2d Quantum Geometry

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Partly based on new work with Timothy Budd

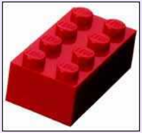
A noble task in ancient, pre-AdS/CFT time, was to to find a non-perturbative definition of Polyakov's bosonic string theory.

$$Z = \int \mathcal{D}[g_{\alpha\beta}]\, e^{-\Lambda \int d^2\xi \sqrt{g}} \int \mathcal{D}_g X_\mu \, e^{-\frac{1}{2\alpha'} \int d^2\xi \sqrt{g} g^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\mu}.$$

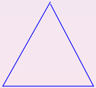
$[g_{\alpha\beta}]$ represents a continuous 2d geometry. Assume that the piece-wise linear geometries one can obtain by gluing together equilateral triangles with the link length a is uniformly dense in the set of continuous 2d geometries when $a \to 0$ (Dynamical Triangulations (DT)). Dimensionless DT bosonic string version:

$$Z(\mu) = \sum_T e^{-\mu N_T} \int \prod_{\triangle \in T,\, \nu}' dx_\nu(\triangle)\, e^{-\frac{1}{2} \sum_{\triangle,\triangle'} (x_\nu(\triangle) - x_\nu(\triangle'))^2}.$$

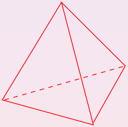
showcasing piecewise linear geometries via building blocks:



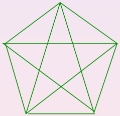
$\longrightarrow$



2d



3d



4d

How to obtain the continuum limit when $a \to 0$.

Free particle as an example. The propagator $G(Y_\nu, Y'_\nu)$

$$G(Y_\nu, Y'_\nu) = \int \mathcal{D}[g] e^{-\Lambda \int d\xi \sqrt{g}} \int \mathcal{D}X_\nu \; e^{-\frac{1}{2\alpha'} \int_0^1 d\xi \sqrt{g} g^{-1} (\partial_\alpha X_\nu)^2}$$

where $X_\nu(0) = Y_\nu$ and $X_\nu(1) = Y'_\nu$. $[g]$ is the geometry of a world line, i.e. $d\ell^2 = g(\xi) d\xi^2$ and $\int d\xi \sqrt{g} = \ell$. We now discretize the path integral by dividing the worldline in n equal steps, and use dimensionless variables: $(x(0) = x, \, x(n) = x')$

$$G(x_\nu, x'_\nu, \mu) = \sum_n e^{-\mu n} \int \prod_{i=1,\nu=1}^{n-1,d} dx_\nu(i) \; e^{-\frac{1}{2} \sum_{i=1}^n (x_\nu(i) - x_\nu(i-1))^2}$$

One can perform the Gaussian integrations:

$$\int \prod_{i=1,\nu=1}^{n-1,d} dx_\nu(i)\; e^{-\frac{1}{2}\sum_{i=1}^{n}(x_\nu(i)-x_\nu(i-1))^2} = \frac{(2\pi)^{nd/2}}{(2\pi n)^{d/2}}\; e^{-\frac{(x_\nu-x'_\nu)^2}{2n}}.$$

Introducing $\mu_c = \log(2\pi)^{d/2}$

$$G(x_\nu,x'_\nu\mu) = \sum_n \frac{1}{(2\pi n)^{d/2}}\; e^{-(\mu-\mu_c)n}\; e^{-\frac{(x_\nu-x'_\nu)^2}{2n}}$$

$$G(x_\nu,x'_\nu\mu) \approx f(|x_\nu - x'_\nu|)\;\; e^{-m(\mu)|x_n-x'_n|}, \qquad m(\mu) \propto \sqrt{\mu-\mu_c}.$$

Scaling : $m^2(\mu) = \mu-\mu_c = m_{ph}^2 a^2,\;\; x\,a = Y,\;\; x'\,a = Y',\;\; t = na^2$

$$G(Y_\nu,Y'_\nu;m) = \lim_{a\to 0} a^{d-2} G(x,x',\mu) = \int_0^\infty \frac{dt}{(2\pi t)^{d/2}}\; e^{-m^2 t - \frac{(Y-Y')^2}{2t}}$$

One can also perform the Gaussian integration in the string case (zero mode fixed, Δ_T compinatorial Laplacian on the duel (ϕ^3) graph to T)

$$\int \prod_{\triangle\in T,\,\nu}' dx_\nu \, e^{-\frac{1}{2}\sum_{\triangle,\triangle'}(x_\nu(\triangle)-x_\nu(\triangle'))^2} = \Big(\det(-\Delta'_{T_N})\Big)^{-d/2}$$

$$Z(N) = \sum_{T_N} \Big(\det(-\Delta'_T)\Big)^{-d/2} = e^{\mu_c N} N^{\gamma(d)-3}\Big(1+0\Big(\frac{1}{N^2}\Big)\Big)$$

$$Z(\mu) = \sum_N e^{-\mu N} Z(N) = \sum_N e^{-(\mu-\mu_c)N} N^{\gamma(d)-3}\Big(1+0\Big(\frac{1}{N^2}\Big)\Big)$$

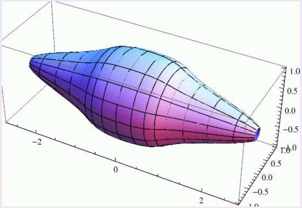
Scaling limit: $\mu \to \mu_c$

$$\mu - \mu_c = \Lambda a^2, \quad (\mu-\mu_c)N_T = \Lambda \int d^2\xi\sqrt{g}.$$

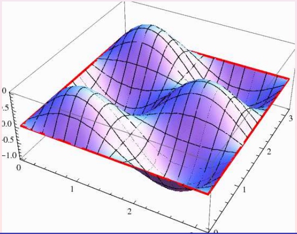
From Z to Green functions with n boundaries, $G(\gamma_1, \ldots, \gamma_n; \mu)$. In particular $G(x_1, \ldots, x_n; \mu)$ (Fourier transform: vertex functions), and $G(\gamma_{L_1 \times L_2}, \mu)$ (a planar "Wilson loop").

Properties (subadditivity):

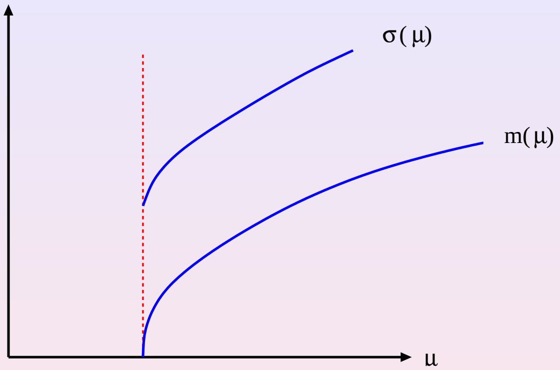
$$G(x, y, \mu) \leq e^{-m(\mu)|x-y|}, \quad m(\mu) \geq 0.$$



$$G(\gamma_{L_1 \times L_2}, \mu) \leq e^{-\sigma(\mu) A(\gamma_{L_1 \times L_2})}, \quad \sigma(\mu) \geq 0.$$



However the dominant worldsheet surfaces look completely different.

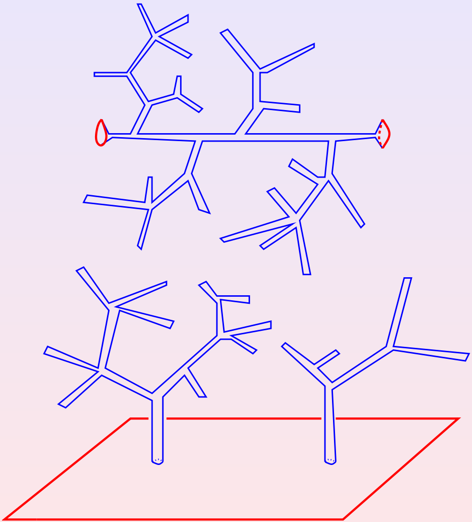


$$m(\mu) = (\mu - \mu_c)^\nu = m_{ph}\, a^\nu, \quad \sigma(\mu) = \sigma_{ph}\, a^{2\nu}, \quad \sigma_{ph} \to \infty.$$

The dominance of Branched Polymer

$G(x, x', \mu)$

$G(\gamma_{L_1 \times L_2})$



A bosonic string theory, originating from a random surface theory, the random surfaces having positive weight, does not exist.

However, viewing the string world sheet as 2d space-time, we can view Polyakov's bosonic string theory as 2d gravity coupled d massless scalar fields, i.e. to a conformal field theory with central charge $c = d$, and we can instead study 2d quantum gravity coupled to (conformal) field theories. Surprisingly this theory has a rich structure as long as the central charge $c \leq 1$.

$$Z = \sum_N \mathrm{e}^{-\mu N} \sum_{T_N} Z_{T_N}(\mathrm{matter}).$$

$$\sum_{T_N} Z_{T_N}(\mathrm{matter}) = \mathrm{e}^{\mu_c(\beta)N} N^{\gamma(\beta)-3}\Big(1 + O(N^{-2})\Big).$$

Typical example: Ising model coupled to DT (V. kazakov).

$$Z_{T_N}(\beta) = \sum_{\sigma_\triangle} e^{\beta \sum_{\langle \triangle, \triangle' \rangle} \sigma_\triangle \sigma_{\triangle'}}$$

$$Z_N(\beta) = \sum_{T_N} Z_{T_N}, \quad Z_N(\beta) = e^{\mu_c(\beta) N} N^{\gamma(\beta)-3}(1 + \cdots),$$

$$Z(\beta) = \sum_N e^{-\mu N} Z_N(\beta) = \sum_N e^{-(\mu - \mu_c(\beta))N} N^{\gamma(\beta)-3}(1 + \cdots)$$

$\mu_c(\beta)$ is the free energy density of spins. The model has a phase transition at a critical β_c, the transition being third order rather than the standard second order phase transition. At the transition point $\gamma(\beta)$ jumps from -1/2 to -1/3.
Interpretation: at the transition point the system describes the continuum conformal field theory of central charge $c = 1/2$ coupled to 2d quantum gravity. Away from β_c the continuum limit is that of pure 2d gravity (the lattice spins couple weakly).

Continuum formulation:

$$Z = \int \mathcal{D}g_{\alpha\beta}\, e^{-\mu A(g)} \int \mathcal{D}_g \psi\, e^{-S(\psi,g)}, \quad A(g) = \int d^2\xi \sqrt{g}$$

Partially gauge fixing: $g_{\alpha\beta} = e^{\phi}\hat{g}(\tau_i)$

$$Z(\hat{g}) = \int \mathcal{D}_{\hat{g}}\phi\, e^{-S_L(\phi,\hat{g})},$$

Independence of $Z(\hat{g})$ of $\hat{g}$ leads to

$$S_L(\phi,\hat{g}) = \frac{1}{4\pi}\int \sqrt{\hat{g}}\, \Big((\partial_\alpha\phi)^2 + Q\,\hat{R}\,\phi + \mu\, e^{2\beta\phi}\Big)$$

$$Q = \sqrt{(25-c)/6}, \quad Q = 1/\beta + \beta.$$

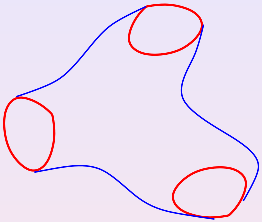
Even for $c = 0$ we have a non-trivial theory.

The $c = 0$ theory can be solved explicitly, even at a regularized level, by counting triangulations (Tutte, 1962). Slightly non-trivial structure can be imposed by n boundaries of lengths ℓ_n.

$$W(\ell_1, \ldots, \ell_n, V) = \int_{\ell_1,\ldots,\ell_n} \mathcal{D}[g_{\alpha\beta}]\, \delta(A(g) - V)$$

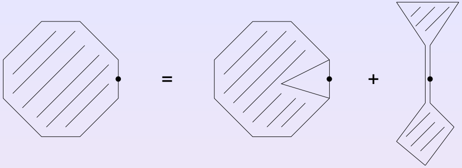
$$W(\ell_1, \ldots, \ell_n, \Lambda) = \int_{\ell_1,\ldots,\ell_n} \mathcal{D}[g_{\alpha\beta}]\, \mathrm{e}^{-\Lambda A(g)}$$

$$W(\Lambda_1^B, \ldots, \Lambda_n^B, \Lambda) = \int \mathcal{D}[g_{\alpha\beta}]\, \mathrm{e}^{-\Lambda A(g) - \sum_i \Lambda_i^B \ell_i(g)}$$

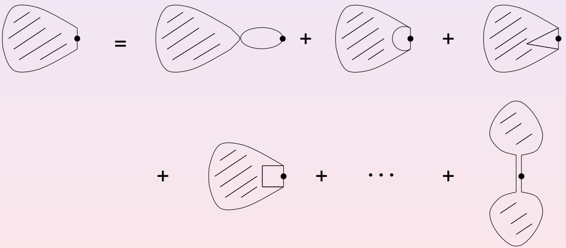


$$W(z, g) = \frac{1}{z} \sum_{n,l} C_{n,l}\, g^n z^{-l}, \quad g = \mathrm{e}^{-\mu},\; z = \mathrm{e}^{\lambda_i}$$

$W(z, g)$ the generation function for $C_{n,l}$, the number of triangulations with n triangles and a boundary of l links.



$$W(z,g) = gz \cdot W(z,g) + \frac{W^2(z,g)}{z}$$ except b.c.



$$W(z,g) = g(\frac{t_1}{z} + t_2 + t_3 z + t_4 z^2 + \cdots) W(z,g) + \frac{W^2(z,g)}{z}$$ except b.c.

One can from $W(z,g)$ calculate $W(\Lambda_B,\Lambda)$ and by inverse Laplace transformation $W(\ell,V)$ and also the more general objects $W(\ell_1,\ldots,\ell_n,V)$ (AJM):
a generalized Hartle-Hawking wavefunction of 2d QG

$$W(\ell_1,\ldots,\ell_n,V) = V^{n-7/2}\sqrt{\ell_1\cdots\ell_n}\;\mathrm{e}^{-(\ell_1+\cdots+\ell_n)^2/V}$$

The result is universal: for any non-negative values of the coupling constants $g_k = t_k g$ we obtain the same expression in the continuum limit.

Generalization: allow for negative weights g_k. Then it is possible to obtain new critical behavior by fine-tuning the g_k's. leading to the so-called m^{th} multicritical point, $m = 2, 3, \ldots$ and corresponds to a (p,q)=(2,2m-1) minimal conformal field theory coupled to 2d quantum gravity.

We thus have an amazing realization of the Wilsonian picture: a infinite dimensional coupling constant space $\{g_k\}$, where the critical surface has finite co-dimension and where fine tuning of the bare coupling constants leads to different critical points.

Only one thing is missing: the concept of a divergent correlation length, the key-ingredient in the Wilsonian picture.

But how to define the concept of correlation length in QG ?

Ordinary QFT: Assume the volume V is sufficiently large and rotation and translational invariance except for boundary effects. ($S(R)$ "area" of spherical shell)

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{Z_V}\int \mathcal{D}\phi\, e^{-S[\phi]}\,\phi(x)\phi(y); \quad Z_V = \int \mathcal{D}\phi\, e^{-S[\phi]}$$

$$\langle \phi\phi(R)\rangle_V \equiv \frac{1}{V\,Z_V\,S(R)}\int \mathcal{D}\phi\, e^{-S[\phi]} \iint dx\,dy\;\phi(x)\phi(y)\;\delta(R-|x-y|).$$

$$\langle \phi\phi(R)\rangle_V \sim \frac{1}{R^{2\Delta_0}}, \qquad R \ll \frac{1}{m_{ph}}, \quad [\phi] = \Delta_0$$

$$\langle \phi\phi(R)\rangle_V \sim R^{-\alpha}e^{-m_{ph}R} \qquad \frac{1}{m_{ph}} \ll R \ll \frac{1}{V^{1/d}}$$

Generalization to a diffeomorphism invariant, metric theory

$$\langle \phi\phi(R)\rangle_V \equiv \frac{1}{V\, Z_V} \int \mathcal{D}[g]\, \delta\Big(A(g)-V\Big) \int \mathcal{D}_g\phi\, \mathrm{e}^{-S[g,\phi]}$$
$$\iint dx\, dy\, \frac{\sqrt{g(x)}\,\sqrt{g(y)}}{S_g(y,R)}\, \phi(x)\phi(y)\, \delta(R - D_g(x,y)).$$

$D_g(x,y)$ is the geodesic distance between x and y.

Main questions: does it make sense to think about a "diffeomorphism invariant" correlation length, does it make sense in a lattice theory ? (and if so, do the exponents Δ_0 of flat space change ?)

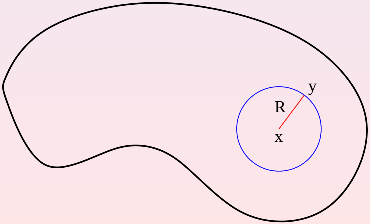
If not, it is difficult to believe that one can apply any Wilsonian way of thinking...

Let us first discuss the quantity $S_g(x,R)$. The "non-classical" behavior of $S_g(x,R)$ will be the reason for the change $\Delta_0 \to \Delta$ when going from flat space to the fluctuating geometries of 2d quantum gravity.

$$S_V(R) = \frac{1}{V}\left\langle \int dx\,\sqrt{g(x)}\; S_V(x,g)\right\rangle_V$$

$$S_\Lambda(R) = \left\langle \int dx\,\sqrt{g(x)}\; S(x,g)\right\rangle_\Lambda$$

$$Z_\Lambda S_\Lambda(R) = \int_0^\infty dA\; e^{-\Lambda A}\; V Z_V S_V(R)$$



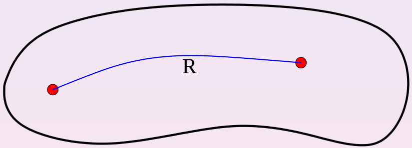
Hausdorff dimension d_h:

$$S_V(R) = R^{d_h-1} F\left(\frac{R}{V^{1/d_h}}\right)$$

$$G_\Lambda(R) \equiv Z_\Lambda S_\Lambda(R) =$$

$$\iint \mathcal{D}[g]\mathcal{D}\psi \; e^{-\Lambda A_g - S[\psi,g]} \iint dxdy \sqrt{g(x)g(y)}\; \delta(D_g(x,y) - R)$$

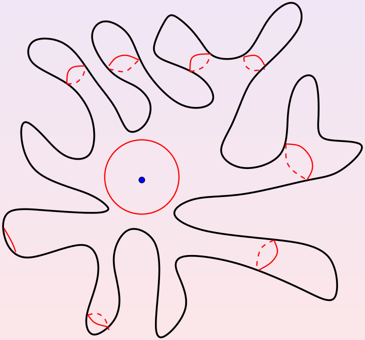
The partition function for universes with two marked points separated a geodesic distance R



For $c = 0$: combinatorial problem. Continuum limit:

$$G_\Lambda(R) = \Lambda^{3/4} \frac{\cosh(\sqrt[4]{\Lambda}\, R)}{\sinh^3(\sqrt[4]{\Lambda}\, R)} \qquad S(R)_V = R^3 F\left(\frac{R}{V^{1/4}}\right),$$

This presumably implies the following mathematical statement: there exists a measure on the set of continuous 2d geometries (of fixed topology) such that by probability one the Hausdorff dimension of a randomly chosen geometry is four.
A situation very similar to the one for the ordinary random work where the Hausdorff dimension of the walk is two, not one. But how is it possible when there is no extrinsic target space ?



$$S(R)_V = \frac{1}{VZ(V)} \int \mathcal{D}[g]\, \delta(A(g) - V)$$
$$\int dxdy \sqrt{g(x)}\sqrt{g(y)}\, \delta(D_g(x,y) - R)$$

R is an external parameter setting a scale

Thus we expect the following behavior for a conformal theory coupled 2d Euclidean QG:

$$\langle \phi\phi(R) \rangle_V = R^{-d_h \Delta}\, F\left(\frac{R}{V^{1/d_h}}\right),$$

$$\langle \phi\phi(R) \rangle_V = V^{-\Delta}\, \frac{F(x)}{x^{d_h \Delta}}, \qquad x = \frac{R}{V^{1/d_h}}$$

Here $F(0) = const. > 0$, and $F(x)$ falls of at least exponentially fast for $x > 1$.

$$\Delta = 2\,\frac{\sqrt{1-c+12\Delta_0} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}, \qquad \text{KPZ – DDK scaling.}$$

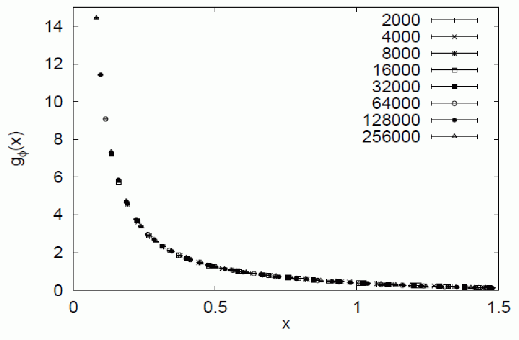
$$c \to -\infty \qquad \Delta \to \Delta_0 \qquad \text{Does } d_h \to 2?$$

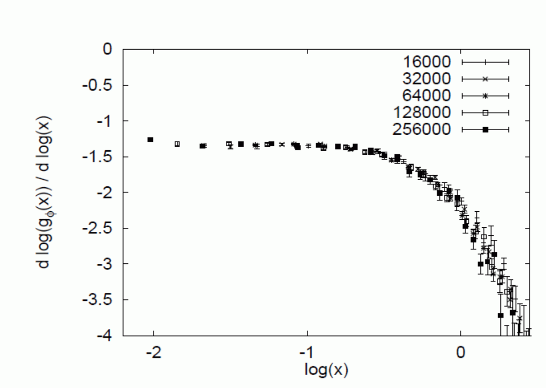
Test this in DT

$V \sim N_T$. Geodesic distance $R \sim \ell$ the link distance between two vertices.

$$\langle \phi\phi(\ell) \rangle_N = N^{-\Delta} \; \frac{F(x)}{x^{d_h \Delta}} \qquad x = \frac{\ell}{N^{1/d_h}}$$

FSS!





Finite Size Scaling allows us to determine Δ and $d_h\Delta$

Theory for Ising model ($c = 1/2$):

$$\Delta_0 = \frac{1}{8} \to \Delta = \frac{1}{3}, \qquad 2\Delta_0 = \frac{1}{4} \to d_h\Delta = 1.40\ldots \;\; d_h = 4.2..$$

In general we thus have

$$\langle S(R)\rangle_V \sim R^{d_h(c)-1}, \qquad R \ll V^{1/d_h(c)},$$

Is there a general formula for $d_h(c)$? Yes (Watabiki):

$$d_h(c) = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}, \qquad d_h(0)=4, \quad d_h(-\infty)=2$$

Let $\Phi_n[g]$ be invariant under diffeomorphisms and assume $\Phi_n[\lambda g] = \lambda^{-n}\Phi[g]$ classically for constant λ. Then the quantum average satisfies (generalized KPZ-DDK)

$$\langle \Phi[g]\rangle_{\lambda V} = \lambda^{-\alpha_{-n}/\alpha_1}\langle \Phi[g]\rangle_V, \qquad \alpha_n = \frac{2n}{1+\sqrt{\frac{25-c-24n}{25-c}}}$$

$$\Phi_1[g] = \int dx\sqrt{g}\, \Delta_g(x)\, \delta_g(x, x_0)|_{x=x_0}, \quad \Phi_1[\lambda g] = \lambda^{-1}\Phi_1[g]$$

This operator appear when we study diffusion on a smooth manifold with metric $g_{\mu\nu}$. The diffusion kernel is

$$K(x, x_0; t) = e^{t\Delta_g} K(x, x_0; t), \quad K(x, x_0; 0) = \delta_g(x, x_0)$$

The short distance behavior is

$$K(x, x_0; t) \sim \frac{e^{-D^2(x,x_0)/2t}}{t^{d/2}}(1 + O(t)), \quad \langle D(x, x_0; t)^2\rangle \sim t + 0(t^2)$$

The return probability is

$$\begin{aligned} P(t) &= \frac{1}{V}\int dx\sqrt{g}\, K(x, x; t) \\ &= \frac{1}{V}\int dx\sqrt{g}\, (1 + t\Delta_g + \cdots)\, \delta_g(x - x_0)|_{x=x_0} \\ &= 1 + t\,\Phi_1[g] + O(t^2) \end{aligned}$$

For the Hausdorff dimension we have (declaring Dim[V] = 2)

$$\langle V \rangle_R = R^{d_h}, \qquad \mathrm{Dim}[R] = \frac{2}{d_h}$$

From the diffusion equation

$$\mathrm{Dim}[D(x,x_0)] = -\frac{1}{2}\mathrm{Dim}[\Phi[g]]$$

Taken the quantum average, using KPZ scaling:

$$\mathrm{Dim}[\langle D(x,x_0)\rangle] = -\frac{1}{2}\mathrm{Dim}[\langle \Phi[g]\rangle] = -\frac{\alpha_{-1}}{\alpha_1}$$

Thus

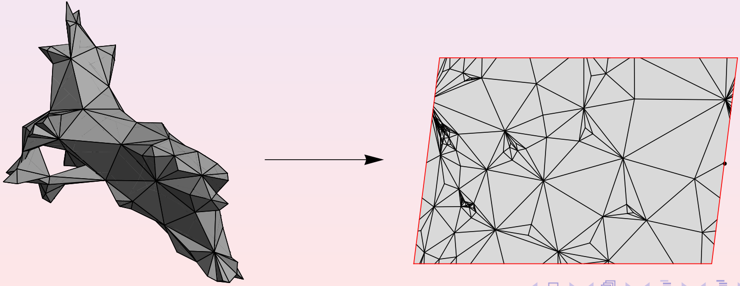
$$d_h = \frac{-2\alpha_1}{\alpha_{-1}} = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}$$ Correct?

Test the formula in the case of toroidal topology.

Virtues:

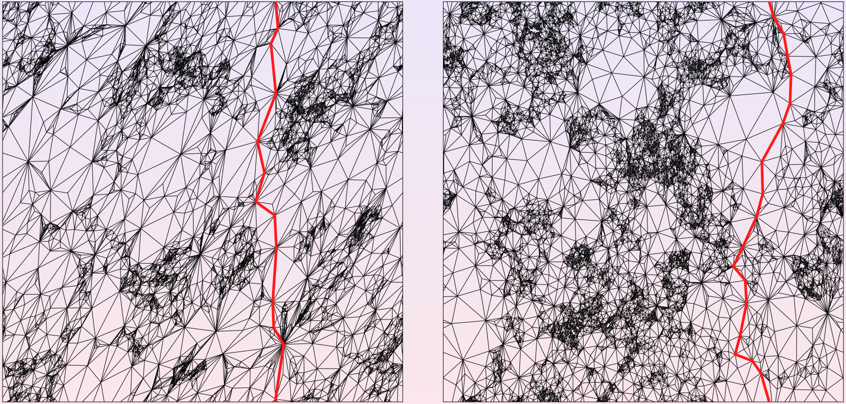
(1) the shortest non-contractable loop is automatically a geodesic curve. Thus in the discretized case we only have to look for such loops.

(2) If the manifold is analytic we have harmonic forms which have very nice discretized analogies, and we can use the these to construct a conformal mapping from the abstract triangulation to the complex plane.



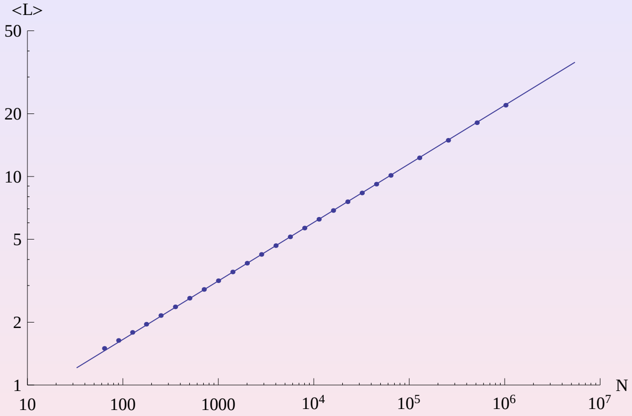
Since the shortest contractable loop is a geodesic we expect

$$\langle L \rangle_N \sim N^{1/d_h(c)}$$



left figure $c = 0$, i.e. $d_h = 4$, right figure $c = -2$, $d_h = 3.56$

Quantitative check of $\langle L \rangle_N \sim N^{1/d_h}$ for $c = -2$



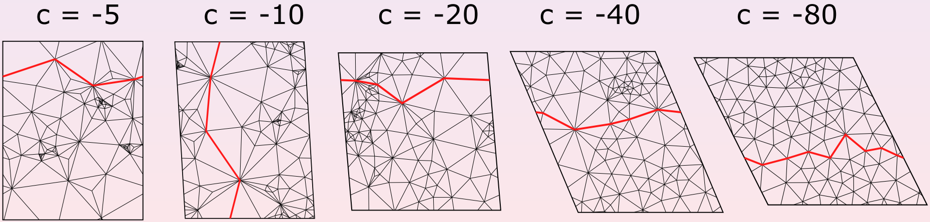
Straight line: $\langle L \rangle_n = 0.45\, N^{1/3.56}$.

Recall the (regularized) bosonic string $c = d$:

$$Z(\mu) = \sum_T e^{-\mu N_T} \int \prod_{\triangle \in T,\, \nu}' dx_\nu(\triangle)\, e^{-\frac{1}{2}\sum_{\triangle,\triangle'}(x_\nu(\triangle)-x_\nu(\triangle'))^2}.$$

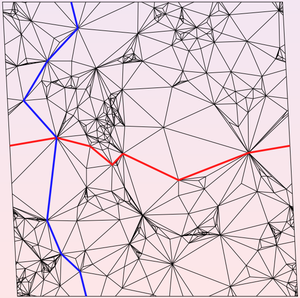
$$Z(\mu) = \sum_N e^{-\mu N_T} Z(N), \quad Z(N) = \sum_{T_N} \Big(\det(-\Delta'_{T_N})\Big)^{-d/2}$$

(Note that $d = -2$ is special.)



However, the situation for $c > 0$ more difficult and until recently numerical simulations could not really determine $d_h(c)$ for $c > 0$. Matter correlation functions gave agreement with Watabiki's formula, but geometric measurements agreed better with $d_h = 4$ for $0 < c < 1$.

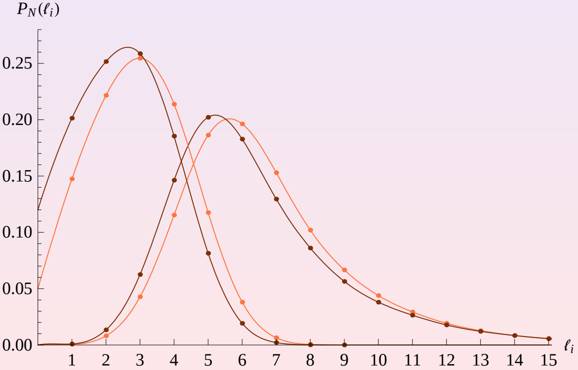
Using simulations of the DT-torus with Ising spin (c=1/2) and 3-state Pott's model (c=4/5), and analyzing the second shortest (independent) loop, one obtains data with little discretization "noise".

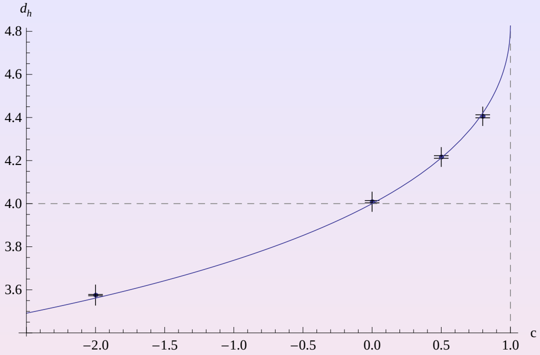


The probability distributions for homotopy classes Γ_i of simple connected, non-contractable loops:

$$P_N^{(i)}(\ell_i) = N^{1/d_h} F_i(x_i) \qquad x_i = \frac{\ell_i}{N^{1/d_h}}$$

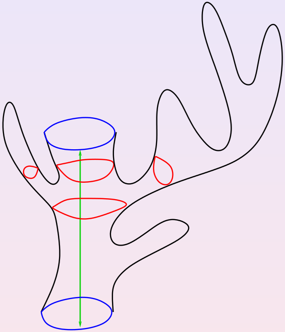
Refence loop distributions for $N = 8000$:





c	d_h (by fit)	d_h (theoretical)
-2	3.575 ± 0.003	3.562
0	4.009 ± 0.005	4.000
$1/2$	4.217 ± 0.006	4.212
$4/5$	4.406 ± 0.007	4.421

The geodesic distance R is a natural “time” parameter of the quantum universe, and one can indeed talk about the “propagation” of “space” in a proper time $R \equiv t$:



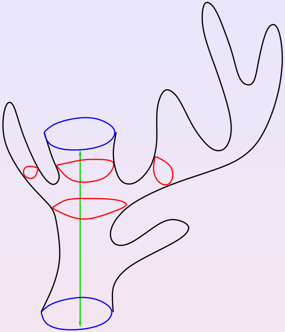
$$G(\ell_1, \ell_2; t) = \langle \ell_2 | e^{-\hat{H} t} | \ell_1 \rangle$$

$$G(x, y, t) = \int d\ell_1 d\ell_2 \, e^{-(x\ell_1 + y\ell_2)} G(\ell_1, \ell_2, t)$$

$$\frac{\partial G(x, y, t)}{\partial t} = \frac{\partial (W(x) G(x, y, t))}{\partial x}$$

$\hat{H}$ cannot be Hermitian

The branching off of baby universes dominates everything and is responsible for the fractal dimension different from 2 as well as $\hat{H}$ being non-Hermitian. Can this be “cured”?



Introduce a penalty for branching by attaching a coupling constant g to each creation (or disappearance) of a baby universe. Keeping track of the cut-off a, assuming ℓ has dimension $[a]$, x dimension $-[a]$, t dimension $\varepsilon[a]$, and $W(x)$ dimension $-\eta[a]$ one obtains:

$$a^{\varepsilon}\frac{\partial}{\partial t}G(x,y,t) = \frac{\partial}{\partial x}\Big((a(x^2-\lambda)+2ga^{\eta-1}W(x))\,G(x,y,t)\Big)$$

For Euclidean quantum gravity $\eta = 3/2$ and baby universe creation dominates and geodesic distance t scales anomalously ($\varepsilon = 1/2$).

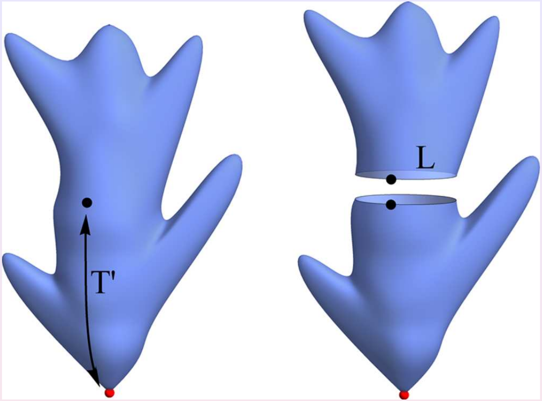
However, one would naturally expect $W(x)$ to scale as $\eta = -1$. In this case one can obtain $\varepsilon = 1$ by choosing

$$g = G_s a^3 \quad \text{or} \quad g = 0.$$

This results in a continuum theory governed by the loop-loop equation

$$\frac{\partial}{\partial t} G(x,y,t) = \frac{\partial}{\partial x} \Big(\big((x^2 - \lambda) + 2G_s W(x)\big) \, G(x,y,t) \Big)$$

One can again make an exact counting of triangulations where each baby universe has a weight g and where the two boundary are separted a geodesic distance t and in this way derive the above equation. The equation also selfconsistently determines the disk amplitude $W(x)$.



One can use $G(x,y,t)$ and $W(x)$ to find the two-point function:

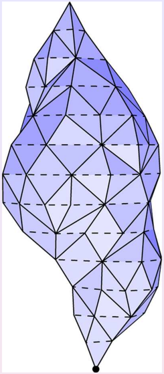
Take $t = R$ to agree with former notation:

$$G_{\Lambda,G_s}(R) = \Sigma^3 \frac{\cosh(\Sigma R + \theta)}{\sinh^3(\Sigma R + \theta)},$$

$$\tanh\theta = \frac{\Sigma}{\alpha} = \sqrt{1 - \frac{G_s}{2\alpha^3}}, \quad \alpha^3 - \lambda\alpha + G_s = 0, \quad \lambda = \Lambda + 3\left(\frac{G_s}{2}\right)^{2/3}.$$

Note: R has canonical dimension and number of baby universes is finite. so it is really a different class of surfaces

$$G_{\Lambda,G_s\to\infty}(R) \propto G_\Lambda(R'), \quad R' = 3^{1/4}\left(\frac{G_s}{2}\right)^{1/6} R.$$



In the limit $G_s \to 0$, even the finite number of baby universes are suppressed. No baby universes means we can talk about a time foliation, R being proper time.

$$G_{\Lambda, G_s \to 0}(R) \propto e^{-2\sqrt{\Lambda} R}, \qquad \text{CDT}$$

CDT is then governed by the Hermitean Hamiltonian associated with

$$\frac{\partial}{\partial t} G(x,y,t) = \frac{\partial}{\partial x}\Big((x^2 - \Lambda)\, G(x,y,t) \Big)$$

$$\hat{H} = -L \frac{d^2}{dL^2} + \Lambda L.$$

The same Hamiltonian is obtained by quantizing projectable 2d HL gravity without higher derivative terms.