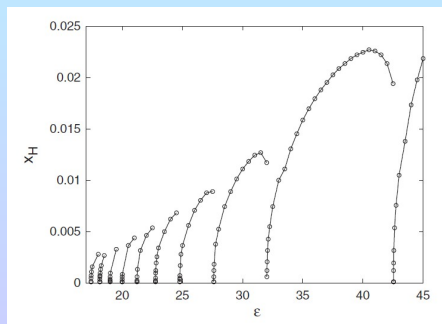
- Scalar field collapse
- BH formation and ringdown
- Low order QNMs \sim perturbative studies, but mode coupling
- CFT stress-energy tensor consistent with thermalized $\mathcal{N} = 4$ SYM fluid
- Difference of CFT $T_{\theta\theta}$ and hydro (+1st, 2nd corr.)



3.5 Fundamental properties of BHs

Stability of AdS

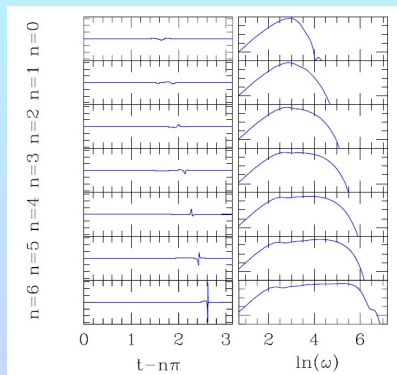
- $m = 0$ scalar field in as. flat spacetimes Choptuik, PRL **70** (1993) 9
 $p > p^* \Rightarrow \text{BH}$, $p < p^* \Rightarrow \text{flat}$
- $m = 0$ scalar field in as. AdS Bizoń & Rostworowski, PRL **107** (2011) 031102
- Similar behaviour for “Geons”
Dias, Horowitz & Santos '11
- $D > 4$ dimensions
Jałmużna et al, PRD **84** (2011) 085021
- $D = 3$: Mass gap: smooth solutions
Bizoń & Jałmużna, arXiv:1306.0317



Stability of AdS

- Pulses narrow under successive reflections

Buchel et al, PRD **86** (2012) 123011



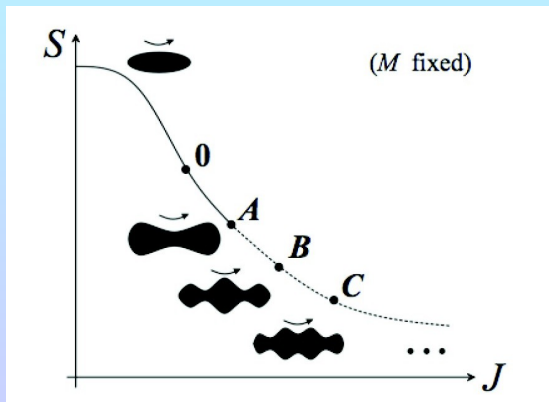
- \exists Non-linearly stable solutions in AdS

Dias et al, CQG **29** (2012) 235019, Buchel et al, arXiv:1304.4166,

Maliborski & Rostworowski arXiv:1303.3186

Bar mode instability of Myers-Perry BH

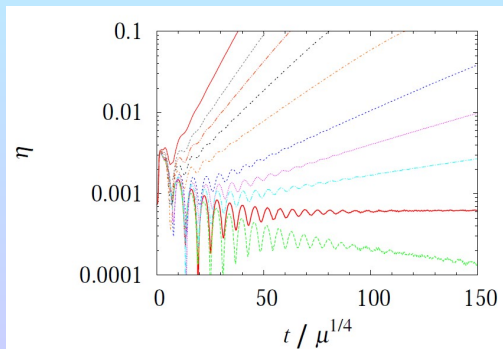
- MP BHs (with single ang.mom.) should be unstable.
- Linearized analysis Dias et al, PRD **80** (2009) 111701(R)



Non-linear analysis of MP instability

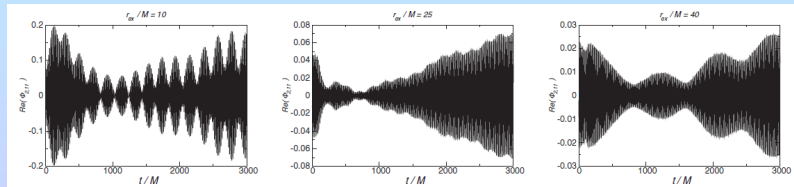
Shibata & Yoshino, PRD **81** (2010) 104035

- Myers-Perry metric; transformed to Puncture like coordinate
- Add small bar-mode perturbation
- Deformation $\eta := \frac{2\sqrt{(l_0 - l_{\pi/2})^2 + (l_{\pi/4} - l_{3\pi/4})^2}}{l_0 + l_{\pi/2}}$



Superradiant instability

- Scattering of waves with $\text{Re}[\omega]$ off BH with ang. horizon velocity Ω_H
 \Rightarrow amplification $\Leftrightarrow \text{Re}[\omega] < m\Omega_H$
- Measure photon mass? Pani et al, PRL **109** (2012) 131102
- Numerical simulations Dolan, arXiv:1212.1477 Witek et al
- Instability of spinning BHs, Beating effects

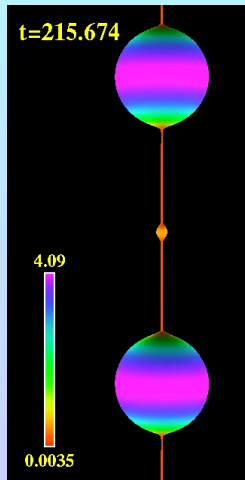


Witek et al, PRD **87** (2013) 043513

Cosmic Censorship in $D = 5$

Pretorius & Lehner, PRL **105** (2010) 101102

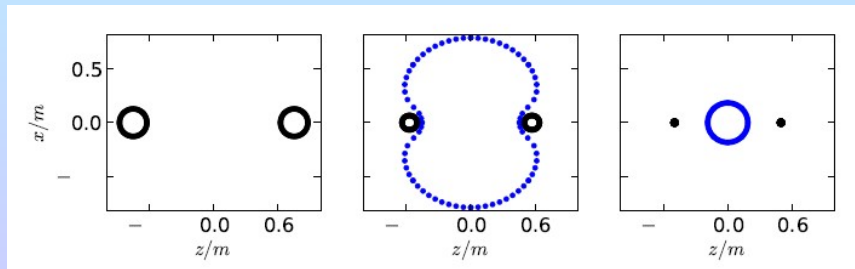
- Axisymmetric code
- Evolution of black string...
- Gregory-Laflamme instability cascades down in finite time until string has zero width \Rightarrow naked singularity



Cosmic Censorship in $D = 4$ de Sitter

Zilhão et al, PRD **85** (2012) 124062

- Two parameters: MH , d
- Initial data: McVittie type binaries McVittie, MNRAS **93** (1933) 325
- “Small BHs”: $d < d_{crit} \Rightarrow$ merger
 $d > d_{crit} \Rightarrow$ no common AH
- “Large” holes at small d : Cosmic Censorship holds



Further reading

- **Reviews about numerical relativity**

Centrella et al, Rev. Mod. Phys. **82** (2010) 3069

Pretorius, arXiv:0710.1338

Sperhake et al, arXiv:1107.2819

Pfeiffer, CQG **29** (2012) 124004

Hannam, CQG **26** (2009) 114001

Sperhake, IJMPD **22** (2013) 1330005