Loop Quantum Gravity & Cosmology: a primer

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Why Quantum Gravity?

- Two pillars of modern theoretical physics:
  - General Relativity:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{2} \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

  - Quantum Mechanics (& Field Theory):

$$-i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle$$

- Mutually incompatible:
  - Mutually excluding principles.
  - But: Do their domains of applicability intersect?

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Why Quantum Gravity?

- General Relativity:
    - Basic principles:
        - Matter described by classical fields.
        - Matter content and geometry interact.
        - Physics does not depend on the method of descibing the system (coordinate system).
    - Domain of applicability:
        - Large scale (astrophysical, cosmological).
        - Strong gravitational fields.

Why Quantum Gravity?

- Quantum Mechanics (& Field Theory):
  - Basic principles:
    - Matter described via wave functions & states, not classical fields
    - Fixed background spacetime.
    - Coordinates play crucial role in qunatization process.
  - Domain of applicability:
    - Small (microscopic) scale.
    - Weak gravitational fields.

Why Quantum Gravity?

- The problem:
  In the (history of the) Universe there exist physical processes where the domains of applicability (of need) do intersect:
  - large energies (quantum effects)
  - large gravitational fields (GR effects)
- These are:
  - Early Universe evolution (close to Big Bang).
  - Black hole interiors.
- Need unification of both GR and QM/QFT !!
  - need to include both types of effects.
- Various approaches

Approaches to QG

- String theory: (in context of AdS/CFT: T. Wiseman & seminars)
  - Main idea:
    - particle approach to gravity (graviton)
    - Nambu-Goto/Polyakov action on flat spacetime
    - high dimension spacetime, $4D$ spacetime emergent
- Noncommutative Geometry: (A. Sitarz)
  - Main idea:
    - Ordinary Riemannian (spin-)geometries are described by a commutative C^* algebras
    - Spacetime is emergent (spectrum)
    - Many geometry objects well defined upon generalization to noncommutative C^* algebra
    - Classical approach but expected to include quantum effects

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Approaches to QG

- Conformal cyclic cosmology: (sir R. Penrose)
  - Main idea:
    - Restoring conformal invariance in some epoch of Universe evolution allows to extend the evolution through Big Bang singularity without taking into account quantum gravity effects.
- "Discrete" approaches:
  Based on division of / representation of spacetime by discrete structures:
  - Causal Dynamical triangulation
  - Simplicial gravity
  - Loop Quantum gravity (canonical & Spin Foams)
  - Loop Quantum Cosmology

Discrete QG

Main principle:

⋆ background independence – no underlying metric,

⋆ geometry structures emergent

- Causal Dynamical triangulation: (R. Loll)
  - Main principle: discrete time slices, space decomposed onto symplexes, evolution governed by axiomatic rules implementing causality.
  - Predictions: spacetime dimensionality (scale varying).
- Simplicial gravity: (J.Jurkiewicz)
  - Main principle: Path integral approach with discretization of spacetime (usually via decomposition onto symplexes).
- Loop Quantum gravity/Cosmology/SF

LQG/SF/LQC

- Main principle:
  - Explicit background independence: geometry represented by objects (labelled graphs) embedded in manifold without metric
  - Explicit (strict) diffeomorphism invarince.
  - Non-standard quantum representation.
- Main (independent) branches:
  - Loop Quantum gravity: (canonical) (T. Thiemann, A. Ashtekar)
  - Spin Foams (E. Bianchi)
  - Loop Quantum Cosmology (A. Ashtekar, T. Pawlowski (cont))

Canonical LQG

See lecture by T. Thiemann.

- Main properties:
  - Canonical: based on $3+1$ canonical splitting of the spacetime
  - Basic objects: parallel transports (holonomies) and analogs of electric fluxes.
  - Unique representation of the holonomy-flux algebra (LOST)
  - States spanned by labelled graphs: spin-networks

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Canonical LQG

- Main achievements/predictions:
  - Precise mathematical framework on the diffeomporphism-invariant level
  - Discrete spectra of geometric (diff-invariant) operators: area, volume.
  - Well defined (diff-invariant) coherent states (preservation by dynamics unknown)
  - Reproduced Bekenstein-Hawking formula + quantum corrections to black hole entropy.
  - In specific frameworks (wrt. matter content not symmetry) quantization program completed.
  - No explicit dynamical calculations as of yet.

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Spin Foams

See lecture by E. Bianchi

- Main properties:
  - Covariant approach, constructed to mimic the path integral of LQG spin networks.
  - Basic objects – histries of LQG spin networks, same structure of quantum labeling.
  - Not a path integral folulation of LQG: practical constructions resemble the simplicial gravity approaches.
- Main achievements/predictions:
  - Calculted graviton propagator in low field regime.
  - Reproduced Newton gravity law.

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Loop Quantum Cosmology

See lecture by A. Ashtekar

- Main properties:
  - Application of methods of LQG to cosmological models:
    - Early stage: symmetry reduced models
    - Current stage: division onto quasi-global degrees of freedom including homogeneous “background” ones.
  - Not derived as symmetry-reduction of LQG.
  - For many scenarios precise and complete quantum frameworks.
- Main achievements/predictions:
  - Explicit calculation of the quantum universe dynamics in simple (homogeneous) scenarios.
  - Early Universe paradigm shift: Big Bang $\longrightarrow$ Big Bounce.
  - Predictions of primordial perturbations structure.

The intersection

Models originally "independent" but precise bridges are being constructed.

- LQG↔SF:
  - Feynman-diagramatic approach to SF (Lewandowski, Puchta, ...). SF can be formulated as Feynman diagrams of evolving LQG spin networks.
  - Path-integral formulation of LQG (specific Hamiltonian) (Alesci, Thiemann, Zipfel)
- LQG↔LQC:
  - Approximate cosmologies from SF symplexes (Rovelli, Vidotto, Garay, ...).
  - Evolution eq. of cosmological DOF resemble LQC one but due to simplifications known only qualitatively.

The intersection

- SF$\leftrightarrow$LQC:
  - Systematic extraction of the cosmological degrees of freedom and their dynamics from specific construction of LQG Hamiltonian. (Alesci, Gianfrani)
  - Evolution eq. of cosmological DOF resemble LQC one but due to simplifications known only qualitatively.

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LQG - classical framework

- Action: gravity coupled to matter

$$\frac{1}{4G}\int \mathrm{d}^4x\sqrt{-g}R \ + \ S_{\mathrm{SM}}+\text{boundary term}$$

- $3+1$ splitting

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ab}(N^a\mathrm{d}t + \mathrm{d}x^a)(N^b\mathrm{d}t + \mathrm{d}x^b)$$

- Ashtekar-Barbero canonical variables: densitized triad E^a_i and $SU(2)$ valued connection A^i_a

$$A^i_a = \Gamma^i_a(E) + \gamma K^i_a \qquad \{A^i_a(x), E^b_j(y)\} = \delta^i_j\delta^a_b\delta(x,y)$$

LQG - classical framework

- Classical constraints (grav. part):
  - Gauss: $\mathcal{G}_i = \partial_a E^a_i + \epsilon^k{}_{ij} A^j_a E^a_k$
  - Diffeomorphism: $C^G_a = E^b_i F^i_{ab} - A^i_a \mathcal{G}_i$
  - Hamiltonian:

$$\mathcal{H}_G = \frac{\gamma^2}{2\sqrt{\det E}} E^a_i E^b_j \left(\epsilon^{ij}{}_k F^k_{ab} + 2(1-\gamma^2) K^i_{[a} K^j_{b]} \right)$$

- Constraints form Dirac algebra $\rightarrow$ Dirac quantization program
  - Quantize system ignoring constraints
  - Express constraints as quantum operators
  - Physical states: annihilated by constraints
- Basic variables for quantization: holonomies and fluxes:

$$U_\gamma(A) \equiv P \exp \int_\gamma A^i_a \tau^i \mathrm{d}x^a \qquad K^i = \int_S E^{ai} \mathrm{d}\sigma_a$$

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LQG - kinematics

GNS quantization of the holonomy-flux algebra + the Dirac program for the constraints. Many authors, over 25 years of development.

- Kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$: spanned by the spin-network states:
  - Embedded graph with oriented edges, (topology fixed but not restricted)
  - spin labels j on its edges, (allow for $j = 0$)
  - intertwiners I on vertices,
- Solution to Gauss constraint (Thiemann 1993)
  - Spin labels restricted by angular momentum addition rules,
  - Intertwiners $\longrightarrow$ (vertex valence dependent) discrete set encoding addition order,
- Representation is unique (LOST theorem).

LQG: Diff-invariant sector

- Till recently no diffeo generator in LQG.
- Group averaging over finite (exponentiated) diffeomorphisms (Marolf at al 1995).
- The result: For fixed graph topology, on sufficiently large class of graphs (lattices, etc.) the embedded graph lifted to abstract one.
  - Statement not true for general graphs.

The Hamiltonian constraint

- Regularization as proposed by Thiemann: reexpression in terms of holonomies and volume operator
  - Fundamental representation for holonomies: $U_\gamma^{1/2}$ (following results by Perez)
- The result: quite complicated combinatorial operators coupling j-labels of the adjacent edges.
  - Depending on the formulation Hamiltonian constraint may add new edges to the graph.

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The difficulty

- Hamiltonian constraint too complicated to find explicitly its kernel.
- The solutions:
  - The Master program (Dittrich, Thiemann). Form one constraint using Feynmann trick

    “ $\hat{M} = \int \mathrm{d}^3x[\eta^{ij}\mathcal{G}_i^\dagger \mathcal{G}_i + {}^o g^{ab} C_a^\dagger C_b + \mathcal{H}^\dagger \mathcal{H}]$ ”
  - Difficulty: kernel elements again impossible to find. Existence of approximate solutions proven (Dittrich, Thiemann).
  - An alternative: the deparametrization.

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Deparametrization

- Idea: Couple gravity to matter fields. Use them as the frame.
  - Separation of the Hamiltonian constraint

$$\boxed{H = 0 \ \Leftrightarrow \ p_T^n = \tilde{H}}, \ n = 1, 2$$

  (T, p_T) - canonical "time" field pair.

- Several frames used:
  - Dust: (J Brown, K Kuchar, 1995, Phys.Rev.**D51** 5600-5629)
  - Tetrad of massless scalar fields.
- Quantization: applying LQG formalism, two programs:
  - Gravity + dust in Algebraic LQG framework: K Giesel, T Thiemann, 2010, Class.Quant.Grav.**27** 175009
  - Massless scalar fields in LQG: M Domagala, K Giesel, W Kaminski, L Lewandowski, 2010, Phys.Rev.**D82** 104038

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Simple example of depar.

(Husain, TP)

- Synthesis of several components:
  - Specific matter choice: Coupling to the irrotational dust only.
    - Provides just time instead of full frame.
    - Classically considered by Kuchar, Torre 1991
  - Natural time gauge: Proper time of the dust “particles”
    - Slight step away from principles of LQG.
  - Diffeo-invariant formalism of the conservative LQG.
    - Construction of the space of diffeo-invariant states $\mathcal{H}_{\rm diff}$
    - Graph preserving form of the original Hamiltonian regularized a la Thiemann defined on $\mathcal{H}_{\rm diff}$ (action of components may differ)
    - Known diffeo-invariant geometric observables.

Gravity + irr. dust

- Gravity coupled to irrotational dust:

$$S = \frac{1}{4G}\int \mathrm{d}^4x\sqrt{-g}R \; - \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}_m +$$

$$-\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}M(g^{ab}\partial_a T\partial_b T + 1)$$

T - dust potential, M - Lagrange multiplier

- The stress energy tensor: $U_a := \partial_a T$

$$T^{ab} = MU^aU^b + (M/2)g^{ab}(g_{cd}U^cU^d + 1) \quad (*)$$

- Standard canonical formalism:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ab}(N^a\mathrm{d}t + \mathrm{d}x^a)(N^b\mathrm{d}t + \mathrm{d}x^b)$$

- Dust component of canonical action

$$S_D = \int \mathrm{d}t\mathrm{d}^3x[p_T\dot{T} - N\mathcal{H}_D - N^aC_a^D]$$

$$\mathcal{H}_D = \frac{1}{2}\left[\frac{p_T^2}{M\sqrt{q}} + \frac{M\sqrt{q}}{p_T^2}(p_T^2 + q^{ab}C_a^DC_b^D)\right] \qquad C_a^D = -p_T\partial_a T$$

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Classical deparametrization

- Relation from equation of motion for M:

$$M^2 = [q]^{-1} P_T^4 (p_T^2 + q^{ab} C_a^D C_b^D)^{-1}$$

- Hamiltonian constraint (density)

$$\mathrm{sgn}(M) \sqrt{p_T^2 + q^{ab} C_a^D C_b^D} + \mathcal{H}_G + \mathcal{H}_m = 0$$

- Gauge fixing by proper time of dust particles: $T = t$

$$C_a^D = 0 \quad \Rightarrow \quad C^a = 0 \;\Leftrightarrow\; C_a^G + C_a^m = 0$$

(diffeo constraint like without the dust)

- The deparametrization: physical Hamiltonian density:

$$\tilde{H} = -p_T = \mathcal{H}_G + \mathcal{H}_m$$

No $|\cdot|$ due to M role in $(*)$ and (indep.) def. of $p_T = \sqrt{q}\frac{M}{N}\dot{T}$.

- Suitable for ANY quantization framework!

Summary of properties

- System with true physical Hamiltonian.
- Hamiltonian not of the square root form.
  - Defined directly on $\mathcal{H}_{\rm diff}$.
  - Its action is explicitly known.
- Physical Hilbert space known explicitly: $\mathcal{H}_{\rm phy} = \mathcal{H}_{\rm diff}$
- All known kinematical diffeo-invariant observables now become physical.
- Evolution is governed by time independent Schrödinger equation which action on $\mathcal{H}_{\rm phy}$ is explicitly known.

$$i\frac{\partial \Psi}{\partial t} = [\hat{H}_G + \hat{H}_m]\Psi$$

- States can be evolved numerically.

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Open issues

Applying practically even the simplest framework requires taking care of some issues:

- Specific constructions of the Hamiltonian:
  - Many ambiguities of the construction: factor ordering, alternative regularizations, choice of component operators.
  - Open question: which construction gives consistent dynamical picture
    - Lesson from LQC: answer to this nontrivial and important.
- Preservation of the coherent states by the dynamics:
  - Serious applications require semiclassical treatment. For that sufficiently well behaved semiclassical states are necessary.
  - Existing prescriptions never dynamically tested.

Loop Quantum Cosmology

LQC: Application of LQG methods to models with quasi-global degrees of freedom (symmetric spacetimes, perturbative frameworks,...)

- Basic formalism on FRW example
  - How LQG methods work in simplest scenarios
- Singularity resolution
- Inclusion of inhomogeneities

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FRW universe

Isotropic RW cosmological model

- Spacetime: manifold $M \times \mathbb{R}$ where $M = \mathbb{R}^3$
  $M \times \{t\}$ (where $t \in \mathbb{R}$) – homogeneous slices.
- Metric: $g_{\mu\nu} = -(\nabla_\mu t)(\nabla_\nu t) + a(t)(\pi^\star {}^o q)_{\mu\nu}$
  ${}^o q_{ab}$ - flat fiducial metric $(dx^2 + dy^2 + dz^2)$.
    - System is gauge-fixed! Also some background structure present!
- Triad formalism: ${}^o e^\mu_i$, ${}^o \omega^i_\mu$ – constant orthonormal triad/cotriad associated with ${}^o q_{ab}$.
- Ashtekar-Barbero canonical variables: also subject to symmetries
    - Unique class with el. of the form:

$$A^i_a = \tilde{c}\, {}^o\omega^i_a \qquad E^a_i = \tilde{p}\sqrt{{}^o q}\, {}^o e^a_i$$

- Constraint algebra: Since Gauss and Diffeomorphism constraint are automatically satisfied the Hamiltonian one is the only constraint.

Infrared regulator

- Global degrees of freedom: canonical pair $\tilde{c}, \tilde{p}$
- Infinity problem: $S = \int_M \mathrm{d}^4 x \mathcal{L} = \infty$ due to homogeneity.
- Solution: Restrict to a box (fiducil cell) $\mathcal{V}$ of volume V_o.
  - Role of the infrared regulator: Final theory has to be well defined in the regulator removal limit.
- Cell dependence in the symplectic structure

$$\{A_a^i, E_i^a\} = 8\pi G\gamma \qquad \Rightarrow \qquad \{\tilde{c}, \tilde{p}\} = 8\pi G\gamma / 3V_o$$

- Rescaling to remove the dependence:

$$c := V_o^{\frac{1}{3}} \tilde{c} \quad p := V_o^{\frac{2}{3}} \tilde{p} \qquad \Rightarrow \qquad \{\tilde{c}, \tilde{p}\} = 8\pi G\gamma / 3$$

- Final variables:

$$A_a^i = V_o^{-\frac{1}{3}} c\, {}^o\omega_a^i \qquad E_i^a = V_o^{-\frac{2}{3}} p \sqrt{{}^o q}\, {}^o e_i^a$$

Classical Hamiltonian constr.

- Euclidean and Lorentzian component:

$$H_g = \int_M \mathrm{d}^3x e^{-1}[\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab} - 2(1+\gamma^2) E^a_i E^b_j K^i_{[a} K^j_{b]}]$$

where $e = \sqrt{|\det E|}$ and $K^i_a = K_a{}^{bo}\omega^i_b$.

- Using $A^i_a = \Gamma^i_a + \gamma K^i_a$ we express the Lorentzian term in terms of field strength F^k_{ab} and curvature of spin connection Γ^i_a

$$F^k_{ab} := 2\partial_a A^k_b + \epsilon^k{}_{ij} A^i_a A^j_b \quad \Omega^k_{ab} := 2\partial_a \Gamma^k_b + \epsilon^k{}_{ij} \Gamma^i_a \Gamma^j_b$$

$$E^a_i E^b_j K^i_{[a} K^j_{b]} = \frac{1}{2\gamma^2} \epsilon^{ij}{}_k E^a_i E^b_j (F^k_{ab} - \Omega^k_{ab})$$

where for flat model $\Omega^k_{ab} = 0$.

- Final form of the (gravitational part of the) constraint:

$$H_g = -\frac{1}{\gamma^2} \int_M \mathrm{d}^3x \epsilon^{ij}{}_k e^{-1} E^a_i E^b_j F^k_{ab}$$

LQC quantization: kinematics

Direct application of the LQG quantization algorithm:

- Holonomies along integral curves ${}^{o}e^{a}_{i}$ suffice to separate homogeneous, isotropic connections.
- Holonomy along the edge in direction of ${}^{o}e^{a}_{i}$ of length $\lambda V_o^{\frac{1}{3}}$

$$h^{i}_{(\lambda)} = \cos(\lambda c/2)\mathbb{I} + 2\sin(\lambda c/2)\tau^{i} \qquad 2i\tau^{k} = \sigma^{k}$$

- In consequence an equivalent of holonomy algebra in LQG is generated by almost periodic functions: $N_{(\lambda)}(c) := \exp(i\lambda c/2)$
- The Gel'fand spectrum of this algebra (support of the elements of $\mathcal{H}^{\text{grav}}_{\text{kin}}$) analog of is the Bohr compactification of real line $\bar{\mathcal{R}}_{\text{Bohr}}$.
- Basic operators: $\hat{p}$, $\hat{N}_{(\lambda)}$.
- Analog of LQG unique state ("vacuum") is $+$ve linear functional f

$$f(\hat{N}_{(\lambda)}) = \delta_{\lambda,0} \quad \text{and} \quad f(\hat{p}) = 0.$$

LQC quantization: kinematics

- Final results: The GNS construction leads to Gravitational kinematical Hilbert space $\mathcal{H}_{\rm kin}^{\rm grav} = L^2(\bar{\mathcal{R}}_{\rm Bohr}, {\rm d}\mu_{\rm Haar})$.
- Bohr compactification: Space of almost periodic functions $\lambda \mapsto N_{(\lambda)}(c)$. The scalar product

$$\langle f_1 | f_2 \rangle = \lim_{L\to\infty} (1/2L) \int_{-L}^{L} \bar{f}_1(c) f_2(c)$$

- Representation of states in which operator $\hat{p}$ is diagonal. Eigenstates of $\hat{p}$ labeled by μ satisfy

$$\langle \mu_1 | \mu_2 \rangle = \delta_{\mu_1,\mu_2}$$

- Action of fundamental operators:

$$\hat{p}\,|\mu\rangle = \tfrac{4}{3}\pi\gamma\ell_{\rm Pl}^2 \mu\,|\mu\rangle \qquad \exp(i\lambda c/2)\,|\mu\rangle = |\mu+\lambda\rangle$$

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Hamilt. constr. regularization

Expression in terms of holonomies and fluxes. (Thiemann)

- The term $e^{-1}EE$

$$\epsilon_{ijk}e^{-1}E^{aj}E^{bk} = \sum_k \frac{\mathrm{sgn}(p)}{2\pi\gamma G\lambda V_o^{1/3}} {}^o\epsilon^{abc}\,{}^o\omega_c^k \,\mathrm{Tr}(h_k^{(\lambda)}\{h_k^{(\lambda)-1}, V\}\tau_i)$$

- The field strength operator
  - Given a square in $i-j$ plane of the side length $\lambda V_o^{\frac{1}{3}}$ wrt. ${}^oq_{ab}$

$$F_{ab}^i = -i \lim_{\mathrm{Ar}_\square \to 0} \frac{1}{\lambda^2 V_o^{2/3}} \mathrm{Tr}\left(h^\lambda_{\square_{ij}} - 1\right)\sigma^k \,{}^o\omega_a^i \,{}^o\omega_b^j$$

where $h^\lambda_{\square_{ij}} := h_i^{(\lambda)}h_j^{(\lambda)}(h_i^{(\lambda)})^{-1}(h_j^{(\lambda)})^{-1}$.

  - Problem: the limit $\mathrm{Ar}_\square \to 0$ of above operator doesn't exist!
  - Solution: We take $\mathrm{Ar}_\square = \Delta$, where Δ – smallest non-zero eigenvalue of area operator in full LQG. $\lambda^2|p| = \Delta = 4\sqrt{3}\pi\gamma\ell_{\mathrm{Pl}}^2$.
  - Consequence: λ is function of μ:

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Holonomy component operator

- Relevant holonomy:

$$h^i = \frac{1}{2}[N + N^{-1}]\mathbb{I} - i[N - N^{-1}]\tau^i \quad N = e^{i\lambda(\mu)\mu c/2}$$

  - $\hat{h}^i$ can be expressed in termns of $\hat{N}$ (new basic operator).

- Action of the component operator:
  - Exponentiated $d/d\mu$ is well defined

$$\hat{N}\Psi(\mu) = \exp[i\lambda(\mu)(\mathrm{d}/\mathrm{d}\mu)]\Psi(\mu)$$

  - The affine parameter along $\lambda(\mu)(\mathrm{d}/\mathrm{d}\mu)$ is given by $v = K\,\mathrm{sgn}(\mu)|\mu|^{3/2}$, where K – const.

- Convenient reparametrization: $(c, p) \longrightarrow (b, v)$

$$v = K\,\mathrm{sgn}(\mu)|\mu|^{3/2} \qquad \{v, b\} = 2$$

- Action of basic operators:

$$\hat{N}|v\rangle = |v+1\rangle \qquad \hat{p}|v\rangle = (2\pi\gamma\sqrt{\Delta})^{2/3}\,v\,|v\rangle$$

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LQC Hamiltonian constraint

- The final form: (symmetric ordering)

$$\hat{H}_g = \frac{3\pi G}{8\alpha}\sqrt{|\hat{v}|}(\hat{N}^2 - \hat{N}^{-2})\sqrt{|\hat{v}|}$$

where $\alpha = 2\pi\gamma\sqrt{\Delta}\ell_{\rm Pl}^2 \approx 1.35\ell_{\rm Pl}^3$

- Basic properties:
    - Is essentially self-adjoint.
    - Non-positive definite.

Matter coupling

- To build nontrivial system we have to introduce some matter content.
- Several possibilities:
  - Massless scalar field originally considered in LQC (see talk by A. Ashtekar).
  - Other matter with quadratic kinetic term.
  - Application of the irrotational dust frame from LQG convenient for demonstration.
- Dust time frame: for gravity + dust
  - $\hat{H}_G$ becomes physical Hamiltonian.
  - $\mathcal{H}_g = L^2(\mathcal{R}_{\rm Bohr}, {\rm d}\mu_{\rm Haar})$ becomes physical Hilbert space.
  - Evolution: Schrödinger equation $-i\hbar_t \partial \Psi(v,t) = \hat{H}_g \Psi(v,t)$
  - Since lapse $N = 1$ presence of singularities related to extendability of evolution for all $t \in \mathbb{R}$ (see talk by H. Ringstrom).

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Digression: Geometrodynamics

Wheeler-DeWitt quantization program for flat FRW with dust.

- Flat FRW metric: $g = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$
- Geometry variables: analogous to LQC:

$$v = \alpha^{-1} a^3, \quad \alpha \approx 1.35 \ell_{\mathrm{Pl}}^3 \qquad \{v, b\} = 2$$

- Hamiltonian: $H_G = -\frac{3\pi G}{2\alpha} b^2 |v|$
- Schrödinger quantization:
  - Hilbert space: $\mathcal{H}_G = L^2_s(\mathbb{R}, \mathrm{d}v)$
  - Hamiltonian: $\hat{H}_G = -\frac{3\pi G}{2\alpha} \sqrt{|\hat{v}|} \hat{b}^2 \sqrt{|v|}$

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WDW Hamiltonian properties

- Negative definite
- Self-adjointness:
  - $\hat{H}_G$ defined on domain

    $$\mathcal{D} = \{\psi \in \mathcal{S}, \psi(0) = \partial_v \psi(0) = 0\}$$

    where $\mathcal{S}$ is Schwartz space.
  - Deficiency subspaces $\mathcal{K}_\pm$: spaces of normalizable solutions $\varphi_\pm$ to

    $$\langle \varphi_\pm | \hat{H}_G^\star \mp iI |\psi\rangle = 0, \qquad \psi \in \mathcal{D}$$

  - If $\dim(\mathcal{K}_+) = \dim(\mathcal{K}_-) \neq 0$ domain of $\hat{H}_G$ has many extensions. All of them are defined by unitary transformations $U_\beta : \mathcal{K}_+ \to \mathcal{K}_-$:

    $$\mathcal{D}_\beta = \{\psi + a(\varphi_+ + U_\beta(\varphi_+)); \psi \in \mathcal{D}, a \in \mathbb{C}\}$$

  - Deficiency eq solvable: $\dim(\mathcal{K}_+) = \dim(\mathcal{K}_-) = 1$.
    - 1-parameter family of self-adjoint extensions $\mathcal{D}_\beta$.

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Auxiliary space

In original repsresentation difficult ot solve

- Auxiliary Hamiltonian: $\tilde{H}_G = 3i\pi\alpha^{-1}\ell_{\rm Pl}^2\hat{b}^2\partial_b$
- There exist invertible maps $P_\beta : \mathcal{H}_\beta \to \tilde{\mathcal{H}}$ such that

$$-P_\beta^{-1}[\tilde{H}_G]^+P_\beta = \hat{H}_\beta.$$

- Configuration variable: $x = 1/b \propto 1/H$
- Physical state:

$$\tilde{\mathcal{H}} \ni P_\beta\Psi(x) = \int_0^\infty \mathrm{d}k\tilde{\Psi}(k)[\theta(x)e^{ikx} + \theta(-x)e^{i\delta\beta}e^{ikx}]$$

- The evolution: $\Psi(x,t) = e^{i\omega(k)(t-t_o)}\Psi(x,t_o), \quad \omega(k) = 3\pi\ell_{\rm Pl}^2\alpha^{-1}k$
  - Free propagating wave packet with extenson dependent phase change at the singularity.

WDW dynamics

- In auxiliary space the observable $\hat{V} = |\hat{a}|^3$ has simple form.
- the quantum trajectory:

$$\langle\hat{V}\rangle(t) = V(t) = 6\pi\ell_{\rm Pl}^2\langle -\hat{H}_G\rangle(t-t_o)^2 + 2\alpha\sigma_x^2$$

  where t_o - point where $\langle\Psi : i\partial_x\hat{x} : \Psi\rangle = 0$

- The consequances:
  - Additional boundary data needed at the singularity $x = 0$.
  - At $t = t_o$ $V = 0$ up to variance.
  - Singularity not resolved in any sense (deterministic or dynamical).

LQC dynamics

- Auxiliary Hamiltonian per analogy to WDW:

$$\tilde{H}_G = P\hat{H}_G P^{-1} = -[3i\pi\ell_{\mathrm{Pl}}^2\alpha^{-1}\sin^2(b)\partial_b]^+$$

  where P - invertible mapping like for WDW.

- The time evolution: $x = -\cot(b)$

$$\tilde{\mathcal{H}} \ni P\Psi(x) = \int_0^\infty \mathrm{d}k\tilde{\Psi}(k)e^{i(kx+\omega(k)t)}$$

  - Freely propagating wave packet.

- the trajectory: (t_o - point where $\langle\Psi : i\partial_x\hat{x} : \Psi\rangle = 0$)

$$\langle\hat{V}\rangle(t) = V(t) = 6\pi\ell_{\mathrm{Pl}}^2\langle-\hat{H}_G\rangle(t-t_o)^2 + \frac{\alpha^2}{3\pi\ell_{\mathrm{Pl}}^2}\langle-\hat{H}_G\rangle + 2\alpha\sigma_x^2$$

- Consequences:
  - Evolution unique (self-adjointness).
  - Minimal V well separated from 0 - Big Bounce.
  - Singularity resolved deterministically and dynamically.

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The comparizon

True for all values of cosmological constant

- Geometrodynamics (WDW)
  - Lack of singularity resolution:
    - additional boundary data at the singularity
    - minimal volume comparable to dispersions
- Loop Quantum Cosmology
  - Dynamical singularity resolution
    - unique unitary evolution
    - minimal volume well isolated from $V = 0$