Discrete Wheeler DeWitt Equations

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Based on:

H. Hamber and R. Williams :*"Discrete Wheeler DeWitt Equations"* arxiv: 1109.2530
Phys. Rev. D.84. 104033(2012)

H. Hamber, R. Toriumi, and R. Williams: *"Wheeler DeWitt Equation in 2+1 Dimensions"* arxiv: 1207.3759
Phys. Rev. D. 86, 084010 (2012)

H. Hamber, R. Toriumi, and R. Williams: *"Wheeler DeWitt Equation in 3+1 Dimensions"* arxiv: 1207.3759

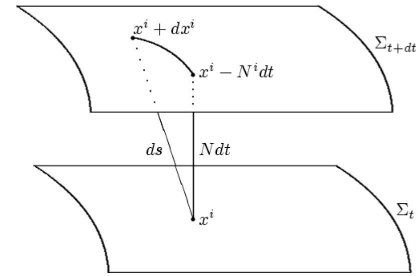
Motivation for Discretization

No reason not to try quantizing discrete spacetime.
May even be certain advantages
e.g., natural cut off (minimum length)
finite number of variables

At the Planck scale, spacetime may be discrete anyway, with rapidly changing topologies
(spacetime foam, Wheeler)

Canonical Formulations

ADM (Arnowitt, Deser, Misner) Formalism
(*continuum* classical canonical formalism)



Introducing a time-slicing of space-time, by introducing a sequence of space like hypersurfaces.

⇓

Wheeler DeWitt Equation (*continuum* quantum canonical)

⇓

Discrete Wheeler DeWitt Equation (*discrete* quantum canonical)

The lack of covariance of the canonical ADM approach has not gone away, and is therefore still part of the present formalism.

Continuum Quantum Canonical Formulation

Energy constraint $\hat{H}\,|\Psi\rangle \;=\; 0$

Wheeler De Witt Eq. in *d+1* dim.

$$\left\{ -\,(16\pi G)^2\, G_{ij,kl}\,(x)\, \frac{\delta^2}{\delta g_{ij}\,(x)\;\delta g_{kl}\,(x)} - \sqrt{g\,(x)}\left({}^{(d)}R\,(x) - 2\,\lambda \right) \right\} \Psi\,[g_{ij}\,(x)] = 0$$

Supermetric: $G_{ij,kl} = \frac{1}{2}\, g^{-1/2}\; (g_{jk}\; g_{jl} + g_{il}\; g_{jk} -\; \alpha\; g_{ij}\; g_{kl})$

Momentum constraint $\hat{H}_i\,|\Psi\rangle \;=\; 0$

$$\left\{ 2\,i\,g_{ij}(\mathbf{x})\,\nabla_k(\mathbf{x})\,\frac{\delta}{\delta g_{jk}(\mathbf{x})} \right\}\;\Psi[g_{ij}(\mathbf{x})] \;=\; 0$$

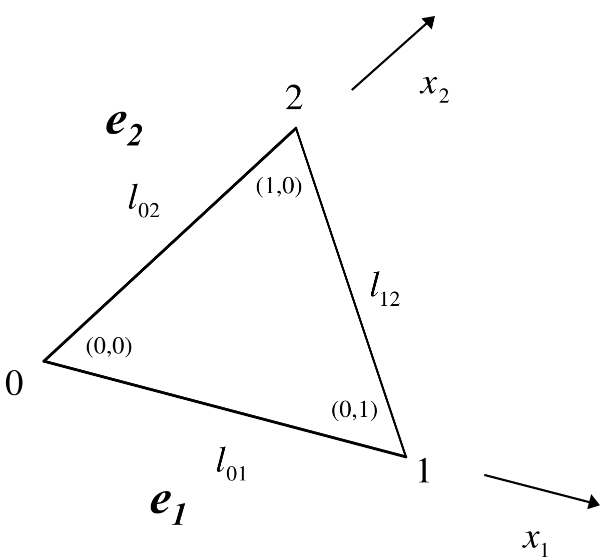
Regge Lattice *Discretization*:

Regge 1961

In constructing a *discrete* Hamiltonian for gravity one has to decide what degrees of freedom one should retain on the lattice.

→Use geometric *Regge lattice discretization* for gravity, with *edge lengths* suitably defined on a random lattice as *the primary dynamical variables*.

Degrees of freedom for edges and metric tensor are both $D(D+1)/2$ in D dimensions.



$$g_{ij}(\sigma) = e_i \cdot e_j \qquad l_{ij} = |e_i - e_j|$$

$$g_{ij}(\sigma) \;=\; \frac{1}{2}\left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2\right)$$

↑ simplex

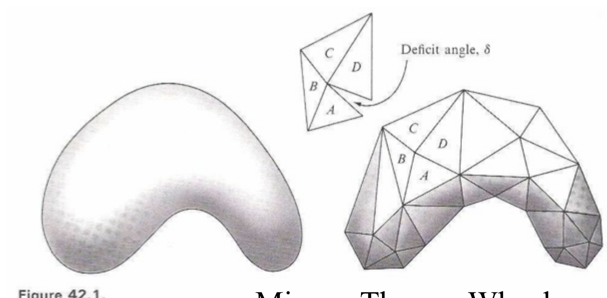
Regge Formulation: Constituents

Curved space(time)s are piece-wise linear.

Flat building blocks are D-dim. Simplices $^{(D)}$

Point (0 –simplex) in 0-dim
Line (1 –simplex) in 1-dim
Triangle (2 –simplex) in 2-dim
Tetrahedron (3 –simplex) in 3-dim
all "flat"

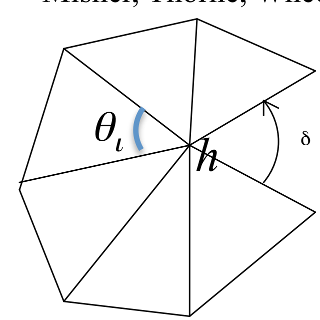
Deficit angle → Curvature (defined at a hinge$^{(D\text{-}2)}$
at a vertex for 2-dim,
at an edge for 3-dim,
at a triangle for 4-dim)



Misner, Thorne, Wheeler

$$\delta(h) = 2\pi - \sum_{\sigma \supset h} \theta(\sigma, h)$$

sum over dihedral angles θ extends over all simplices σ meeting on hinge h.



$d+1$- Dimensional *Discrete* Wheeler DeWitt equation

$$\left\{ -(16\pi G)^2 \, G_{ij,kl}(x) \, \frac{\delta^2}{\delta g_{ij}(x) \, \delta g_{kl}(x)} - \sqrt{g(x)} \left({}^{(d)}R(x) - 2\lambda \right) \right\} \Psi\left[g_{ij}(x) \right] = 0$$ *Continuum*

$$g_{ij}(x)$$

$$\Downarrow$$

$$g_{ij}(\sigma) = \frac{1}{2} \left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2 \right)$$

$$\left\{ -(16\pi G)^2 \, G_{ij}(l^2) \, \frac{\partial^2}{\partial l_i^2 \, \partial l_j^2} - \sqrt{g(l^2)} \left({}^{(d)}R(l^2) - 2\lambda \right) \right\} \Psi\left[l^2 \right] = 0$$ *Discrete*

Kinetic term — Curvature term — Cosmological constant term

Both equations are defined at each "point" in space.

Discrete WDW eq.: one eq. for each simplex

Kinetic Term

$$\left\{ -(16\pi G)^2\, G_{ij}\,(l^2)\, \frac{\partial^2}{\partial l_i^2\, \partial l_j^2} - \sqrt{g\,(l^2)} \left({}^{(d)}R\,(l^2) - 2\,\lambda \right) \right\} \Psi\,[l^2] = 0$$

$$G^{ij}(l^2) \;=\; -\,d! \sum_{\sigma} \frac{1}{V(\sigma)}\, \frac{\partial^2\, V^2(\sigma)}{\partial l_i^2\, \partial l_j^2}$$ (geometric) by Regge and Lund

w/ $V^2(\sigma) \;=\; \left(\frac{1}{d!}\right)^2 \det g_{ij}(l^2(\sigma))$

Only involves the variables within one simplex.

Curvature Term

$$\left\{ -(16\pi G)^2\, G_{ij}\,(l^2)\, \frac{\partial^2}{\partial l_i^2\, \partial l_j^2} - \sqrt{g\,(l^2)} \left({}^{(d)}R\,(l^2) - 2\,\lambda \right) \right\} \Psi\,[l^2] = 0$$

$$\sqrt{g}\ \ {}^{(d)}R = \frac{2}{q} \sum_{h \subset \sigma} \delta_h\ \ {}^{(d-2)}V_h$$

q : coordination number

Involves the variables of the neighbor simplices of a simplex.

Discrete WDW eqns in *2+1 dim.*

$\Psi[l^2]$ is a function of the whole simplicial geometry (overall *geometry* of the manifold), due to the built-in *diffeomorphism invariance*.

$\Psi[l^2]$ depends collectively on all the edge lengths in the lattice.

Therefore even though we have one equation for each simplex, there should be *one wave function that satisfies all the equations* for each simplices in one configuration.

Exact Solution for A Single Triangle *(2+1 dim.)*

A single triangle:

- Curvature term is absent in this configuration.
- as a starting point for the strong coupling expansion in *1/G*.
- should show the physical nature of the wavefunction solution deep in the strong coupling regime.

$$(16\pi G \to G\ )$$

WDW eq: $$\left\{ G^2 \, 4 A_\triangle \left(\frac{\partial^2}{\partial a \, \partial b} + \frac{\partial^2}{\partial b \, \partial c} + \frac{\partial^2}{\partial c \, \partial a} \right) + 2 \lambda A_\triangle \right\} \Psi[\, a, \; b, \; c\,] \; = \; 0$$

$$\Psi\,[\, a, \; b, \; c\,] = \quad \mathcal{N} \, \frac{J_{1/2}\left(\frac{2\sqrt{2}\,\lambda}{G} \boxed{A_\Delta} \right)}{\left(\frac{2\sqrt{2}\,\lambda}{G} \boxed{A_\Delta} \right)^{1/2}} \qquad \left(= \tilde{\mathcal{N}} \, \frac{\sin\left(\frac{2\sqrt{2}\,\lambda}{G} \, A_\triangle \right)}{A_\triangle} \right)$$

Normalization constant fixed by the standard rule of quantum mechanics:

$$\int_0^\infty dA_\Delta \, | \, \Psi(A_\Delta) \, |^2 \; = \; 1$$

Significance of Single Triangle Solution *(2+1 dim.)*

nontrivial result $$\Psi\left[\,a,\ b,\ c\,\right] = \Psi\left[A_{\triangle}\right]$$

Since a discretization of space-time breaks the diffeomorphism invariance, it raises the question of whether and in what form part of the diffeomorphism symmetry can still be realized at the discrete level.

The solution only depends on geometry
i.e., spatial diffeomorphism is retained.

Problem Set-up *(2+1 dim.)*

In principle, any solution of the Wheeler-DeWitt equation corresponds to a possible quantum state of the universe.

The boundary conditions on the wavefunction will act to restrict the class of possible solutions;

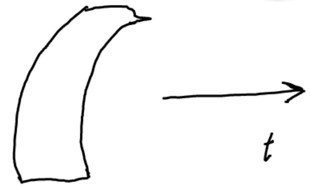
in ordinary quantum mechanics, they are determined by the physical context of the problem and some set of external conditions.

In our analytical calculations, we used *spherical boundary conditions for the spatial manifold*,
further, regular polyhedra approximations to a *2*-sphere.

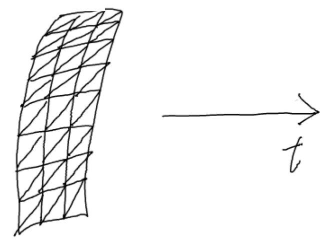
Problem Set-up *(2+1 dim.)*

The idea:

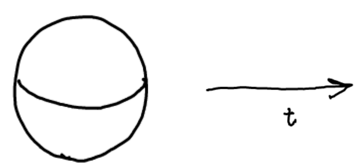
$$R \otimes S^2$$

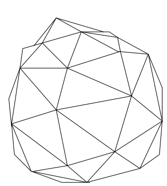


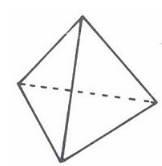
Latticize



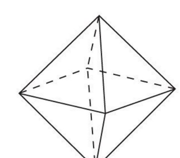
Boundary condition is such that:



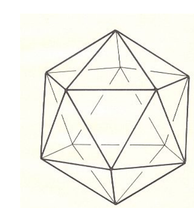
(More precisely  , further more



tetrahedron



octahedron



icosahedron)

With Curvature, and Equilateral *(2+1 dim.)*

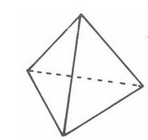
Equilateral (edges fluctuating together)

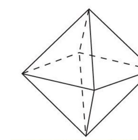
WDW eq. $$\Psi'' + \frac{2n+1}{x}\Psi' - \frac{2\beta}{x}\Psi + \Psi = 0$$

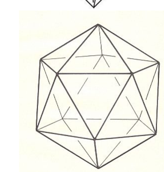
$$x = \frac{\sqrt{2\ \lambda}}{G} A_{tot}$$

$$n = \tfrac{1}{4}(N_\Delta - 2)$$

$$\beta = \frac{2\sqrt{2}\ \pi\ G}{\sqrt{\lambda}}$$

$n_{tetra} = \frac{1}{2}$ 

$n_{octa} = \frac{3}{2}$ 

$n_{icosa} = \frac{9}{2}$ 

Regular solution: $$\Psi(x) = \frac{F_l(\beta, x)}{x^{n+\frac{1}{2}}}$$ ← Coulomb wave function

$$l = n - \tfrac{1}{2}$$

→ $\Psi(x) = \mathcal{N}\frac{J_n(x)}{x^n}$ (without curvature)

Curvature and Euler Characteristics *(2+1 dim.)*

in *2* dimensions $\int d^2x \sqrt{g}\, R = 4\pi\,\chi$ (Gauss Bonnet theorem)

χ : Euler characteristics of the manifold $\begin{cases} \chi = 2 & \text{(sphere)} \\ \chi = 0 & \text{(torus)} \end{cases}$

On a discrete manifold in two dimensions

$$\chi = N_0 - N_1 + N_2$$

N_i : number of simplices of dimension i
N_0 : sites (vertices)
N_1 : edges
N_2 : faces

$$\beta = \frac{\sqrt{2}\,\pi\,\chi}{\sqrt{\lambda}\,G}$$

β's dependence on boundary conditions becomes explicit.

Key Results *(2+1 dim.)* so far from tetra, octa, and icosahedra

-The solution is in totally a generalized form, $\Psi(x) = \frac{F_l(\beta, x)}{x^{n+\frac{1}{2}}}$

$i.e.,\ \Psi$ [topology, total area, number of triangles]

- The solution only depends on the geometric quantities such as total areas in $2 + 1$ dimensions.

(does not just depend on quantities like edge lengths which are not diffeomorphism invariant)

Key Quantities Associated with Phase Transitions

-Universal Exponent ν $\quad \nu^{-1} = -\,\beta\,'\,(G_c)$ *cutoff-independent* quantity

-Averages $\quad < V > \sim \dfrac{\partial}{\partial \lambda_0} \ln Z_{latt}$

-Fluctuations $\quad \chi_V \sim \dfrac{\partial^2}{\partial \lambda_0^2} \ln Z_{latt}$

A divergence or non-analyticity in Z, as caused for example by a phase transition, is expected to show up in these *local averages* as well.

Scaling Assumption

Correlation length is given by $\xi \sim |g - g_c|^{-\nu}$

e.g., Parisi, Cardy

A divergence of correlation length signals the presence of transition and leads to the appearance of singularity in free energy. $F \sim \frac{1}{V} lnZ$

Scaling Assumption: $F_{sing} \sim \xi^{-d}$ therefore $F_{sing} \sim |g - g_c|^{d\,\nu}$

$$\chi \sim \frac{1}{V}\frac{\partial^2}{\partial g^2} lnZ \sim |g - g_c|^{d\,\nu - 2} \sim \xi^{\frac{2 - d\,\nu}{\nu}}$$

For g close to the critical point g_c, the correlation length saturates to its maximum value $\xi \sim L$.

knowing $L \sim \sqrt{N_\triangle} \sim \sqrt{n}$

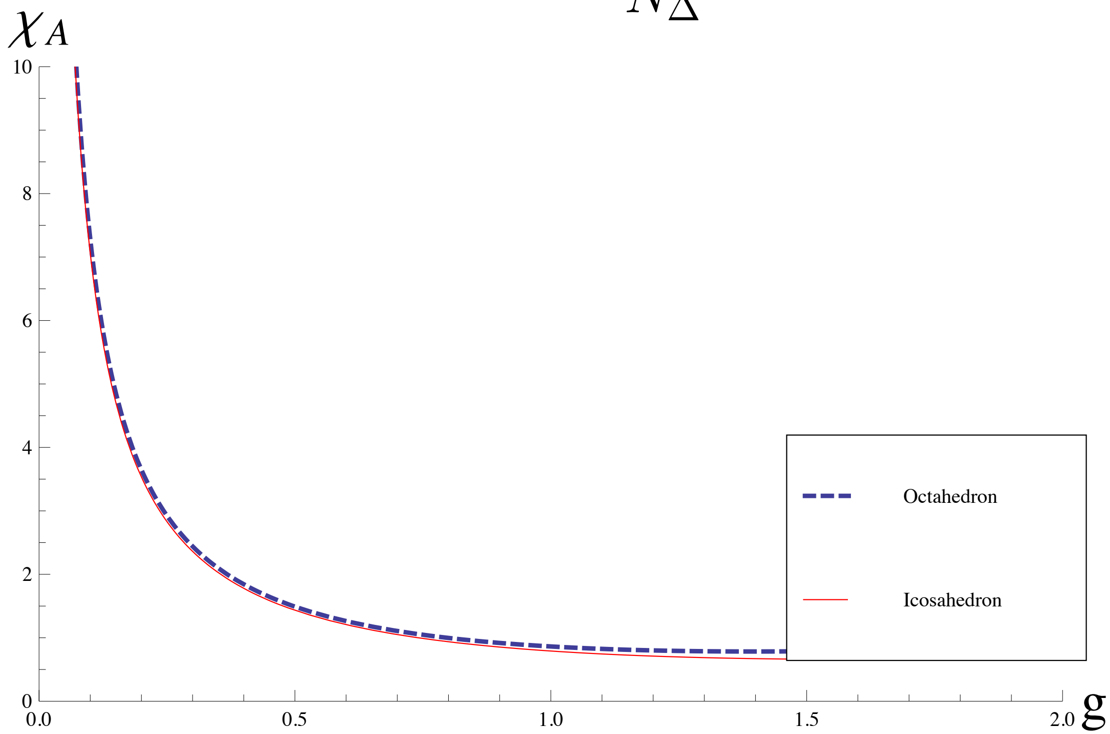
$$\therefore \boxed{\chi_A \underset{g \to g_c}{\sim} \xi^{\frac{2 - 3\,\nu}{\nu}} \sim n^{\frac{1}{\nu} - \frac{3}{2}}} \quad \text{(Fluctuation)}$$

n-dependence of χ provides a way to estimate the exponent ν directly.

Numerically Computed χ_A *(2+1 dim.)*

Using $$\Psi(x) = \frac{F_l(\beta, x)}{x^{n+\frac{1}{2}}}$$

Numerically computed $$\chi_A = \frac{1}{N_\Delta}\{< (A_{tot})^2 > - < A_{tot} >^2\}$$



$(g \equiv \sqrt{G})$

χ_A diverges as $g \to 0$ signaling phase transition

Analytical Expression *(2+1 dim.)*

$$\Psi(x) = \frac{F_l(\beta, x)}{x^{n+\frac{1}{2}}} \qquad l = n - \frac{1}{2}$$

Coulomb wave function

$$F_l(\beta, x) = \frac{2^{l+1}}{\sqrt{\pi}} \Gamma\left(l + \frac{3}{2}\right) C_l(\beta)\; x \sqrt{\frac{\pi}{2\,x}} \left\{ \sum_{k=l}^{\infty} b_k(\beta)\; J_{k+\frac{1}{2}}(x) \right\}$$

with $\{...\} \sim$

$$J_{l+\frac{1}{2}}(x) + \frac{2\,l+3}{l+1}\beta J_{l+\frac{3}{2}}(x) + \frac{2\,l+5}{l+1}\beta^2 J_{l+\frac{5}{2}}(x)$$

$$+ \left\{ -\frac{(2\,l+7)}{3\;(l+2)}\beta + \frac{(2\,l+7)\,(2\,l+5)}{3\;(l+2)\,(l+1)}\beta^3 \right\} J_{l+\frac{7}{2}}(x) \cdots$$

with more terms linear in β appearing in higher orders of J

Including infinite orders of β means including infinitely many orders of Bessel functions in the expansion, therefore means obtaining exact coulomb wave function.

Analytical Asymptotic Result

Critical Exponent ν

β^m : Send $m \longrightarrow \infty$

Require $< A_\Delta > \sim \dfrac{1}{g^{3m-1}\, n^{\frac{m+1}{2}}}$ to be finite as n is large (thermodynamic limit)

$$\therefore \quad g(n) \sim \frac{1}{n^{\frac{m+1}{2(3m-1)}}}$$

Then in turn $\chi_A \sim \dfrac{1}{g^{3m-2}\, n^{\frac{m}{2}}}$ $\longrightarrow$ $\chi_A \underset{m\to\infty}{\sim} n^{\frac{1}{3}}$

But know from scaling argument $\chi_A \sim n^{\frac{1}{\nu}-\frac{3}{2}}$

$$\therefore \quad \nu = \frac{6}{11} = 0.5454...$$

Conclusions *(2+1 dim.)*

$$\nu = \frac{6}{11} = 0.5454...$$ for *2 + 1* dimension, Lorenzian

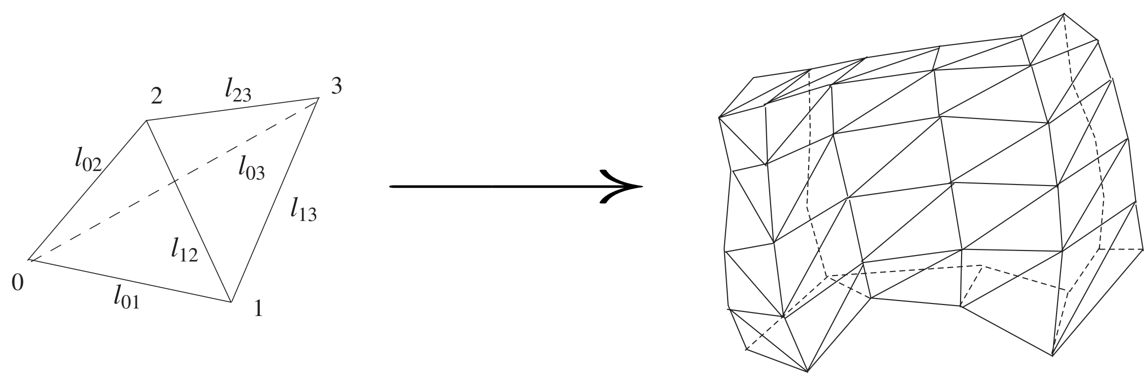
- Does not seem to depend on Euler characteristic χ, and therefore on the boundary conditions.

- Compare with the numerically exact Euclidean three-dimensional quantum gravity result obtained in Hamber and Williams Phys. Rev. D47, 510 (1993), $\nu \sim 0.59(2)$. The exponent ν is expected to represent a universal quantity, independent of short distance regularization details. Therefore, it should apply to both the Lorentzian and Euclidean formulation, and our results are consistent with this conclusion.

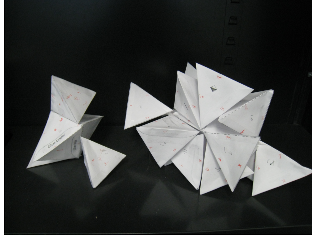
- $Gc \rightarrow 0$, indicating that weak coupling is not present at all.

Discrete Wheeler DeWitt equations in $3 + 1$ dimensions

Building blocks are tetrahedra.

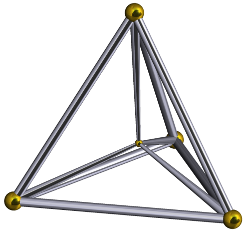


Regular Triangulations (*3* + *1* dim.)

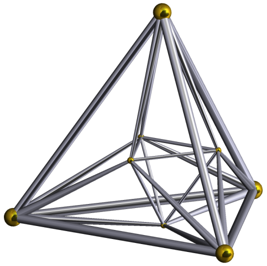


Regular triangulations of 3-sphere

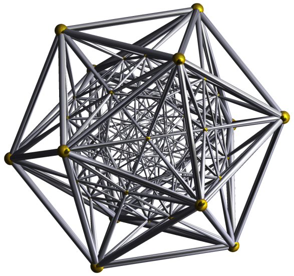
5 cell ($q = 3$)
5 tetrahedra glued together
"Hyper-tetrahedron"



16 cell ($q = 4$)
16 tetrahedra glued together
"Hyper-octahedron"



600 cell ($q = 5$)
600 tetrahedra glued together
"Hyper-icosahedron"



Schlegel diagrams

Discrete Wheeler DeWitt Equation (*3* + *1* dim.)

$$\left\{ - (16\pi G)^2 \sum_{i,j \subset \sigma} G_{ij}(\sigma) \frac{\partial^2}{\partial l_i^2 \, \partial l_j^2} - \frac{2}{q} \sum_{h \subset \sigma} l_h \, \delta_h + 2 \, \lambda \, V_\sigma \right\} \Psi \left[l^2 \right] = 0$$

$$R_{tot} \equiv 2 \sum_{h \subset \sum \sigma} \delta_h \, l_h \longleftrightarrow \int \sqrt{g} \, R$$

→ $\psi(R_{tot}, V_{tot})$

Differential eq. (*3 + 1* dim.)

WDW: $$\frac{\partial^2\psi}{\partial V^2} + c_V \frac{\partial \psi}{\partial V} + c_R \frac{\partial \psi}{\partial R} + c_{VR} \frac{\partial^2 \psi}{\partial V \partial R} + c_{RR} \frac{\partial^2 \psi}{\partial R^2} + c_\lambda \psi + c_{curv} \psi = 0$$

$$\left\{ \begin{aligned}
c_V &= \frac{11 + 9\,q}{2\,q^2} \cdot \frac{N_3}{V} = \frac{11 + 9\,q_0}{2\,q_0^2} \cdot \frac{N_3}{V} + \frac{22 + 9\,q_0}{48\sqrt{2}\,3^{1/3}\,\pi\,q_0} \cdot \frac{N_3^{1/3}\,R}{V^{4/3}} + \mathcal{O}(R^2) \\
c_R &= -\frac{2}{9}\frac{R}{V^2} + \frac{11 + 9\,q_0}{6\,q_0^2} \cdot \frac{N_3\,R}{V^2} + \mathcal{O}(R^2) \\
c_{VR} &= \frac{2}{3}\frac{R}{V} + \mathcal{O}(R^2) \\
c_{RR} &= \frac{2}{9}\frac{R^2}{V^2} \\
c_\lambda &= \frac{32\,\lambda}{q^2\,G^2} = \frac{32}{G^2\,q_0^2} + \frac{4\sqrt{2}\lambda}{3\,3^{1/3}\,\pi\,q_0\,G} \cdot \frac{R}{N_3^{2/3}\,V^{1/3}} + \mathcal{O}(R^2) \\
c_{curv} &= -\frac{16}{G^2\,q^2} \cdot \frac{R}{V} = -\frac{16}{G^2\,q_0^2} \cdot \frac{R}{V} + \mathcal{O}(R^2)\,.
\end{aligned} \right.$$

q_0: flat (*i.e.*, $R = 0$)

So far we have not been able to find the general solution for the above differential eq. but probably still some type of Bessel function or hypergeometric function.

Simplified Differential eq. (*3* + *1* dim.)

In the limit of the *small curvature* and the *large volume*,

Further, set $c_{VR} = 0$ and keeps only the leading term in c_V

$$\frac{\partial^2 \psi}{\partial V^2} + c_V \frac{\partial \psi}{\partial V} + c_\lambda \psi + c_{curv} \psi = 0$$

$$g = \sqrt{G}$$

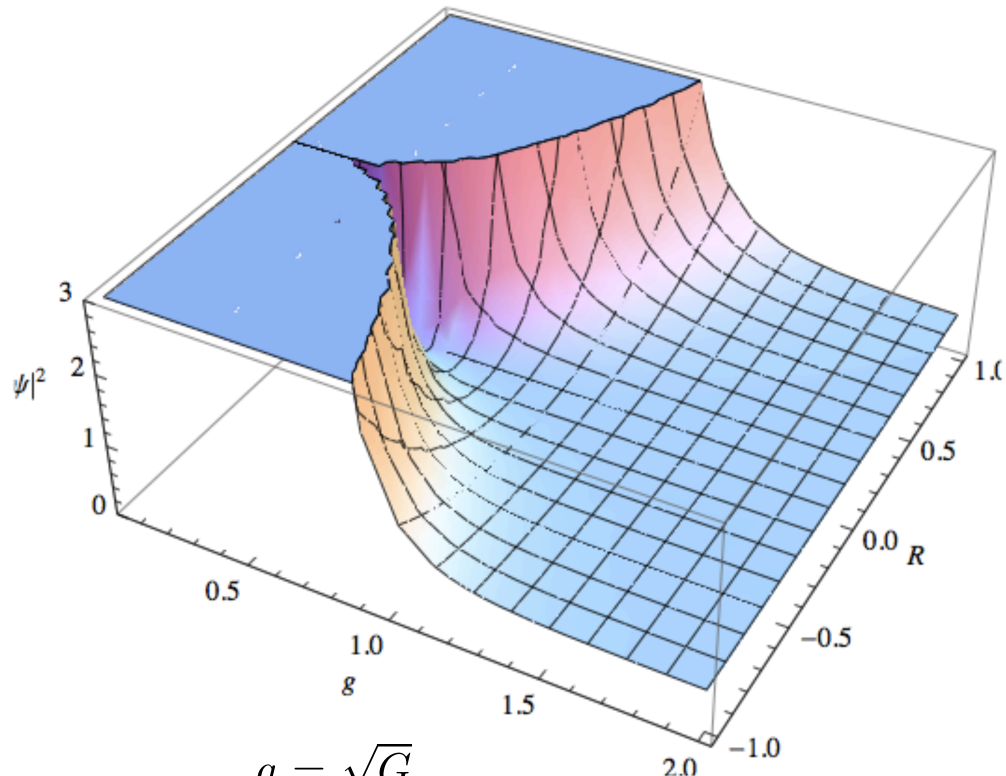
$$\psi(V, R) \simeq e^{-\frac{4 i V}{q_0 g}} \cdot \frac{\Gamma\left(\frac{(11+9 q_0) N_3}{4 q_0^2} + \frac{2 i R}{q_0 g^3}\right)}{\Gamma\left(1 - \frac{(11+9 q_0) N_3}{4 q_0^2} + \frac{2 i R}{q_0 g^3}\right)}$$

$${}_1F_1\left(\frac{(11+9 q_0) N_3}{4 q_0^2} - \frac{2 i R}{q_0 g^3}, \frac{(11+9 q_0) N_3}{2 q_0^2}, \frac{8 i V}{q_0 g}\right)$$

${}_1F_1$: confluent hypergeometric function of first kind

Check: a function of geometric invariants *V* and *R* only.

Probability as a function of G ($3 + 1$ dim.)



$g = \sqrt{G}$

at $N_3 = 10$

For strong coupling, different curvature scales are equally important.

Very small probability at $R \sim 0$ for small G, *so no sensible continuum limit.*

Still work in progress

($3 + 1$ dim.)