On the stability and topology of the universe

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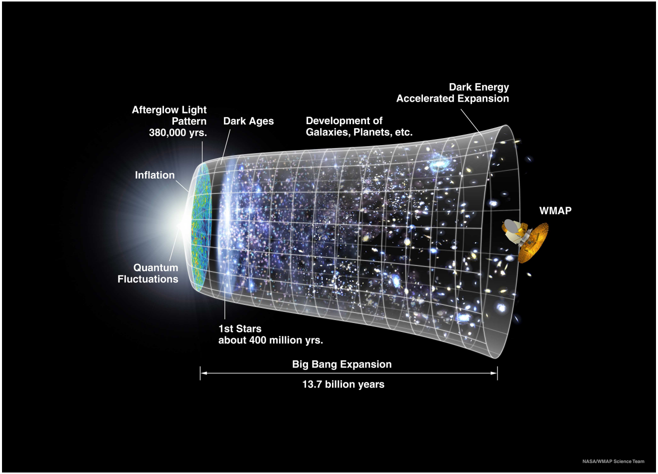
Introduction

The standard models of the universe

- satisfy the **cosmological principle** (i.e., they are spatially homogeneous and isotropic),
- are spatially flat,
- have matter content consisting of ordinary matter, dark matter and dark energy.

Current model of the universe



Current model of the universe: NASA/WMAP Science Team.

Questions

Since the universe is not exactly spatially homogeneous and isotropic, it is natural to ask: **are the standard models future stable?**

Since the assumption of spatial homogeneity and isotropy yields strong restrictions on the allowed topologies, it is also of interest to ask: **what are the restrictions on the global topology of the universe imposed by the constraint that what we observe seems to be close to a standard model?**

Matter models

Perfect fluids: Matter described by energy density ρ and pressure $\mathfrak{p}$. *Dust:* $\mathfrak{p} = 0$. *Radiation:* $\mathfrak{p} = \rho/3$.

Vlasov matter: collection of particles, where

- the particles all have unit mass,
- collisions are neglected,
- the particles follow geodesics,
- collection described statistically by a distribution function.

Stress energy tensor, Vlasov matter

In the Vlasov setting, the relevant mathematical structures are

- the **mass shell** P; the future directed unit timelike vectors in (M, g),
- the **distribution function** $f : P \to [0, \infty)$,
- the **stress energy tensor**

$$T_{\alpha\beta}|_{\xi} = \int_{P_{\xi}} f p_{\alpha} p_{\beta} \mu_{P_{\xi}},$$

- the **Vlasov equation**

$$\mathcal{L} f = 0.$$

The Einstein-Vlasov system

The Einstein-Vlasov system consists of the equations

$$\begin{aligned} G + \Lambda g &= T, \\ \mathcal{L} f &= 0 \end{aligned}$$

for g and f. Note that the second equation corresponds to the requirement that f be constant along timelike geodesics.

Standard models

Spatial homogeneity, isotropy and flatness imply that the metric takes the form

$$g = -dt^2 + a^2(t)\bar{g}.$$

The stress energy tensor of the Vlasov matter then takes perfect fluid form, and the energy density and pressure are given by

$$\begin{aligned} \rho_{\mathrm{Vl}}(t) &= \int_{\mathbb{R}^3} \bar{\mathsf{f}}\,(a(t)\bar{q})\,(1+|\bar{q}|^2)^{1/2}d\bar{q}, \\ \mathfrak{p}_{\mathrm{Vl}}(t) &= \frac{1}{3}\int_{\mathbb{R}^3} \bar{\mathsf{f}}\,(a(t)\bar{q})\,\frac{|\bar{q}|^2}{(1+|\bar{q}|^2)^{1/2}}d\bar{q}, \end{aligned}$$

where $\bar{\mathsf{f}}$, a function on $\mathbb{R}^3$, is the initial datum for the distribution function.

Approximating fluids

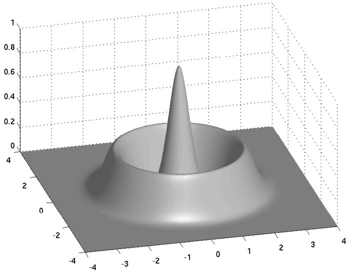


Figure: An illustration of an initial datum for the distribution function which is appropriate when approximating a standard model.

Minkowski space; non-silent causality

Let $\gamma(t) = (t, 0, 0, 0)$. Then γ is an observer in Minkowski space. How much of the $t = 0$ hypersurface does γ see?

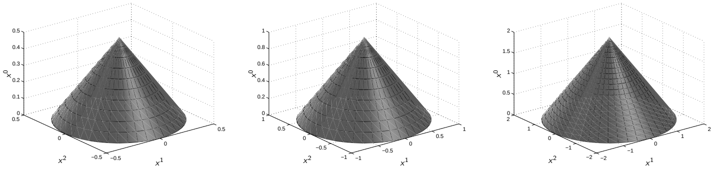


Figure: The causal past of $\gamma(t)$ intersected with the causal future of the $t = 0$ hypersurface for $t = 1/2$, $t = 1$ and $t = 2$.

de Sitter space; silent causality

Consider the metric

$$g = -dt^2 + e^{2t}\bar{g}.$$

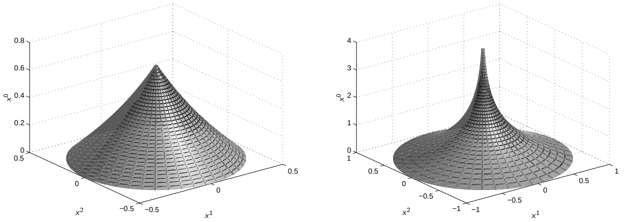


Figure: The causal past of $\gamma(t)$ intersected with the causal future of the $t = 0$ hypersurface for $t = 1/2$ and for all t.

Induced initial data

Let (M, g, f) be a solution and Σ be a spacelike hypersurface in (M, g). Then the **initial data induced on** Σ consist of

- the Riemannian metric induced on Σ by g, say $\bar{g}$,
- the second fundamental form induced on Σ by g, say $\bar{k}$,
- the induced distribution function $\bar{f} : T\Sigma \to [0, \infty)$.

Here

$$\bar{f} = f \circ \mathrm{proj}_{\Sigma}^{-1},$$

where $\mathrm{proj}_{\Sigma} : P_{\Sigma} \to T\Sigma$ represents projection orthogonal to the normal.

Projection to the tangent space

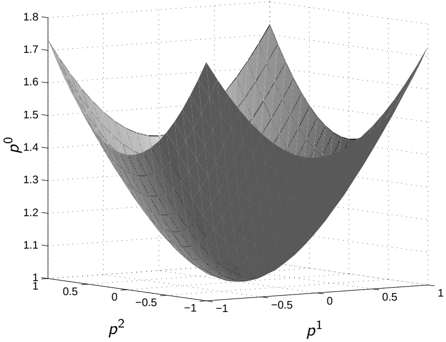


Figure: In suitable coordinates, proj_Σ corresponds to the ordinary projection to the p^1p^2-plane in the figure.

Function spaces

If Σ is a compact manifold, $\bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$ denotes the space of smooth functions $f : T\Sigma \to \mathbb{R}$ such that

$$\|\bar{f}\|_{H^l_{\mathrm{Vl},\mu}}$$

$$= \left(\sum_{i=1}^{j} \sum_{|\alpha|+|\beta|\leq l} \int_{\bar{\mathsf{x}}_i(U_i)\times\mathbb{R}^n} \langle\bar{\varrho}\rangle^{2\mu+2|\beta|} \bar{\chi}_i(\bar{\xi})(\partial^{\alpha}_{\bar{\xi}}\partial^{\beta}_{\bar{\varrho}}\bar{\mathsf{f}}_{\bar{\mathsf{x}}_i})^2(\bar{\xi},\bar{\varrho})d\bar{\xi}d\bar{\varrho} \right)^{1/2}$$

is finite for every $l \geq 0$, where

$$\langle\bar{\varrho}\rangle = (1+|\bar{\varrho}|^2)^{1/2}.$$

Previous results, exponential expansion

- Stability of de Sitter space in $3+1$-dimensions, etc., Helmut Friedrich, '86, '91.
- Stability of even dimensional de Sitter spaces, Michael Anderson '05.
- Stability in the non-linear scalar field case, H.R. '08.
- Einstein-Euler with a positive cosmological constant, Igor Rodnianski and Jared Speck, *preprint* '09.

Future stability, assumptions

Let $(G, \bar{g}_{\mathrm{bg}}, \bar{k}_{\mathrm{bg}}, \bar{f}_{\mathrm{bg}})$ be initial data corresponding to an expanding spatially homogeneous solution to the Einstein–Vlasov system. Assume, that

- the solution is neither of Bianchi type IX nor of Kantowski Sachs type,
- there is a cocompact subgroup Γ of the isometry group of the initial data.

Let $\Sigma = G/\Gamma$ be the compact quotient.

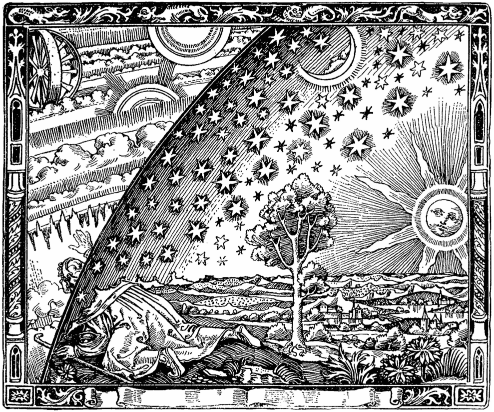
Future stability, conclusions

Then there is an $\epsilon > 0$ such that if $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ are initial data satisfying

$$\|\bar{g} - \bar{g}_{\mathrm{bg}}\|_{H^5} + \|\bar{k} - \bar{k}_{\mathrm{bg}}\|_{H^4} + \|\bar{f} - \bar{f}_{\mathrm{bg}}\|_{H^4_{\mathrm{Vl},\mu}} \leq \epsilon,$$

then the maximal Cauchy development of $(\Sigma, \bar{g}, \bar{k}, \bar{f})$ is future causally geodesically complete. Moreover, the solution is asymptotically de Sitter like.

What is the shape of the universe?



Question, topology

Assume that

- the observational data indicate that, to our past, the universe is well approximated by one of the standard models,
- interpreting the data in this model, we only have information concerning the universe for $t \geq t_0$,
- there is a big bang,
- analogous statements apply to all observers in the universe (with the same t_0).

The question is then: what conclusions are we allowed to draw concerning the global spatial topology of the universe?

Ingredients

Assume we are given

- a standard model, characterised by an existence interval I, a scale factor a etc.,
- a $t_0 \in I$, which represents the time to the future of which we wish the approximation to be valid,
- an $l \in \mathbb{N}$, specifying the norm with respect to which we measure proximity to the standard model,
- an $\epsilon > 0$, characterising the size of the distance,
- a closed 3-manifold Σ.

Construction

There is a solution (M, g, f) with the following properties:

- (M, g, f) is a maximal Cauchy development,
- (M, g) is future causally geodesically complete,
- there is a Cauchy hypersurface, say $\bar{S}$, in (M, g), diffeomorphic to Σ,
- given an observer γ in (M, g), there is a neighbourhood, say U, of

  $$J^-(\gamma) \cap J^+(\bar{S})$$

  such that the solution in U is ϵ-close to the standard model in a solid cylinder of the form $[t_0, \infty) \times \bar{B}_R(0)$,
- all timelike geodesics in (M, g) are past incomplete,
- the solution is stable with these properties.

Questions

- **Strong cosmic censorship;** do initial data generically yield an inextendible maximal Cauchy development?
- **Curvature blow up;** does the curvature blow up in the incomplete directions of causal geodesics (in the maximal Cauchy development)?
- **Stability;** given a certain behaviour at the singularity or in the expanding direction, is it stable under perturbations?

Asymptotic aspects

- **Silence;** do horizons form in the direction of the singularity? Do observers loose the ability to communicate in the expanding direction?
- **Isotropy;** Does the solution isotropize or not?
- **Convergence;** Is the behaviour convergent or oscillatory?

Expanding vacuum solutions; a non-silent situation

Let Σ be a closed hyperbolic manifold and $\bar{g}_H$ be a (suitable) hyperbolic metric on Σ. Then

$$g = -dt^2 + t^2 \bar{g}_H$$

is a solution to Einstein's vacuum equations.

This solution is future stable; Andersson and Moncrief '04.

Hyperbolic dominance/isotropization

Due to geometrization, it is possible to **cut up** a closed 3-manifold and to endow the pieces with preferred geometries.

There is a general conjecture relating the asymptotic behaviour of vacuum solutions to such a division; due to *Moncrief, Fischer and Anderson*.

Rough conjecture: all the volume is in the hyperbolic pieces asymptotically. The solution isotropizes on the hyperbolic pieces.

Geometric decomposition

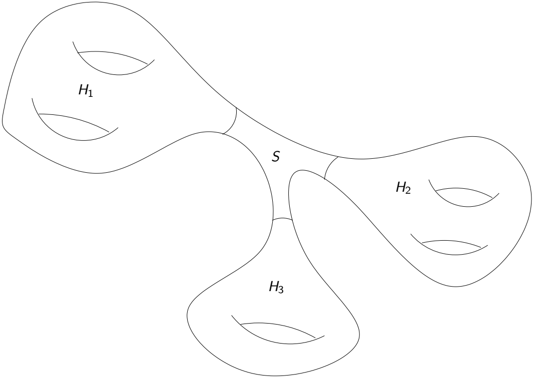


Figure: A schematic depiction of a geometric decomposition; H_1, H_2 and H_3 represent the hyperbolic pieces and S the Seifert manifold pieces.

Support

- Andersson and Moncrief '05,
- Choquet-Bruhat and Moncrief '01,
- ...,
- Anderson '01,
- Reiris '10,
- ...

Horizons and CMB

Misner '69:

> *. . . if the $3^\circ K$ background radiation were last scattered at a redshift $z = 7$, then the radiation coming to us from two directions in the sky separated by more than about 30° was last scattered by regions of plasma whose prior histories had no causal relationship. [. . .] Robertson-Walker models therefore give no insight into why the observed microwave radiation from widely different angles in the sky has very precisely ($\lesssim 0.2\%$) the same temperature.*

Mixmaster and BKL

Misner: Mixmaster solutions (Bianchi IX) do not form horizons.

BKL: In generic collapse horizons form and the local behaviour is well described by Bianchi IX solutions (which are oscillatory).

Caveat: Convergent behaviour for scalar fields, stiff fluids and in higher dimensions.

Support

Oscillatory:

- Belinskii, Khalatnikov, Lifschitz (BKL), '70, '82,
- Misner '69,
- Chitre '72,
- Damour, Henneaux, Nicolai '03,
- Heinzle, Uggla, Rohr '09,
- ...

Non-oscillatory:

- Andersson, Rendall '01.
- Damour, Henneaux, Rendall, Weaver '02.

Hierarchy

- Silent, convergent and isotropic,
- Silent, convergent and anisotropic,
- Non-silent, partially convergent and partially isotropic,
- Silent (?), oscillatory and anisotropic.

Thank you!