### Discrete Wheeler DeWitt Equations

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Based on:

H. Hamber and R. Williams:*"Discrete Wheeler DeWitt Equations"* arxiv: 1109.2530 Phys. Rev. D.84. 104033(2012)

H. Hamber, R. Toriumi, and R. Williams: *"Wheeler DeWitt Equation in 2+1 Dimensions"* arxiv: 1207.3759 Phys. Rev. D. 86, 084010 (2012)

H. Hamber, R. Toriumi, and R. Williams: "Wheeler DeWitt Equation in 3+1 Dimensions" arxiv: 1207.3759

# Motivation for Discretization

No reason not to try quantizing discrete spacetime. May even be certain advantages

*e.g.*, natural cut off (minimum length) finite number of variables

At the Planck scale, spacetime may be discrete anyway, with rapidly changing topologies (spacetime foam, Wheeler)

## Canonical Formulations

ADM (Arnowitt, Deser, Misner) Formalism (*continuum* classical canonical formalism)



Introducing a time-slicing of spacetime, by introducing a sequence of space like hypersurfaces.

Wheeler DeWitt Equation (*continuum* quantum canonical)

 $\hat{D}$ *iscrete* Wheeler DeWitt Equation (*discrete* quantum canonical)

The lack of covariance of the canonical ADM approach has not gone away, and is therefore still part of the present formalism.

### avenues for discretization are possible. One could discretize the theory from the very beginning, Formulation, and it a lapse and a shift function, and it a lapse state  $\mathbb{P}^1$ extrinsic and intrinsic discrete curvatures etc. Alternatively one could try to discretize the contin- $\theta$  and the momentum density Hi to  $\theta$  and  $\theta$  if and  $\theta$ Continuum Quantum Canonical words and the control of the control<br>→ Control of the = 3 ⌫ 2 (44)  $\overline{\mathbf{u}}$  ( $\overline{\mathbf{v}}$

Energy constraint  $\hat{H} | \Psi \rangle = 0$  $\mathbf{y}$ 

Wheeler De Witt Eq. in  $d+1$  dim. where *be* when  $pq$ . Wheeler De Witt Eq. in *d+1* dim.  $\frac{1}{\sqrt{1 + \sum_{i=1}^{n} x_i}}$  $g \to Eq.$  in  $d+1$  dim.

$$
\left\{ - (16\pi G)^2 G_{ij,kl}(x) \frac{\delta^2}{\delta g_{ij}(x) \delta g_{kl}(x)} - \sqrt{g(x)} \left( {}^{(d)}R(x) - 2\lambda \right) \right\} \Psi \left[ g_{ij}(x) \right] = 0
$$
  
Supermetric:  $G_{ij,kl} = \frac{1}{2} g^{-1/2} (g_{jk} g_{jl} + g_{il} g_{jk} - \alpha g_{ij} g_{kl})$ 

Momentum constraint  $\hat{H}_i |\Psi\rangle = 0$  $\bigg\{ \, 2 \, i \, g_{ij}(\mathbf{x}) \, \nabla_k(\mathbf{x}) \, \frac{\delta}{\delta g_{jk}(\mathbf{x})}$  $\mathcal{L}$  $\left\{ 2 i g_{ij}(\mathbf{x}) \nabla_k(\mathbf{x}) \frac{\partial}{\partial g_{ik}(\mathbf{x})} \right\} \Psi[g_{ij}(\mathbf{x})] = 0$  $\overline{\phantom{a}}$  $\textsf{constraint} \;\; \hat{H}_i \ket{\Psi} \; :$ *<sup>i</sup>* @*l* 2 *j*  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$  $\mathcal{L}$ @2 @*l*<sup>12</sup> @*l*<sup>01</sup> ◆

### $\mathbf{C}$ Note that the state that the state is the state  $\mathsf{Re}{\tt q}$  at every x, each of the form  $\mathsf{Re}{\tt q}$  at  $\mathsf{Re}{\tt q}$ Regge Lattice Discretization:  $\bigcap_{i=1}^n$  is a gauge invariance  $\bigcup_{i=1}^n$  in the  $\bigcup_{i=1}^n$  in the  $\bigcup_{i=1}^n$  in the  $\bigcup_{i=1}^n$

*h* + 2<sup>*A*</sup> + 2<sup>*A*</sub></sup> [ *s* ]=0*,* (24) Regge!1961!! $\eta$  theory theory theory is the  $\eta$  support in  $\eta$  and the need for  $\eta$  and the need for  $(0.161)$ 

In constructing a *discrete* Hamiltonian for gravity one has to decide what degrees of freedom one should retain on the lattice. ⇣ 2 <sup>p</sup><sup>2</sup> degrees of freedom one should retain on the lattice.

 $\rightarrow$  Use geometric  ${Regge}$  *lattice discretization* for gravity, with edge lengths suitably defined on a random lattice as the primary *)))dynamical)variables*. studing are linearly related to deformations of the induced metric. In a given simplex of the n-dimensional simplex  $\ell$ lattice discretiza *C A A A A A A A A*  $\imath$  for gravity, with simplex with vertices 1,2, 3 ... n + 1 and square edge lengths 122  $=$  121  $\pm$  122  $\pm$  121  $\pm$ 

σ, take coordinates based at a vertex 0, with a vertex 0, with axes along the edges from 0. The other vertices<br>Το οποίο του στην συνεχία Degrees of freedom for edges and metric tensor are both  $D(D+1)/2$  in  $D$  dimensions. for edges and metric tensor are both  $D(D+1)/2$  in  $D$  dimensions.<br>, egrees of freedom for edges and metric tensor are both  $D(D+1)/2$  in D dimensions.



$$
g_{ij}(\sigma) = e_i \cdot e_j \qquad l_{ij} = |\mathbf{e}_i - \mathbf{e}_j|
$$
  
\n
$$
g_{ij}(\sigma) = \frac{1}{2} \left( l_{0i}^2 + l_{0j}^2 - l_{ij}^2 \right)
$$
  
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\mathbf{e}_1 \qquad \qquad l_{01}
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$$
\mathbf{e}_1 \qquad \qquad l_{12}
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\mathbf{e}_2 \qquad \qquad l_{01}
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\mathbf{e}_3 \qquad \qquad l_{13}
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\mathbf{e}_3 \qquad \qquad l_{13}
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\mathbf{e}_4 \qquad \qquad l_{13}
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\mathbf{e}_2 \qquad \qquad l_{15}
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$$
\mathbf{e}_3 \qquad \q
$$

## **Regge Formulation:** Constituents

Curved space(time)s are piece-wise linear.

*Flat* building blocks are D-dim. Simplices  $(D)$ 

Point (0 -simplex) in 0-dim Line  $(1 - \text{simplex})$  in 1-dim Triangle (2 -simplex) in 2-dim Tetrahedron (3 -simplex) in 3-dim all "flat"

Deficit angle  $\rightarrow$  Curvature (defined at a hinge( $(D-2)$ ) at a vertex for 2-dim, at an edge for 3-dim, at a triangle for 4-dim)

$$
\begin{array}{|c|c|}\n\hline\nc & D & Deficit angle, s \\
\hline\n\end{array}
$$

Figure 42.1.

Misner, Thorne, Wheeler

$$
\left(\begin{array}{c}\n\cdot \\
\cdot \\
\cdot\n\end{array}\right)
$$

$$
\delta(h) = 2\pi - \sum_{\sigma \supset h} \theta(\sigma, h)
$$

sum over dihedral angles  $\theta$  extends over all simplices  $\sigma$  meeting on hinge h.

### $J+1$  Dimonsional  $d+1$ - Dimensional  $\alpha$  is creater  $\lambda$  lattice  $\alpha$  Dal $\lambda$  ittle counting *Discrete* wheeler Dewitt equation  $\sqcup$  $Discrete$  Wheeler DeWitt equation (16⇡*G*)  $D$ *iscrete* v  $\mathcal{L}(\mathcal{L} \times \mathcal{L}) = \mathcal{L}(\mathcal{L} \times \mathcal{L})$  $\mathbf{q}$ [*gij* (*x*)] = 0 (46)

⇢

$$
\left\{ - (16\pi G)^2 G_{ij,kl}(x) \frac{\delta^2}{\delta g_{ij}(x) \delta g_{kl}(x)} - \sqrt{g(x)} \left( \frac{d}{dR(x)} - 2\lambda \right) \right\} \Psi \left[ g_{ij}(x) \right] = 0 \quad \text{Continuum}
$$
\n
$$
g_{ij}(x)
$$
\n
$$
g_{ij}(x)
$$
\n
$$
g_{ij}(\sigma) = \frac{1}{2} \left( l_{0i}^2 + l_{0j}^2 - l_{ij}^2 \right)
$$
\n
$$
\left\{ - (16\pi G)^2 G_{ij} \left( l^2 \right) \frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \sqrt{g(l^2)} \left( \frac{d}{dR} \left( l^2 \right) - 2\lambda \right) \right\} \Psi \left[ l^2 \right] = 0 \quad \text{Discrete}
$$
\n
$$
\text{Kinetic term}
$$
\n
$$
\text{Curvature term}
$$
\n
$$
\text{Cosmological constant term}
$$

are defined  $\overline{a}$ at each "noint" in snace Both equations are defined at each "point" in space.

### Discrete WDW eq.: one eq. for each simplex *i*⇢4 e Discrete WDW eq.: one eq. for each simplex *<sup>G</sup>i,j* () @<sup>2</sup> @*l* 2 *<sup>i</sup>* @*l* 2 *j* 2 *q h*⇢ *l<sup>h</sup> <sup>h</sup>* + 2 *V*  $\mathbf{R}$ ⇥ *l* = 0 (51)

### ( Kinetic Term ⇠ *<sup>g</sup>*<sup>0</sup> (43) **16** Kinet @*l* 2 *<sup>i</sup>* @*l* 2 <sup>p</sup>*<sup>g</sup>* (*l*2) ⇣ (*d*1)*R l* 2 i<br>∂la i ∂la je na je n The right hand side of this equation contains precisely the expression appearing in the continuum  $i$   $\blacksquare$   $\sim$  1 CM and for the following processes for the supermetric intervals processes for the supermetric intervals for the supermetric intervals for the supermetric intervals for the supermetric intervals for the supermetric

$$
\left\{ \frac{- (16\pi G)^2 G_{ij} (l^2) \frac{\partial^2}{\partial l_i^2 \partial l_j^2}}{-\sqrt{g(l^2)} \left( \frac{d^2 R}{l^2} - 2\lambda \right)} \right\} \Psi [l^2] = 0
$$
\n
$$
G^{ij}(l^2) = -d! \sum_{\sigma} \frac{1}{V(\sigma)} \frac{\partial^2 V^2(\sigma)}{\partial l_i^2 \partial l_j^2}
$$
 (geometric) by Regge and Lund\n
$$
W / V^2(\sigma) = \left(\frac{1}{d!}\right)^2 \det g_{ij}(l^2(\sigma))
$$
\n*Only involves the variables within one simplex.*

**Curvature Term**  
\n
$$
\left\{-(16\pi G)^2 G_{ij} (l^2) \frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \frac{\sqrt{g(l^2)} (d^d R(l^2) - 2\lambda)}{\sqrt{g}} \right\} \Psi [l^2] = 0
$$
\n
$$
\sqrt{g} (d^d R) = \frac{2}{q} \sum_{h \subset \sigma} \delta_h (d-2) V_h
$$

 $q:$  coordination number  $\frac{q}{q}$  . cool a 2) given above. The range of the summation over it and summation over it and support in an over  $\alpha$ 

9

*Involves the variables of the neighbor simplices of a simplex.*   $2 <sup>2</sup>$ *i,j*⇢  $\frac{d}{dx}$ he va *j ria*<br>ria  $\mathbf{L}$ *h*⇢  $\int f \, h \, a \, u \, a$ <u>s'</u>  $\lambda$  bou giuindicas of  $\alpha$ involves the variables of the heighbor simplices of a simplex.

9

12

# $D$ *iscrete* WDW eqns in  $2+1$  dim.

 $\Psi[l^2]$  is a function of the whole simplicial geometry (overall *geometry* of the manifold), due to the built-in diffeomorphism invariance.

 $\Psi[l^2]$  depends collectively on all the edge lengths in the lattice.

Therefore even though we have one equation for each simplex, there should be one wave function that satisfies all the equations for each simplices in one configuration.

### $C<sub>i</sub>$   $\alpha$ ,  $\alpha$ ,  $\alpha$ *A* ngle Triangle  $\sqrt{2}$  [ *a, b, c* ] = <sup>1</sup>  $A$  is the 2+1 case discussed in the previous section,  $\omega$ Exact Solution for A Single Triangle @*<sup>a</sup>* @*<sup>b</sup>* <sup>+</sup> @2 @*b* @*c*  $\overline{ }$ @2 @*c* @*a*  $(2+1)$  dim.) 2  $\frac{1}{2}$ A single triangle:  $(2+1)$  dim. oolution for A Single-Friangie  $\ddot{\phantom{a}}$ **c**<sup>1</sup> cos <del>∪</del> <sup>+</sup> *<sup>c</sup>*<sup>2</sup> sin ✓ @*<sup>a</sup>* @*<sup>b</sup>* <sup>+</sup> @*b* @*c*  $\overline{a}$ @*c* @*a*  $2f$ *q h*  $h \circ \phi$  for  $\Lambda$  Cingle Trionale  $(2+1$  aim.) *<sup>h</sup>* + 2*A*<sup>0</sup> *h* Supply 1 2000 11 2000 11 2000 11 2000 11 2000 11 2000 11 2000 11 2000 11 2000 11 2000 11 2000 11 2000 11 2000 1 @*<sup>a</sup>* @*<sup>b</sup>* <sup>+</sup> @2  $\mathbf{L}$ + .<br>22 @*c* @*a* ◆ 2 Exact Solution for A Single Triangle  $(2+1)$  dim.) exact solution for A single Triangle *A 2* + *L c*  $\bm{J}$  $\bm{l}$  $+1$  dim.) ◆ *.* (90) **Fx**  $\overline{2}$ *q*  $\mathbb{R}$ *h*  $\Gamma$  $z$ le: *<sup>h</sup>* + 2*A*<sup>0</sup> [ *l* 2 <sup>2</sup> ]=0*,* (25)  $\ddot{\phantom{0}}$  2 *h*  $(2 + 1 + 2i)$ @2 @*c* @*a*  $(2 +$  $\mathcal{I}$ *d* 1*m*.) <sup>2</sup> ]=0*,* (25) Exact Solution for A Single! A!single!triangle:!! If one sets act Solution for A Single Triangle  $(2+1)$  dim.)<sup>2</sup> (16⇡*G*) <sup>2</sup> 4*A*<sup>0</sup>  $(2+1\dim)$  $\frac{1}{2}$ @*c* @*a <sup>h</sup>* + 2*A*<sup>0</sup>  $\overline{\mathbf{f}}$  $U$ <sup> $\sim$ </sup> $A$ <sup> $\sim$ </sup>  $\ln$ @*<sup>a</sup>* @*<sup>b</sup>* <sup>+</sup> @2 @*b* @*c*  $\blacksquare$ ngle @*c* @*a*  $\sim$   $\sim$   $\sim$  $\text{F}$

A single triangle:  $\overline{\phantom{a}}$ ngle tr  $\sim$ A single <sup>-</sup> @*b* @*c* ria

- Curvature term is absent in this configuration.  $\lambda$  absent in this corr  $,$ u $\overline{\phantom{a}}$ enge dialigie.<br>Curvature term is absent in this configuration. *reader* and a set the coupling of the configuration o  $N = 1$  note the remarkable, result that the wavefunction only depends on  $\mathcal{O}(n)$ re term is abse ر<br>at in tl @*<sup>a</sup>* @*<sup>b</sup>* <sup>+</sup>  $\overline{a}$  $\overline{\mathsf{C}}$ reside change. coupling!regime.! <u>m is absent in this configuration</u> A single triangle.<br>• Curvature term is absent in this configuration  $\epsilon$ *q* • Curvature term is absent in this co  $\frac{1}{2}$ <u>ature tern</u><br>tarting n  $\overline{a}$  $\mathbf{m}$  $\frac{1}{2}$ nfigurati<br>..  $\overline{a}$  $\frac{1}{\pi}$  come is absent in this configuration.  $\overline{1}$ m this con<br>a*trong co* @2 n.  $\overline{\phantom{a}}$ s configurat  $\mathsf{on}$ .
	- as a starting point for the strong coupling expansion in  $1/G$ . be the strong coupling expansion in  $1/G$ .<br>ar the strong coupling expansion in  $1/G$ .  $\mathfrak{c}$  m in  $1/\mathfrak{c}$ • Curvature term is absent in this configuration.<br>• as a starting point for the strong coupling expansion in 1 *<sup>h</sup>* + 2*A*<sup>0</sup> ration.<br>ng expans  $\ddot{\phantom{1}}$ 16*A* as a starting point for the strong coupling expansion in 1/G. a in  $1/G$ . lfiguration.<br>bunling expansion in *1 /G*  $\bullet$  as a starting point <u>bsent i</u>n th<br>or the str<mark>c</mark>  $\overline{z}$ dpling exp • as a starting point for the strong coupling expansion in  $1/G$ . @2  $\bullet$   $\epsilon$  [ *a, b, c* ]=0*,* (26)  $\ddot{\cdot}$ nguration.<br>upling expansion in *1/G*. 2<br>2<br>2 on in
- should show the physical nature of the wavefunction solution deep in the should show the privilent nature of<br>strong coupling regime. of the wavefunction as the area of the triangle approaches zero, *A* ! 0, requires for the constant should show the physical nature of the wavefunction solution deep in the<br>strong coupling rogimo  $(16\pi G \rightarrow G\,)$  [ *a, b, c* ]=0*,* (26)  $c_1$  regime. Therefore the correct  $c_1$  and  $c_2$  and  $c_3$  determined,  $c_4$  and  $c_5$  determined,  $c_6$  and  $c_7$  and  $c_8$  and  $c_9$  and @*c* @*a*  $\frac{3}{4}$ <br>
ahould show the physical nature of the wavefunction solution deep in the  $\overline{O}$ pung expans<br>:he wavefun<mark>c</mark> • should show the physical nature of the wavefunction solution deep in the *c*<sup>1</sup> = 0. Therefore the correct quantum-mechanical solution is unambiguously determined,  $(16\pi G \rightarrow G)$ @*c* @*a* th read in the set of the s *n* and the wavefunction solution deep  $\frac{1}{2}$  hould show the physical nature of the wa ong coupling re ling regime. @*c* @*a* coupling!regime.! @*<sup>a</sup>* @*<sup>b</sup>* <sup>=</sup> <sup>1</sup> (16*A*)<sup>2</sup> (*<sup>b</sup>* <sup>+</sup> *<sup>c</sup> <sup>a</sup>*) (*<sup>a</sup>* <sup>+</sup> *<sup>c</sup> <sup>b</sup>*) *d*.<br>.. on deep ir *d dA*  $S_{\rm eff}$  the partial derivatives leads to the equation  $\epsilon$ ow the physical nature of the wavefunction solution deep in the<br>… I'<u>n a we tive</u> o of the wavefunction as the area of the triangle approaches zero, *A* ! 0, requires for the constant [ *a, b, c* ]=0*,* (26) piing expans<br>bo wayofung  $\frac{1}{2}$  o.<br>Dh solution de @*c* @*a*  $\ddot{\phantom{0}}$ ◆

 $(10 \text{A} \rightarrow G)$  $\left(\frac{\partial}{\partial a}\right) + 2\lambda A_{\triangle}\right\} \Psi$  $\overline{a}$ 1  $[c] = ($  $\overline{0}$  $\frac{1}{\sqrt{2}}$ WDW eq:  $\sqrt{ }$  $G^2$  4 $A_\triangle$  $\int \theta^2$  $\frac{\partial}{\partial a} \frac{\partial}{\partial b}$  +  $\partial^2$  $\partial b$   $\partial c$  $+$  $\partial^2$  $\partial c \ \partial a$ ◆  $+ 2\lambda A_{\triangle}$  $\mathcal{L}$  $\Psi [ \, a, \; b, \; c \, ] \; = \; 0,$ *dA*<sup>2</sup>  $\frac{1}{2}$ *dA* + 16 ˜ *A* = 0 *.* (87)  $\sqrt{a}$   $\theta$   $\theta$   $\theta$   $\theta$   $\theta$  $\partial^2$   $\Big\}$  = 2)  $2\lambda A_\triangle$  $\downarrow \Psi$ sin ✓ 4*A* ˜  $\mathcal{L}_{\mathcal{A}}$ ◆<br>◆<br>◆  $\{2\lambda A_\triangle\} \Psi[a, b, c] = 0.$  $\int_{\mathcal{A}}^2 44 \left( \frac{\partial^2}{\partial^2} + \frac{\partial^2}{\partial^2} + \frac{\partial^2}{\partial^2} \right) + 94 \left[ \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \theta^2} \right]$  $\partial^2$   $\left.\right)$   $\left.\right.$   $\left.\right)$   $\left.\right)$   $\left.\right]$   $\left.\right]$   $\left.\right]$   $\left.\right.$   $\left.\right]$   $\left.\right.$   $\left.\right]$   $\left.\right.$   $\left.\right]$   $\left.\right.$   $\left.\right]$   $\left.\right.$   $\left.\right.$  $\left(\frac{\partial^2}{\partial t} + \frac{\partial^2}{\partial t}\right) + 2\lambda A_0$  (*y* [a, b, c] = 0  $\overline{\phantom{a}}$  $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$   $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$   $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *d*<sup>2</sup> *d* + 8  $\int \theta^2$   $\theta^2$  $\frac{1}{\partial c}$  -*A*  $\begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$  $\frac{d\mathbf{a}}{d\mathbf{b}}\frac{\partial \mathbf{c}}{\partial \mathbf{c}} + \frac{\partial \mathbf{c}}{\partial \mathbf{c}}\frac{\partial \mathbf{a}}{\partial \mathbf{a}} + 2\lambda A_{\Delta} \nbrace \Psi[a, b, c] = 0$  $\int_{\Omega}^{2} 44 \left( \frac{\partial^{2}}{\partial^{2}} + \frac{\partial^{2}}{\partial^{2}} \right) + 94 \right] dx$  $\begin{pmatrix} \frac{\partial^2}{\partial^2} & \frac{\partial^2}{\partial^2} \end{pmatrix}$  $4A_{\wedge}$   $\left(\begin{array}{ccc} \frac{\partial^2}{\partial} & + \end{array}\begin{array}{ccc} \frac{\partial^2}{\partial} & + \end{array}\begin{array}{ccc} \frac{\partial^2}{\partial} & + \end{array}\begin{array}{ccc} 2\lambda A_{\wedge} & \lambda \end{array}\begin{array}{ccc} \lambda \end{array}\begin{array}{ccc} \Psi(a, b, c) & = \end{array}\right)$ *d*2  $\frac{1}{2}$  $\uparrow$ *dA*  $+ \left(\frac{\partial^2}{\partial c \partial a}\right) + 2\lambda A_\triangle \bigg\{\Psi[a, b, c] = 0\bigg\}$  $(10 \pi G \rightarrow G)$  $\frac{\partial^2}{\partial^2} + \frac{\partial}{\partial^2}$  $\Delta$  $+2$ @2  $\set{\triangle^F}$  $\overline{f}$ @*c* @*a*  $= 0$ @*c* @*a*  $\int \Psi[ a, b, c ] = 0$ 

$$
\Psi \left[ \, a, \, b, \, c \, \right] = \;\; \mathcal{N} \, \frac{J_{1/2} \left( \frac{2 \sqrt{2 \, \lambda} \, \, A_\Delta}{G} \right)}{\left( \frac{2 \sqrt{2 \, \lambda} \, \, A_\Delta}{A_\Delta} \right)^{1/2}} \quad \left[ = \tilde{\mathcal{N}} \, \frac{\sin \left( \frac{2 \sqrt{2 \, \lambda} \, \, A_\Delta}{G} \, A_\Delta \right)}{A_\Delta} \, \right]
$$

Normalization constant fixed by the standard rule of quantum mechanics:  $\mathsf{H}$ ˜ Normalization constant fixed by the daru rule or quantum mechanics.  $\bm{v}_0$ dard rule of quantum mechanics: dard rule of quantum mechanics:<br>——————————————————— *l* = <sup>0</sup> *simplices* ˜ malization const  $\overline{a}$  $\int$  ant fixed by the Moreover we have the subset of the standard rule of quantum mechanics:  $\int_0^a dA_\Delta |\Psi(A_\Delta)|^2 = 1$ **12** Normalization constant fixed by the  $\int_{a}^{\infty} dA \cdot d\vec{v}$ *nianzation* constant inted by the<br>idard rule of quantum mechanics: it fixed by the<br><sup>Lum mochanics</sub>:</sup> *Ilatt*

 $\int_0^\infty$  $\int_0^{\pi} dA_{\Delta} |\Psi(A_{\Delta})|^2 = 1$  $\overline{\phantom{a}}$  $\int^{\infty} dA \wedge |\Psi(A_{\Lambda})|^2 = 1$  $\int_0^{\frac{\pi}{2}}$  $\int_0^\infty$  $\int_{0}$   $\mu A$  $\int_{0}^{\infty} dA \cdot | \Psi(A_+) |^2 = 1$ <sup>2</sup> = 1 *.* (94) **P**<br>**CS:**  $\int_0^1 dA_\Delta |\Psi(A_\Delta)|^2 = 1$  $=$  1 *h*  $\int_0^{\pi}$  (*d*)  $|\Psi(A_{\Delta})|$ <sup>2</sup>

*<sup>h</sup>* + 2*A*<sup>0</sup>

 $\rightarrow G$  )

## Significance of Single Triangle Solution  $(2+1)$  dim.)

nontrivial result

$$
\Psi \left[\,a,\;b,\;c\,\right]=\Psi \left[A_\triangle \right]
$$

Since a discretization of space-time breaks the diffeomorphism invariance, it raises the question of whether and in what form part of the diffeomorphism symmetry can still be realized at the discrete level.

The solution only depends on geometry *i.e.*, spatial diffeomorphism is retained.

## Problem Set-up  $(2+1)$  dim.)

In principle, any solution of the Wheeler-DeWitt equation corresponds to a possible quantum state of the universe.

> The boundary conditions on the wavefunction will act to restrict the class of possible solutions;

> in ordinary quantum mechanics, they are determined by the physical context of the problem and some set of external conditions.

In our analytical calculations, we used *spherical boundary* conditions for the spatial manifold,

further, regular polyhedra approximations to a 2-sphere.

## Problem Set-up  $(2+1)$  dim.) The idea:  $\bigwedge$  More precisely  $\bigotimes$  , further more Latticize Boundary condition is such that:  $R \bigotimes S^2$  (1)  $\begin{CD} \longrightarrow \\ \downarrow \downarrow \downarrow \downarrow \end{CD}$ <sup>1</sup> *kr*<sup>2</sup> <sup>+</sup> *<sup>r</sup>*<sup>2</sup> *<sup>d</sup>*✓<sup>2</sup> <sup>+</sup> *<sup>r</sup>*<sup>2</sup> sin<sup>2</sup> ✓ *<sup>d</sup>*<sup>2</sup>  $k$   $\mapsto$ **k**  $\left\{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\right\}$ tetrahedron octahedron <sub>icosahedron</sub>

**n**  $\frac{1}{2}$  **1** (6) **n**  $\frac{1}{2}$  **1** (6) **n**  $\frac{1}{2}$  **1** (6) **n**  $\frac{1}{2}$  **1** (6) **n**  $\frac{1}{2}$ 

### It is rather remarkable that all of the previous cases (except the single) cases (except the single) can be described the single triangle) can be described the single triangle of the single single  $\alpha$ by one single set of interpolating functions, where the interpolating variable is related to the overall lattice size (the number of triangles).  $F_{\rm eff}$  equilateral triangles and in the absence of curvature, the wave function  $V_{\rm eff}$  $\mathbf{b}$ **VVILII CUI VALUI C, AI**  $(2+1$  dim.  $\overline{a}$ with carvature, and equilities in  $\epsilon$  $2 \cdot 1$  ullio,  $\blacksquare$  $\left($  $\mathbf{f}$  function and properties of and properties of arguments in the confluent hypergeometric hypergeometric hypergeometric hypergeometric hypergeometric hypergeometric hypergeometric hypergeometric hypergeometric hyper With Curvature, and Equilateral  $(2+1)$  dim.) With C  $\int$ 2 p*a* p*c* ◆ tur ✓*<sup>a</sup>* <sup>+</sup> *<sup>d</sup> <sup>e</sup>* , dl *d* ◆ 1 F ( ✓*<sup>c</sup>* <sup>+</sup> *<sup>d</sup> <sup>f</sup>* 2 p*c* p *d* ◆ (63) ◆  $2<sup>1</sup>$ . *l dim* 2 p*a* p*e*  $(2+1$  dim.)  $-h \cap$  $100/2$ UP TO THE V  $E$ onilatoral  $\blacksquare$ Lyundtel di regular near the origin.  $\int$ It is rather remarkable that all of the previous cases (except the single) cases (except the single) can be described the single triangle) can be described the single of the single single single single single single singl by one single set of interpolating functions, where the interpolating variable is related to the overall  $\mathbf{b}$ *v* Villi Cui valui C<sub>i</sub> and both Curvature and Equilate (b) Equilateral Case with Curvature Term (" = 0)  $(2 \cdot 1 \cdot \text{with})$ **With Curvature, and Equilateral coordinates in Section 20, 29** "  $(2+1)$  dim therefore leads to a perfectly acceptable, normalizable solution.  $(2+1)$  *dim.*)  $101 \, \text{C}$ rvac 2 p*a*  $\mathbf t$  $\overline{\phantom{a}}$ ✓*<sup>a</sup>* <sup>+</sup> *<sup>d</sup> <sup>e</sup>* p ◆  $-9$ ✓*<sup>c</sup>* <sup>+</sup> *<sup>d</sup> <sup>f</sup>* p*c* p h Curvature, and Equilate ✓*<sup>a</sup>* <sup>+</sup> *<sup>b</sup> <sup>c</sup>* ◆ ✓*<sup>a</sup>* <sup>+</sup> *<sup>e</sup> <sup>d</sup>* cos<sup>1</sup> cos<br>1919 - Costal Jrv 'ature, a  $f(x) = \frac{1}{2} \int_{0}^{x} f(x) \, dx$ solutions for this case, as well as the two corresponding Hankel (H) functions. Nevertheless, only the solution associated with the Bessel J function is

 $\mathcal{L}_{q}$ cases is described by the following equation expression  $\chi$ equilateral (edges fluctuating together) Equilateral (edges fluctuating together)  $\mathbb{R}$  and  $\mathbb{R}$  and  $\mathbb{R}$  and  $\mathbb{R}$  at each vertex  $\mathbb{R}$  $t$ her) the curvature contribution for each equilateral equilater equilateral (edges fluctuating together)<br>Equilateral (edges fluctuating together) equilateral (edges fluctuating together)<br>
ρεταιρισμού του Εραιλία του Εραιλία του Εραιλία του Εραιλία του Κατατισμού του Εραιλία του Εραιλία του Εραιλί Fouilateral (edges fluctuating together)  $F_{\text{max}}$ ilateral (edges fluctuating together) 2 p*a b* 2 p*a* p*e* The discussion proceeds in a way to the octahedron proceeds in a way to the octahedron proceeds in a way to the  $\frac{1}{\sqrt{2}}$  $\mathcal{S}_{\mathcal{S}}$  , and  $\mathcal{S}_{\mathcal{S}}$  at each vertex  $\mathcal{S}_{\mathcal{S}}$ 

WDW eq.

\n
$$
\Psi'' + \frac{2n+1}{x} \Psi' - \frac{2\beta}{x} \Psi + \Psi = 0
$$
\n
$$
x = \frac{\sqrt{2\lambda}}{G} A_{tot}
$$
\n
$$
n = \frac{1}{4} (N_{\Delta} - 2)
$$
\n
$$
\beta = \frac{2\sqrt{2}\pi G}{\sqrt{\lambda}}
$$
\nEquation (1) and (2) and (3) are

\n
$$
n_{\text{total}} = \frac{3}{2}
$$
\n
$$
n_{\text{icosa}} = \frac{9}{2}
$$
\n
$$
F_I(\beta, x) \leftarrow \text{Coulomb } \text{W2VQ function}
$$

Regular solution: 
$$
\Psi(x) = \frac{F_l(\beta, x)}{x^{n + \frac{1}{2}}} \leftarrow \text{Coulomb wave function}
$$
  

$$
l = n - \frac{1}{2}
$$

$$
\Rightarrow \Psi(x) = \mathcal{N} \frac{J_n(x)}{x^n} \quad \text{(without curvature)}
$$

### Curvature and Euler Characteristics  $\zeta$  and  $\zeta$  arbitrary topology one has by the Gauss-Bonnet with arbitrary topology one has by the Gauss-Bonnet with arbitrary topology one has by the Gauss-Bonnet with arbitrary topology one has by the Gauss-Bonne  $\z$  $(2+1)$  dim.)  $\subset$ ✓*<sup>a</sup>* <sup>+</sup> *<sup>c</sup> <sup>b</sup>* 2 p*a* p*c*  $\mathbf{A}^{\dagger}$ ure i ✓*<sup>a</sup>* <sup>+</sup> *<sup>d</sup> <sup>e</sup>* າa E *d* er C ✓*<sup>c</sup>* <sup>+</sup> *<sup>d</sup> <sup>f</sup>* 2 p*c* p *d*  $\mathbf{r}$ racteristics 2 2 p*a* p*e* ◆ aim ✓*<sup>b</sup>* <sup>+</sup> *<sup>e</sup> <sup>f</sup>* tori teristics racteristics p2<br>P2 September = p *<sup>G</sup>* (69)

in 2 dimensions  $\int d^2x \sqrt{g} R = 4\pi \chi$  (Gauss Bonnet theorem) ✓*<sup>b</sup>* <sup>+</sup> *<sup>c</sup> <sup>a</sup>*  $\sum_{i=1}^{n}$  $\int$  $c \sqrt{g} K = 4\pi$  $\mathcal{L}$ ✓*<sup>b</sup>* <sup>+</sup> *<sup>f</sup> <sup>e</sup>*  $\overline{\mathsf{m}}$ net theorem)

 $\gamma$  is the characteristics of the manifold.  $\int \chi = 2$  (sphere)  $\begin{bmatrix} \lambda & 0 \end{bmatrix}$  $\chi \colon \;$  Euler characteristics of the manifold 2 *b* p*c* 2 p*c* p*f* 2 *b* p*f*  $\overline{\phantom{a}}$  $\chi = 0$  (torus)  $\chi = 2$  (sphere)  $\lambda$   $\sim$  (10.25)  $\Omega$  ( $\overline{1}$ )  $\chi = 0$  (torus) (sphere) (torus)

> $\frac{1}{2}$ **On a discrete manifold in two dimensions** *G* wo dime<br>wo dime  $\frac{1}{\sqrt{2}}$

$$
\chi = N_0 - N_1 + N_2
$$

 $\chi=N_0-N_1+N_2 \qquad N_i$  : number of simplices of dimension  $i$  $N<sub>I</sub>$  : edges  $N_0$ : sites (vertices )  $N<sub>I</sub>$ : edges  $N<sub>2</sub>$ : faces  $N_f$ : edges<br> $N_f$ : faces

$$
\beta = \frac{\sqrt{2} \pi \chi}{\sqrt{\lambda} G}
$$

 $\beta$  's dependence on boundary conditions becomes explicit.

### Key Results (2+1 dim.) so far from tetra, octa, and icosahedra  $\mu$  nesults  $\mu$  + 1 dimensions, nsahedra neara with parameter  $n$ n = <sup>1</sup> a octa and icosahedra The non-singular, normalizable solution is now given by which now describes the radial wavefunction for a quantum particle in D  $=$ utta, and itosancula  $\ell_{\text{e}}$   $\sum_{i=1}^n$  is the constant  $\sum_{i=1}^n$  is given the proportionality constant Cl in Eq. (149) is given by given the proportion of  $\sum_{i=1}^n$ **O**  $22$  $2 + 1 - 2 + 1 - 2$ **osahedra** mechanical three-dimensional Coulomb problem in spherical coordinates [28, 29]. The latter is a  $\mathbf{F}_{\text{max}}$ function in  $\mathcal{U}(l)$ , allows one to write it as a  $\mathcal{U}(l)$ , and we take function  $\mathcal{U}(l)$  $\mathcal{M}(\mathcal{M})$  denotes the regular Coulomb wave function that arises in the solution of the s  $\blacksquare$   $\blacksquare$   $\blacksquare$  $1$  alm.)  $\frac{1}{2}$ die op die strong coupling region and without the strong coupling region and with the strong coupling region and without the strong coupling region and with the strong coupling region and with the strong coupling region an id icosaneura scaled area variable as

-The solution is in totally a generalized form,  $\,\Psi\,$  $\overline{ }$  $\boldsymbol{\psi}$  $c)$  $F_l(\beta, x)$  $\overline{x}$  $n + \frac{1}{2}$ bagy, total area, number of triangles]  $\mathbf{m}, \Psi(x) = \frac{F_l(\beta, x)}{n + \frac{1}{x}}$  $\Psi(x) = \frac{F_l(\beta, x)}{n + \frac{1}{2}}$  $\alpha$ , nonormalization  $\alpha$  (n,  $\alpha$ ). The normalization constant can be not a  $\alpha$ ).  $E_i(\beta, x)$ , and then has equivalent representation for the regular wavefunction for the regular wavefuncti as  $\overline{\mathbf{v}}$   $\overline{\mathbf{v}}$  $\mathbf{r}$  $\mathfrak{D}$  triangles  $\overline{a}$  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ angles  $\lfloor$  $\frac{1}{1}$  $-\frac{1}{2}$  $\mathbf{v}$ d form  $\mathbf{u}(x) = \frac{F_l(\beta, x)}{T_l(\beta, x)}$  $f(x)$  function  $f(x)$  =  $\overline{\phantom{a}}$ *i.e.*, Ψ [ topology, total area, number of triangles]  $\delta$  **]** zed form. <sub>.</sub>

 $\mathbf{F}(a)$ 

 $\mathbf{F}(s)$ 

- The solution only depends on the geometric quantities such as total areas in  $2 + 1$  dimensions. (does not just depend on quantities like edge lengths which are not diffeomorphism invariant)  $\nu$  depends on the geometric quantities such as nd on quantities like edge lengths which are erphism invariant)<br>The wavefunction V for large x and via large x<br>The wavefunction V for large x and via large x and via large x and via large x and via large  $x = \frac{1}{2}$ ntorus = <sup>1</sup>  $N_{\rm eff}$  of the imaginary part ( $R_{\rm eff}$ ) of the first argument in the confluent hypergeometric function  $\alpha$  $sions$  $I$  view of he previous discussion the parameter n increases as more triangles as more triangles are included in of Eq. (141) depends on the topology, but does not depend on the number of triangles. Note also that, in spite of appearances, the above wavefunction is still real for nonzero β. again up to an overall wavefunction normalization normalization normalization normalization normalization norm<br>Again we note that the thermalization normalization normalization normalization normalization normalization no i Uniy depends On the geometric quantities such as<br>a single by singular for small r, and will not small r, and will not small r ct depend on quantities like edge lengths which are  $N_{\rm eff}$  is the number of triangles on the proportionality constant  $C$ One then has immediately, from Eq. (141), an equivalent representation for the regular wavefunction **JL**  $S$  ienguis which are  $\parallel$ as  $\mathcal{L}^{\text{max}}$  $\mathcal{L}$  $\ddot{\phantom{0}}$  $\overline{\phantom{a}}$ nich are !ðxÞ ¼ <sup>N</sup> <sup>J</sup>3=2ð<sup>2</sup> ffiffiffi  $\frac{1}{\sqrt{2}}$ 

### Key Quantities Associated with Phase Transitions *h* and its fluctuation, defined as **EXECUTE I**<br> **EXECUTE I**<br> **EXECUTE** *<sup>h</sup> Vh*) *<sup>h</sup> V<sup>h</sup> >*<sup>2</sup> and its fluid in the set of the s **2 Transitions** *<* P *<sup>h</sup> <sup>V</sup><sup>h</sup> <sup>&</sup>gt;* (8)

-Universal Exponent  $v$   $v^{-1} = -\beta'(G_c)$  cutoff-independent quantity related to derivatives of *Zlatt* with respect to the bare cosmological constant 0, as for example in

-Averages

-Fluctuations

$$
\langle V \rangle \sim \frac{\partial}{\partial \lambda_0} \ln Z_{latt}
$$

$$
\chi_V \sim \frac{\partial^2}{\partial \lambda_0^2} \ln Z_{latt}
$$

A divergence or non-analyticity in *Z*, as is expected to show up in these local measure. The bare coupling  $\alpha$  and  $\alpha$  and  $\alpha$  in the gravitational in  $\alpha$  in  $\alpha$ caused for example by a phase transition, but one can define the dimensionless ratio  $a$ */*  $a$ */0, and research so as well.* And  $a$ *averages* as well.

### $\sqrt{2}$ ing Assumption **Scaling Assumption** Assumption 2 ⇡ *< Atot >* = **ing Assumpti** on p⇡  $\alpha$  *Accumptio* יז<br>א *)e.g.*,!Parisi,!Cardy!  $Scaling \triangle$ **Scaling Assumption** *g*3 *scaling Assumption* p<sub>ro</sub> (31)  $\frac{1}{2}$  (31)  $\frac{1}{2}$  (31)  $\frac{1}{2}$  (31)  $\frac{1}{2}$  $\mathbf{z}$ ⇠ ⇠ *|g gc| <sup>V</sup> lnZ* (34) **F***s*  $\frac{1}{2}$  (35) *F* (35) *E*  $C$ coling Accumption  $\frac{1}{2}$ *<u><i>g*<sup>2</sup></del></u> **Scaling Assumption** 1 anng Assumption *Scaling Assumption <sup>V</sup> lnZ* (34) **F**  $\frac{1}{2}$   $\frac{1}{2}$ ⌫ (33) and the common the common the set of the set o *< Atot >* = *i* publi  $\mathcal{O}(\mathcal{O}(\log n))$

 $\frac{1}{2}$  **F**  $\frac{1}{2}$ .<br>11  $\int \mathbf{y} \quad \xi \sim |g - g_c|^{-\nu}$  *e.g., Parisi,*  $-\nu$  $\frac{1}{2}$   $\frac{1}{2}$  $\overline{a}$  $|g - g_c|^{-\nu}$  e.g., Parisi. Cardv Correlation length is given by  $\xi \sim |g - g_c|^{-\nu}$ Correlation length is given by  $\frac{1}{\sqrt{1}}$ Correlation length is given by  $\zeta \sim |g - g_c|$  e.g., Parisi, Car  $\log$ *<sup>F</sup>sing* ⇠ ⇠*<sup>d</sup>* ⇠ *|g gc|* orrelation length is given by  $\xi \sim |g - g_c|^{-\nu}$  $\ln 1$ S  $\frac{2}{3}$  $\frac{1}{\sqrt{2}}$ **Plation length is given by**  $\zeta \sim |g - g_c|$  e.g., Parisi, *< O >*⇠ Correlation length is given by  $\xi \sim |g - g|$ *i l length* is given by  $\left| \xi \sim |g - g_c|^{-\nu} \right|$  *e.g.,*  $\overline{\nu}$ 

*e.g.*, Parisi, Cardy arisi, Cardy<br>*i*.i. <sup>*d</sup></sup> <i>diagram*, Parisi, Cardy</sup>

 $F$  conclained religin signals the presence of transition  $F \sim$ 1  $\frac{1}{V}$   $ln Z$ *e.g.*, Parisi, Cardy<br>A divergence of correlation length signals the presence of transition igtn signa<br>gularity ir and leads to the appearance of singularity in free energy.  $F \sim \frac{1}{V} ln Z$  $\overline{1}$ *C*<sup>*L*</sup> (*L C*<sup>*l*</sup> (*L C*<sup>*l*</sup> (*C*)  $\overline{a}$   $\sim$  $ln \angle$ *b<sup>k</sup>* () *Jk*<sup>+</sup> <sup>1</sup> *fl*  $\epsilon$  *ap*  $\epsilon$  *k* $\epsilon$  *ap*  $\epsilon$  $\frac{1}{2}$ ar *l* +  $\dot{e}$  o *C<sup>l</sup>* () *x* ity in free ei  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 &$ A divergence of correlation length signals the presence of transition<br>and leads to the appearance of singularity in free energy,  $F = \frac{1}{\mu} Z$  $\mathcal V$ gularit<sub>.</sub>  $\ddot{c}$   $\ddot{c}$ free energy.  $F \sim \frac{1}{V} ln Z$ *A a nd leads to the appe* arance e of singularity in free energy.  $F \sim \frac{1}{V} ln Z$  $\frac{1}{2}$  *F*  $\sim \frac{1}{2}$  *lnZ*  $\mathfrak{m}$ *n*<sub>1</sub>*y s*<sup>1</sup><sup>*y*</sup> *b s*<sup>1</sup><sup>*y*</sup> *e.g.*, Parisi, Cardy<br>ance of correlation length signals the presence of transition pear  $\overline{1}$ ance of singularity in free energy.  $F \sim \frac{1}{V} ln Z$ *nd leads to the appearance of singularity in free energy.*  $F \sim \frac{1}{V} ln Z$  $\mathbf{1}_{c}$ ds to the appearance of singularity in free energy.  $F \sim \frac{1}{V} ln Z$  $\ddot{\phantom{1}}$ *V*  $\ddot{\phantom{0}}$  $\overline{\mathbf{u}}$  $\frac{1}{\sqrt{2}}$  (36) of the appearance of singularity in free energy.  $F \sim \frac{1}{V} ln Z$ *divergence of correlation length signals*<br>*d* loads to the enpearance of singularity in fr.  $\overline{\mathsf{p}}$  $e.g.,$ f transition  $r$ gy.  $F$  $q^2$ <sup>3</sup>  $\frac{V}{V}$  (112) **F**  $\frac{1}{\sqrt{7}}$ *e.g.,* Parisi, Cardy<br>vergence of correlation length signals the presence of transition appearance of singularity in free energy.  $F \sim \frac{1}{V} ln Z$ *e.g., Parisi, C F* ⇠ and leads to the appearance of singularity in free energy.  $\overline{a}$ and leads to the appearance of singularity in free energy.  $F \sim \frac{1}{\pi} ln Z$ *V* @ *g <sup>F</sup>sing* ⇠ ⇠*<sup>d</sup>* ⇠ *|g gc|*  $\lceil$  singular tx<br>t *V* @ and leads to the appearance of singularity in free energy.  $F \sim \frac{1}{V} ln Z$  $\int$  ce of singular namy  $\ddot{1}$ *l* + .<br>G  $\mathbf{e}$  $2\pi$   $\mathcal{L}$   $\mathcal{L}$  $\overline{F}$  $\frac{1}{\sqrt{1-\lambda}}$  $\mathbf{I} \mathbf{I} \mathbf{Z}$ 

 $\mathbf{p}$  *Fsir*  $\Omega/\xi^{-d}$  thoroforo  $F$   $\qquad \qquad$   $\alpha \frac{d\nu}{d\chi}$  $g - g_c$ <sup>2</sup>  $F_{sing} \sim |g - g_c|^{d \nu}$  $\frac{1}{2}$ of a provides a direct exponent in Eq. (1831). Note that the correlation of the correlation of the correlation o<br>2  $F_{sing} \sim \xi^{-d}$  therefore  $F_{eing} \sim |g - g_c|^{d \nu}$  $\mathbf{a} = \mathbf{b}$  *Fsing*  $\sim \xi^{-d}$  therefore  $F_{sing} \sim |g - g_c|^{d \nu}$ Caling Assumption:  $F_{sing} \sim \zeta$  therefore  $F_{sing} \sim |g - g|$  $S$ caling Assumption:  $F_{sing} \sim \xi^{-d}$  therefore  $F_{sing} \sim |g - g_c|^{d \nu}$  $d \nu$ *< O >*⇠ efor  $\lambda$  *F*sing  $\sim$   $\zeta^{-d}$  therefore  $F_{sing} \sim |g - g_c|^{d \nu}$  $F_{ing} \sim \xi$   $\blacksquare$  there  $\overline{C}$  $\mid F \mid$ *C<sup>l</sup>* () *x*  $-g_c|^{d\nu}$  $m$ <sup>o</sup>  $m$  $H_{sin}$  $\sim$   $\xi^{-d}$  thoroforo  $F$ , algorated  $\frac{1}{2}$ |<br>|- $\overline{I}$  $\log \sim \zeta$  **therefore**  $F_{sing} \sim |g - g_c|^{3/2}$ **201:**  $F_{\text{sing}}$  $\sim$   $\circ$  $\overline{a}$  therefore  $F_{\text{size}} \propto |a-a_{\text{s}}|^{d\nu}$ ገg As **l**nZz (35) and  $\frac{1}{2}$  $\overline{\phantom{a}}$ *Fore*  $F_{sing} \sim |q - q_c|^{d \nu}$  $\xi^{-a}$  $\mathbf{r}$ *V* .<br>. . .  $F_{sing} \sim \xi^{-a}$  therefore  $F_{sing} \sim |g - g_c|^{d \nu}$ Assu **iphon:**  $r_{sing} \sim \zeta$  therefore  $r_{sing} \sim |g - g_c|^{\alpha}$ 

$$
\chi \sim \frac{1}{V} \frac{\partial^2}{\partial g^2} \ln Z \sim |g - g_c|^{d\nu - 2} \sim \xi^{\frac{2 - d\nu}{\nu}}
$$

*V* @ *g* nøth sati  $\sim$   $\sqrt{ }$ knowing  $L \sim \sqrt{N_{\Delta}} \sim \sqrt{n}$ For g close to the critical point  $g_c$ , the correlation length saturates to<br>knowing  $L \sim \sqrt{N_\wedge} \sim \sqrt{n}$ the correlation length saturates to its maximum value  $\xi \sim L$ .  $\frac{1}{2}$ For *g* close to the critical point  $g_c$ , the correlation length saturates to its maximum value  $\xi \sim L$ . nowing  $L \sim \sqrt{N_{\Delta}} \sim \sqrt{n}$ kno  $\cdot$  $\frac{C^2}{I}$  $\frac{d}{dx}$   $\frac{d}{dx}$   $\frac{d}{dx}$   $\frac{d}{dx}$   $\frac{d}{dx}$   $\frac{d}{dx}$  $\sqrt{N_{\wedge}} \sim \sqrt{n}$ *< O >*⇠ to the critical point  $g_c$ , the correlation length saturates to its maximum  $\sqrt{g_c}$ gui saturates to its maximum value  $\zeta \sim L$ .  $N_{\triangle}$  $\sqrt{n}$ *V* @ *g* 2 *d* ⌫ **For** *v* close to the critical point  $g_c$ , the correlation length saturates to its may  $\frac{1}{2}$ relation length saturates to its maximum value  $\zeta \sim L$ .  $\sqrt{N_{\Delta}}$   $\sim$  $\sqrt{n}$  $\frac{1}{2}$  $\alpha$  ose to the critical point  $g_c$ , the correlation length saturates to its maximum v

$$
\therefore \begin{array}{ccc} \lambda A & \sim & \xi^{\frac{2-3\nu}{\nu}} & \sim & n^{\frac{1}{\nu} - \frac{3}{2}} \\ \lambda A & \frac{g \to g_c}{\nu} & \sim & n^{\frac{1}{\nu} - \frac{3}{2}} \end{array} \text{ (Fluctuation)}
$$

 $\left(\frac{\sum_{i=1}^{n} \binom{n}{i}}{\sum_{i=1}^{n} \binom{n}{i}}\right)$ 

 $\frac{1}{n}$  a finite volume with linear lattice dimensions L  $\frac{1}{n}$  N  $\frac{1}{n}$  $\mathfrak o$  and the exponent  $V$  difectly. *n*-dependence of  $\chi$  provides a way to estimate the exponent  $\nu$  directly.



Analytical Expression (2+1 dim.) *<sup>A</sup>* = ✓ <sup>1</sup> <sup>2</sup> ◆ *g*<sup>2</sup> <sup>4</sup> <sup>+</sup> *<sup>O</sup>*  $\mathsf{m.}$ + *<sup>g</sup>*<sup>2</sup> <sup>+</sup> *···* (34) *<sup>A</sup>* = **SIUT** <sup>4</sup> <sup>+</sup> *<sup>O</sup>*  $\frac{1}{2}$  $\overline{L}$ <sup>2</sup> (4 ⇡)  $\overline{2}$  $|$  A  $|$  F  $\times$ *N*<sup>4</sup>  $\ddot{\bullet}$ *< Atot >*  $n(2+1)$  dim. Expression i N  $\blacksquare$  Analytical Expression  $(2 \cdot$ Analytical Expression ( $2+1$ al Expressie :xpression (2+1 dim.)  $\mathsf{cl}$   $\mathsf{r}$  and coordinates  $(2+1, 1)$ .  $s_{\text{max}}$  of the radial differential equation of the radial differential equation of  $\frac{1}{2}$  $\text{HochEvolution}$   $(2+1\text{dim})$  $\mathsf{ucl}$  LAPICSSION (2) Lattley p<br>1 *(*  $\mathcal{L}$  $\mathbf{S}$  such a variable  $\mathbf{S}$ ar Lyhression (  $1$  dim  $\mathbf{v}$  $\overline{a}$ 

$$
\Psi(x) = \frac{F_l(\beta, x)}{x^{n + \frac{1}{2}}} \qquad l = n - \frac{1}{2}
$$

*Atot*

 $\frac{1}{2}$ 

### Coulomb wave function *wave* function 1 **411 CM** *<i><u>n*  $\alpha$  +  $\beta$  +  $\beta$  +  $\beta$  +  $\beta$  +  $\gamma$  $\overrightarrow{1}$ **g**3 (80) **g**  $\frac{1}{2}$  (80) **g** Coulon wave function Coulomb wave function  $\mathbf{R}$  $\mathbf{r}$  $\mathbf{1}$ equation with Eq. (141) one then identifies  $\mathcal{V}$  = 1411) one then identifies  $\mathcal{V}$  $\mathcal C$  actual wavefunction then given by Rl(r)  $\mathcal C$  $WdV$ C TUITCHOII  $\sim$ f  $\mathbf{r}$  $10$  wave function

$$
F_{l}(\beta, x) = \frac{2^{l+1}}{\sqrt{\pi}} \Gamma\left(l + \frac{3}{2}\right) C_{l}(\beta) x \sqrt{\frac{\pi}{2 x}} \left\{ \sum_{k=l}^{\infty} b_{k}(\beta) J_{k+\frac{1}{2}}(x) \right\}
$$
  
with  $\{\dots\} \sim J_{l+\frac{1}{2}}(x) + \frac{2l+3}{l+1} \beta J_{l+\frac{3}{2}}(x) + \frac{2l+5}{l+1} \beta^2 J_{l+\frac{5}{2}}(x) + \left\{ -\frac{(2l+7)}{3(l+2)} \beta + \frac{(2l+7)(2l+5)}{3(l+2)(l+1)} \beta^3 \right\} J_{l+\frac{7}{2}}(x) \cdots$ 

with more terms linear in  $\beta$  appearing in higher orders of *J*<br>dere of *R* means including infinitely many orders  $\overline{1}$ a *l*  $\frac{1}{2}$  ( $\frac{1}{2}$  + 1)  $\frac{1}{2}$ <br>*B* appearing in higher orders of *J <sup>n</sup>octa* <sup>=</sup> <sup>3</sup>  $\overline{I}$ with more terms linear in  $\beta$  appearing in higher orders of  $J$ the simplicial geometry. For the regulations of the sphere the sphere the sphere the sphere the summation  $\sigma$  $\frac{1}{2}$  $\overline{\phantom{a}}$ of  $J$ with more terms linear in  $\beta$  annearing in higher orders of  $I$ 

 $In$  $\mathfrak{g}$  infinite יי aי<br>el fu e orders of  $\beta$  means including infinitely mant *<sup>n</sup>icosa* <sup>=</sup> <sup>9</sup>  $\mathsf{s}$ = *gi j dx<sup>i</sup>* + *N<sup>i</sup> dt dx<sup>j</sup>* + *N<sup>j</sup>* of Bessei functions in the expansion, there Including infinite orders of  $\beta$  means including infinitely many orders | <u>Chael coalente wave fariotion:</u> is easily determined from Eq. (140). For the solution  $\mathcal{L}$  $\left| \frac{1}{2} \right|$  ncluding infinite orders of  $\beta$  means including infinitely many orders of Bessel functions in the expansion, therefore means obtaining and  $\vert$ exact coulomb v including minite orders or  $p$  means including minitery many orders of Bessel functions in the expansion, therefore means obtaining " ite orders of  $\beta$  means including infinitely many orders  $\vert$ be considered here. Further relevant properties of the Coulomb wave function can be found in the found in irregular  $C$ oulomb wavefunction  $\mathcal{C}$  is singular for small r, and will not small r, exact coulomb wave function. <u>[30, 31, 32] allows out to derive the following</u> Including infinite orders of  $\beta$  means including infinitely many orders

### Analytical Asymptotic Result  $\alpha$  in the expansion for the wave function  $\alpha$ onuclie component v  $\gamma$ d F 1 ponei  $\tilde{\phantom{a}}$  $\int f(x)$ Anaiy  $\mathbf{I}$ .icai Asymptotic Result up to the contain the container which contain the limit of  $\mathcal{L}$ Analytical Asymptotic Result χA(n) ∼ n <sup>3</sup>m−<sup>1</sup> . (196) *Critical Exponent* <sup>ν</sup> r2 *n* m−1  $A$ naly *N*<sup>4</sup>  $\overline{D}$ 4*n* + 2 (78)  $\overline{C}$ *g g g g al Exponent v Critical Exponent v* ILSUIL 2 *d* ⌫  $\overline{G}$  is a result 2 *d* ⌫ *L L L <i>L* **p p**<sub>1</sub> *<sup>A</sup>* ⇠*g*!*g<sup>c</sup>* ⇠ Allalytical A.  $1$  Frnonent  $\nu$ are in the expansion of the expansion of the set of the wave function  $\mathcal{L}_{\textit{H}}$

and also up to terms which are less singular in g for small g. The requirement that the average  $\mathcal{A}$ 

 $i \longrightarrow \infty$  $:$  Send  $m \longrightarrow \infty$  $\beta^m$  : Sei

 $\beta^m$  : Send  $m \longrightarrow \infty$  $\langle A_{\Delta} \rangle \sim$ 1  $g^{3m-1} n^{\frac{m+1}{2}}$ 2 Require  $\langle A_\Delta \rangle \sim \frac{1}{a^{3m-1} n^{\frac{m+1}{2}}}$  to be finite as *n* is large  $\cdot$   $\alpha(n)$ ,  $\alpha(n)$  $n^{\frac{1}{2(3m-1)}}$ Then in turn  $\chi_A \sim \frac{1}{\omega^3 m - 2 \omega^{\frac{m}{2}}}$   $\longrightarrow$   $\chi_A \sim \pi^{\frac{1}{3}}$  $g(n)~\sim$ 1  $\overline{n}$  $m+1$  $2(3m-1)$ Then in turn  $1$ , and the contract of the contract of  $\mathcal{L}(\mathcal{L})$ area per triangle be finite as n → ∞ then requires that the coupling g itself should scale with n (thermodynamic limit)  $\ldots$  g(n)  $\sim$  $\chi_A \sim$ 1  $\overline{g^{3m-2}\,n^{\frac{m}{2}}}$ 2  $\longrightarrow$   $\lambda A \sim n^{\frac{1}{3}}$ ν  $\frac{1}{m+1}$ Require  $\langle A_{\Delta} \rangle \sim \frac{1}{g^{3m-1} n^{\frac{m+1}{2}}}$  to be finite as *n* is large to be finite as *n* is large  $\therefore$   $g(n) \sim \frac{1}{n^{\frac{m+1}{2(3m-1)}}}$  (thermodynamic limit)  $n^{2(s)}$  $\longrightarrow$   $\begin{array}{|c|c|c|}\n\hline\n\lambda & \sim & n^{\frac{1}{3}}\n\end{array}$  $\overline{\mathcal{O}}$  can now take the limit m  $\overline{\mathcal{O}}$  functions retained in the expansions retained in the expansion of  $\overline{\mathcal{O}}$  $g^{3m-1}n^{-2}$  (thermodynamic limit)  $\rightarrow \infty$ <br>to be finite as *n* is large  $\overline{3}$  $g^{3m-1} n^{\frac{m+1}{2}}$  (thermodynam)<br>  $\therefore g(n) \sim \frac{1}{m+1}$ 2 *x*  $\int$  (it)  $\longrightarrow$   $\begin{array}{c} \lambda A & \sim \\ \lambda A & \sim \\ m \rightarrow \infty \end{array}$ **A** to be finite as 2 *d* ⌫ to be finite as  $n$  is large  $\frac{1}{x+1}$  *l*  $\frac{1$  $\frac{1}{\sqrt{2}}$  $\overline{A}$   $\overline{A}$  $\sum_{m \to \infty} n^3$  $\sim$   $n^{\frac{1}{3}}$ finite as  $n$  $\overline{\phantom{a}}$ arge  $m \rightarrow \infty$ *n* 1  $\frac{1}{3}$ Requir  $2^{3}$  $\langle A_{\Delta} \rangle \sim \frac{1}{q^{3n}}$  $\frac{1}{1-\frac{m+1}{2}}$  to be finite as *n* is large *A*  $\frac{1}{2}$  $\overline{a}$  $A_{\Delta}$  >  $\sim$ *n* <sup>3</sup> (88)  $\therefore$   $g(n) \sim \frac{1}{\frac{m+1}{2(n-1)}}$ Then in turn Require  $< A_\Delta >$ THEITH CUTT  $\chi_A \sim \frac{1}{g^{3m-2} n^{\frac{m}{2}}}$  $h^{-1} n^{\frac{m+1}{2}}$  (thermodynamic limit)  $n^{\frac{m+1}{2(3m-1)}}$ (1992). The contract of  $\chi_A \sim \frac{n^{\frac{1}{3}}}{\sqrt{n^{\frac{1}{3}}}}$ 

e<br>Grove c  $\mathbf{I}$ ng ara  $\frac{1}{2} - \frac{3}{2}$  $\overline{a}$ But know from scaling argument  $\chi_A \sim n^{\frac{1}{\nu} - \frac{3}{2}}$  $\frac{1}{\nu} - \frac{3}{2}$  $\frac{3}{2}$ But know from scaling argument  $\mathbb{R}^n$ now fro 1 3  $\mathcal{L}$  are three that the coupling g itself showledge itself showl  $\frac{1}{2} - \frac{3}{2}$ 

$$
\therefore \boxed{\nu \, = \, \frac{6}{11} \, = \, 0.5454...}
$$

**Conclusions** 
$$
(2+1 \dim)
$$
  
 $\nu = \frac{6}{11} = 0.5454...$  for  $2 + 1$  dimension, Lorenzian

- Does not seem to depend on Euler characteristic  $\chi$ , and therefore on the boundary conditions.  $\frac{1}{2}$  and the compare the boundary conditions.
- Compare with the numerically exact Euclidean threedimensional quantum gravity result obtained in Hamber and Williams Phys. Rev. D47, 510 (1993),  $v \sim 0.59(2)$ . The exponent v is expected to represent a universal quantity, independent of short distance regularization details. Therefore, it should apply to both the Lorentzian and Euclidean formulation, and our results are consistent with this conclusion. for Compare with the numerically exact Euclidean three-<br>Compare with the numerically exact Euclidean threeexpected to represent a universal quantity, independent or<br>Therefore regularization details and therefore characteristic characteristic characteristic characteristic cha it suits are consistent with this conclusion.
- $Gc \rightarrow 0$ , indicating that weak coupling is not present at all.

D*iscre*t *< A*<sup>4</sup> *>* = *n*<br>212

### Wheeler DeWitt equations  $in 3 + 1$  dimensions *< Atot >* = r2 *n*  $\overline{D}$ *crete*<br>Vitt equations sions  $\frac{3}{2}$

Building blocks are tetrahedra.





# **Regular Triangulations** (*3 + 1* dim.)



### Regular triangulations of 3-sphere

5 cell (*q = 3*) 5 tetrahedra glued together "Hyper-tetrahedron"



16 cell (*q = 4*) 16 tetrahedra glued together "Hyper-octahedron"

600 cell (*q = 5*) 600 tetrahedra glued together "Hyper-icosahedron"



Schlegel diagrams

### *Discrete Wheeler DeWitt Equation* (*3 + 1* dim.)  $s = \sum_{i=1}^{n}$  $\sim$   $\sim$  (100), which is the explicit curvature term now present in the full interest in the full interest in the full interest in the full interest in the full  $\omega$  the other hand, as is already contribute.  $\overline{\phantom{a}}$ Discrete Wh  $\frac{1}{2}$ *neeler DeWitt* **I )**<br>*quatton*  $\bigcup$   $\bigcup$   $\bigcup$   $\bigcup$   $\bigcup$  $\left| \right|$

$$
\left\{ -\left(16\pi G\right)^2 \sum_{i,j\subset\sigma} G_{ij} \left(\sigma\right) \frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \frac{2}{q} \sum_{h\subset\sigma} l_h \,\delta_h + 2\,\lambda \,V_\sigma \right\} \Psi\left[l^2\right] = 0
$$

$$
R_{tot} \equiv 2 \sum_{h \subset \sum \sigma} \delta_h l_h \iff \int \sqrt{g} R
$$

$$
\rightarrow \boxed{\psi(R_{tot},V_{tot})}
$$

$$
\text{WDW:} \quad \frac{\partial^2 \psi}{\partial V^2} + c_V \frac{\partial \psi}{\partial V} + c_R \frac{\partial \psi}{\partial R} + c_{VR} \frac{\partial^2 \psi}{\partial V \partial R} + c_{RR} \frac{\partial^2 \psi}{\partial R^2} + c_{\lambda} \psi + c_{curv} \psi = 0
$$
\n
$$
c_V = \frac{11 + 9q}{2q^2} \cdot \frac{N_3}{V} = \frac{11 + 9q_0}{2q_0^2} \cdot \frac{N_3}{V} + \frac{22 + 9q_0}{48\sqrt{2}3^{1/3} \pi q_0} \cdot \frac{N_3^{1/3} R}{V^{4/3}} + \mathcal{O}(R^2)
$$
\n
$$
c_R = -\frac{2}{9} \frac{R}{V^2} + \frac{11 + 9q_0}{6q_0^2} \cdot \frac{N_3 R}{V^2} + \mathcal{O}(R^2)
$$
\n
$$
c_{VR} = \frac{2}{3} \frac{R}{V} + \mathcal{O}(R^2)
$$
\n
$$
c_{RR} = \frac{2}{9} \frac{R^2}{V^2}
$$
\n
$$
c_{\lambda} = \frac{32\lambda}{q^2 G^2} = \frac{32}{G^2 q_0^2} + \frac{4\sqrt{2}\lambda}{33^{1/3} \pi q_0 G} \cdot \frac{R}{N_3^{2/3} V^{1/3}} + \mathcal{O}(R^2)
$$
\n
$$
c_{curv} = -\frac{16}{G^2 q^2} \cdot \frac{R}{V} = -\frac{16}{G^2 q_0^2} \cdot \frac{R}{V} + \mathcal{O}(R^2).
$$
\n
$$
q_0: \text{flat } (i.e., R = 0)
$$

 $\frac{1}{2}$  and  $\frac{1}{2}$  are called the coefficients can set the coefficients can set the check that  $\frac{1}{2}$ .  $\frac{1}{\sqrt{1}}$  satisfies Eqs. (125) and (125) and (125) and (125), up to terms of order 1/V  $\frac{1}{\sqrt{1}}$  2. Also note that  $\frac{1}{\sqrt{1}}$ So far we have not been able to find the general solution for the above differential eq. but probably still some type of Bessel function or hypergeometric function.

### function of the shifted  $\alpha$  $\sqrt{2}$  + 1 dimensions.  $i = 1$ Simplified Differential eq.  $(3 + 1)$  dim.)  $\mathbf{I}$  $\mathbf{d}$  $\zeta$ <sup>the</sup> various  $\zeta$ הווכ 2 2λ  $\overline{a}$ Differential eq.  $(3 + 1)$  dim the various gamma-function coefficients involving the curvature R, which are entirely absent in the Consequently we will now make the replacement in ψ(V,R)

In the limit of the small curvature and the large volume, Further, set  $c_{VR} = 0$  and keeps only the leading term in  $c_V$ In the mint of the *small* car valure and the *targe volume*, Function, set  $C_{VR} = 0$  and keeps only the reading term in  $C_V$  $\sigma$  inserting the result of  $\sigma$ 

$$
\frac{\partial^2 \psi}{\partial V^2} + c_V \frac{\partial \psi}{\partial V} + c_\lambda \psi + c_{curv} \psi = 0
$$
\n
$$
g = \sqrt{G}
$$

$$
\psi(V, R) \simeq e^{-\frac{4 i V}{q_0 g}} \cdot \frac{\Gamma\left(\frac{(11+9 q_0) N_3}{4 q_0^2} + \frac{2 i R}{q_0 g^3}\right)}{\Gamma\left(1 - \frac{(11+9 q_0) N_3}{4 q_0^2} + \frac{2 i R}{q_0 g^3}\right)}
$$
  

$$
{}_1F_1\left(\frac{(11+9 q_0) N_3}{4 q_0^2} - \frac{2 i R}{q_0 g^3}, \frac{(11+9 q_0) N_3}{2 q_0^2}, \frac{8 i V}{q_0 g}\right)
$$

Check: a function of geometric invariants  $V$  and  $R$  only.  $_1F_1$ : confluent hypergeometric function of first kind  $_1$ F<sub>1</sub>: confluent hypergeometric function of first kind Check: a function of geometric invariants  $V$  and  $R$  only. ent hypergeometric function of first kind  $g(x)$  is the correlation in the integrate  $V$  and  $R$  and  $r$ 

### Probability as a function of G (*3 + 1* dim.) Probability as a function of  $G$  $\left(3 + l \right)$  dim



*at*  $N_3 = 10$ 

Very small probability at  $R \sim 0$  for small *G, so no sensible continuum limit.* flat around R = 0, giving rise to large fluctuations in the curvature. On the other hand, for weak enough coupling g one observes that curvatures coupling probability. The  $\alpha$ distributions shown suggest therefore a clearly pathological ground state for weak enough coupling cui vature stales are equally important.  $U$  for sinal  $U$ , so no sensible continuum timit. For strong coupling, different curvature scales are equally important.

# Still work in progress  $(3 + 1 \dim.)$