Statistical analysis of punctures for Loop Black holes

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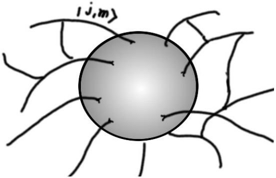
I.Plan

- Punctures as *Quantum Hair*
- Canonical ensemble analysis
- Grand Canonical ensemble analysis
- Correction to the Area law
- Discussion

I. Introduction

- Black hole in LQG : Isolated horizon with punctures. [Eugenio's talk]
- Chern Simons theory on the horizon. Edges of spin network thread the horizon.
- Punctures contribute area elements to the horizon and construct the microstates accounting



for the entropy.

- Area of the horizon is an observable. Statistical analysis for area of the horizon.

I.Introduction

- Microcanonical studies have been done GM, DL, ENP, .. , characterizing the horizon as

$$A = 8\pi\gamma l_P^2 \sum_P \sqrt{j_P(j_P+1)},$$

$$\sum_P m_P = 0.$$

and counting the number of such configurations

$$\Omega \sim \frac{e^{\lambda A}}{\sqrt{A}},$$

and

$$S \sim \lambda A - \frac{1}{2}\log A.$$

- However number of punctures can not be held fixed, horizon can exchange the number of area quanta with the bulk.
- Does this situation corresponds to entropy calculation of a photon gas?

I. Introduction: Punctures as Quantum hair

- Major Class. Quant. Grav. **17** (2000) Ghose and Perez Phys. Rev. Lett. **107** (2011): Punctures as quantum hair.
- Chemical Potential associated with a puncture.
- Horizon as a gas of punctures.
- Microcanonical analysis suggests Bekenstein-Hawking area law recovered.
- Implications for subleading corrections ?

II. Canonical ensemble analysis

- We first fix a graph Γ and calculate the (*canonical*) partition function as first step
- The partition function for this **canonical** ensemble is given as

$$Z_\Gamma(\beta, N) = \sum_{\{n_{jm_j}\}} \frac{N!}{\prod_{jm_j} n_{jm_j}!} \, \delta_{p,0} \, e^{-\beta \sum_{jm_j} n_{jm_j} a_j},$$

with

$$N = \sum_{j,m_j} n_{jm_j}, \qquad \text{and} \qquad 2 \sum_{j,m_j} n_{jm_j} m_j = p.$$

- We use a suitable representation of the delta function to turn the partition function into

$$Z_\Gamma(\beta, N) = \frac{1}{2\pi} \sum_{\{n_{jm_j}\}} \frac{N!}{\prod_{jm_j} n_{jm_j}!} \int_{\mathbf{0}}^{\mathbf{2\pi}} \mathbf{dk} \mathbf{e}^{\mathbf{2ik} \sum_{\mathbf{jm_j}} \mathbf{n_{jm_j}} \mathbf{m_j}} e^{-\beta \sum_{jm_j} n_{jm_j} a_j}.$$

- On simplification,

$$Z_\Gamma(\beta, N) = \frac{1}{2\pi} \int_0^{2\pi} dk \left(\sum_{jm_j} e^{(2ikm_j - \beta a_j)} \right)^N$$

- If we work with Flux area operator [Barbero, Lewandowski,Vilsenor], the Unitary representation of Area operator [Livine], or the semiclassical limit

$$a_j = (j+1) \qquad\qquad j \in \mathbb{N}$$

II.Canonical ensemble analysis

- In this case

$$Z_\Gamma(\beta, N) \approx \frac{1}{2\pi}\int_0^{2\pi} dk \left(\frac{1}{e^{2ik}-1}\sum_{l=1}^{\infty} e^{-\sigma(l+1)}\{e^{ik(l+2)} - e^{-ikl}\}\right)^N$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk \left(\frac{2\cos k - e^{-\sigma}}{e^{2\sigma} - 2e^{\sigma}\cos k + 1},\right)^N,$$

with

$$\sigma = 4\pi\gamma l_p^2\beta,$$

which can be evaluated in the thermodynamic limit $N >> 1$.

- With a transformation

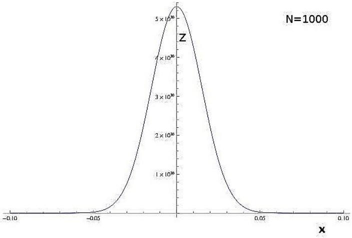
$$k = 2\tan^{-1}(x/2)$$

- the partition function

$$Z_\Gamma(\beta, N) = \int_{-\infty}^{\infty} dx \frac{1}{2\pi(1 + x^2/4)}\left(\frac{2\cos k(x) - e^{-\sigma}}{e^{2\sigma} - 2e^{\sigma}\cos k(x) + 1}\right)^N$$

II. Canonical ensemble analysis

- Partition unction is a unimodal symmetric distribution



- We would like it to approximate as accurately as possible.

II.Canonical ensemble analysis: Approximation schemes

- Moment generating function for a (Non-normalized) Gaussian with a zero mean

$$C \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}\frac{x^2}{\sigma^2}},$$

is given by

$$M(t) = C e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}\frac{(x-t\sigma^2)^2}{\sigma^2}}.$$

- With the substitution $x - t\sigma^2 = x'$ we have

$$\begin{aligned} M(t) &= C e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} dx' e^{-\frac{1}{2}\frac{(x')^2}{\sigma^2}}, \\ &= C e^{\frac{t^2\sigma^2}{2}} \sqrt{2\pi\sigma^2} = A f(i\sigma^2 t), \end{aligned}$$

where $f(x) = C e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$ and $A = \sqrt{2\pi\sigma^2}$.

- In a non-normalized gaussian distribution (with zero mean), the $n-th$ moment is given by

$$\mu_n = \frac{C \int_{-\infty}^{\infty} dx x^n e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}}{C \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}}$$

- Now,

$$M(t) = C\int_{-\infty}^{\infty} dx e^{tx} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$
$$= C\int_{-\infty}^{\infty} dx(1 + tx + \frac{(tx)^2}{2!} + \ldots + \frac{(tx)^n}{n!} + \ldots)e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}.$$

Thus,

$$\mu_n = \frac{M^{(n)}(t)|_0}{M(t)|_0}.$$

- Now,

$$M^{(n)}(t)|_0 = A(i\sigma^2)^n f^{(n)}(i\sigma^2 t)|_0 = A(i\sigma^2)^n f^{(n)}(0),$$
$$M(t)|_0 = Af(i\sigma^2 t)|_0 = Af(0).$$

Therefore the $n - th$ moment is

$$\mu_n = \frac{(i\sigma^2)^n f^{(n)}(0)}{f(0)}.$$

- **Variance**
For second moment

$$\sigma^2 = -\sigma^4 \frac{f''(0)}{f(0)},$$
$$\text{therefore, } \sigma^2 = -\frac{f(0)}{f''(0)},$$

II.Canonical ensemble analysis: Approximation schemes

- **Kurtosis**
  The 4-th moment is again obtained as

$$\mu_4 = \frac{(i\sigma^2)^4 f^{(4)}(0)}{f(0)}.$$

Therefore the kurtosis is given by

$$\beta_2 = \frac{\mu_4}{\sigma^4} = \frac{\left[\frac{(i\sigma^2)^4 f^{(4)}(0)}{f(0)}\right]}{\left[-\sigma^4 \frac{f''(0)}{f(0)}\right]^2},$$

$$\beta_2 = \frac{\mu_4}{\sigma^4} = \frac{f(0) f^{(4)}(0)}{(f''(0))^2}.$$

- The kurtosis for the distribution becomes

$$\frac{\mu_4}{\tilde{\sigma}^4} = \frac{f(x)|_0 f^{(4)}(x)|_0}{[f''(x)|_0]^2} =$$

$$\frac{6[(1-2e^{\sigma})^2(e^{\sigma}-1)^4 + 8e^{3\sigma}(-1+2e^{\sigma}+e^{3\sigma}-e^{2\sigma})N + 8e^{6\sigma}N^2]}{[-1+e^{\sigma}(4+e^{\sigma}(-5+e^{\sigma}(2+4N)))]^2}$$

II.Canonical ensemble analysis: Approximation schemes

- The "excess kurtosis" in the thermodynamic limit vanishes

$$\lim_{N\to\infty} \frac{\mu_4}{\tilde{\sigma}^4} - 3 \to 0.$$

enabling us to approximate the distribution as gaussian and evaluate the partition function as

$$Z_\Gamma(\beta, N) \approx \left[e^{-\sigma} \sqrt{\frac{2\log 4}{N}} \right] \left(\frac{2 - e^{-\sigma}}{(e^{\sigma} - 1)^2} \right)^N .$$

- Corresponding canonical entropy

$$S = \ln Z_\Gamma + \beta A = N[\ln z(\sigma) + \sigma q] - \frac{1}{2}\ln N + \text{const.},$$

with $q = -\partial \log z/\partial\sigma$. The entropy is extremized w.r.t. the number of constituents to get

$$S \approx \frac{\sigma(q_0)A}{4\pi\gamma l_p^2} - \frac{1}{2}\ln\left(\frac{A}{4\pi\gamma l_p^2 q_0}\right) + \text{const.}$$

II.Canonical ensemble analysis

- We recover the B-H area law for the leading order if we take $\gamma = 0.258$

	Analysis	γ
Ghosh et. al. Ghosh, Mitra, Phys. Rev. D. **71** (2005)	Microcanonical LQG	0.274
Ling, Zhang Ling, Zhang, Phys. Rev. D. **68** (2003)	N=1 SUSY LQG	0.247
KL, CV KL & Vaz, Phys. Rev. D. **85** (2012)	Canonical LQG	0.258

- Recent proposals suggest fixation of Immirizi parameter is not core to obtaining the area-law when the problem is posed in terms of local observers [Eugenio's talk].

- We also obtain sub-leading logarithmic corrections with a negative signature.

- Next we allow the number of punctures to vary.

III. Grand-canonical ensemble analysis

- The corresponding *grand-canonical* treatment gives

$$\Xi(\beta,\alpha) = \sum_{N=0}^{\infty} \sum_{n_j=0}^{N} \frac{N!}{\prod_j n_j!} \prod_j (2j+1)^{n_j} e^{-(8\pi\gamma\beta a_j - \alpha) n_j}$$

The average occupation number of punctures in a state j will be

$$\langle n_j \rangle = -\frac{1}{8\pi\gamma\beta} \frac{\partial \ln \Xi}{\partial a_j} = \frac{\lambda(2j+1) e^{-8\pi\gamma\beta a_j}}{1-\lambda z},$$

and the average quantities will be given by

$$\langle N \rangle = \frac{\partial \ln \Xi}{\partial \alpha} = \sum_j \langle n_j \rangle = \frac{\lambda z}{1-\lambda z}.$$

$$A = -\frac{\partial \ln \Xi}{\partial \beta} = \frac{\partial}{\partial \beta} \ln(1-\lambda z) = -N \frac{\partial \ln z}{\partial \beta},$$

where $\lambda(\alpha) = e^{\alpha}$ is the fugacity, and

$$z(\beta) = \sum_j (2j+1) e^{-8\pi\gamma\beta a_j}.$$

III. Grand-canonical ensemble analysis

- Using the relation

$$\Xi = \sum_N Z^N e^{\alpha N}$$

and using the canonical partition function we get

$$\Xi(\sigma, \alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk}{1 - \lambda(\alpha) \sum_{l=1}^{\infty} z_l(\sigma) \left(\frac{\sin k(l+1)}{\sin k} \right)},$$

with $z_l(\sigma) = e^{-\sigma(l+1)}$ and $\lambda(\alpha) = e^{\alpha}$.

- Again, the partition function can be approximated (saddle-point) in the thermodynamic limit

$$\Xi(\sigma, \alpha) \approx \sqrt{2\pi} f(0) \tilde{\sigma} = \frac{1}{\sqrt{\pi \{1 - \lambda z(\sigma)\}\{1 + \lambda b(\sigma)\}}},$$

where

$$z(\sigma) = \sum_{l=1}^{\infty} z_l(\sigma)(l+1)$$

$$b(\sigma) = \sum_{l=1}^{\infty} z_l(\sigma) \left[\frac{2}{3} l^3 + 2l^2 + \frac{1}{3} l - 1 \right].$$

III. Grand-canonical ensemble analysis

- Large N limit is given by
$$\lambda z \to 1$$
- In this limit the ratio A/N depends on the chemical potential and is constant for isothermal cases : Good intensive variable to use.
- Legendre transform of $\ln \Xi$, which is the entropy, becomes
$$S(A,N) = \ln \Xi + \beta A - \alpha N = (N+1)\ln(N+1) - N \ln N + N a \sigma(a) + N \ln z(a)$$
and simplifies, in the limit of large N, to
$$S(A,N) \approx \ln N + N[a\sigma(a) + \ln z(a)] = \frac{\sigma(a)}{\pi\gamma}\frac{A}{4l_p^2} + N \ln z(a) + \ln N.$$
- At some fixed value of the temperature, σ_0, or of the chemical potential, α_0, we find that $a(\sigma_0) = a_0$ then
$$N = \frac{A}{4\pi\gamma l_p^2 a_0}$$
can be used to eliminate N
$$S(A) \approx \frac{1}{\pi\gamma}\left[\sigma_0 + \frac{\ln z(a_0)}{a_0}\right]\frac{A}{4l_p^2} + \ln \frac{A}{4l_p^2} + \text{const.},$$

III. Grand-canonical ensemble analysis

- Inclusion of the projection constraint and the fluctuation in N, in large N limit gives

$$S(A) = \frac{1}{\pi\gamma}\left[\sigma_0 + \frac{\ln z(a_0)}{a_0}\right]\frac{A}{4l_p^2} + \frac{1}{2}\ln\frac{A}{4l_p^2} + \text{const.}$$

- Therefore for isothermal case B-H law is obtained upto fixing the Immirizi parameter.

- For zero chemical potential we reocver the same Immirizi parameter. In general it is chemical potential dependent.

- The logarithmic correction has now become positive signature and differs from microcanonical results.

Discussions

- The B-H area relation can be achieved for isothermal cases in LQG.
- In general, the Immirizi parameter is a function of the temperature/chemical potential.
- Canonical/grand-canonical analysis suggests correction to area law, logarithmic in nature but with opposite signatures.
- Differs from microcanonical analysis
  Barbero, Vilasenor, Class. Quant. Grav. (2011).
- Implications for stability. Energy ensemble in terms of local observers will make the analysis thermal.

Thank you for your attention !