Introduction to Causal Dynamical Triangulations

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Outline

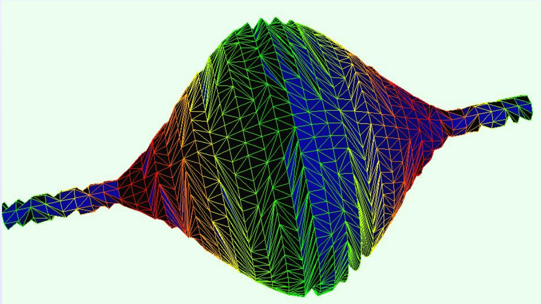
1. Path integral for quantum gravity
2. Introduction to Causal Dynamical Triangulations
3. Phase diagram
4. Emergence of background geometry
5. Quantum fluctuations
6. Effective action

What is Causal Dynamical Triangulation?

Causal Dynamical Triangulation (CDT) is a background independent approach to quantum gravity.

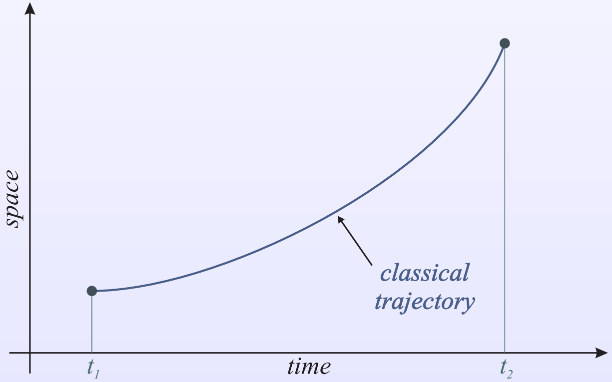
$$\int \mathrm{D}[g] e^{i S^{EH}[g]} \quad \rightarrow \quad \sum_{\mathcal{T}} e^{-S^{R}[\mathcal{T}]}$$

CDT provides a lattice regularization of the formal gravitational path integral via a sum over causal triangulations.



Path integral formulation of quantum mechanics

- A classical particle follows a unique trajectory.
- *Quantum mechanics* can be described by *Path Integrals*: All possible trajectories contribute to the transition amplitude.
- To define the functional integral, we discretize the time coordinate and approximate each path by linear pieces.



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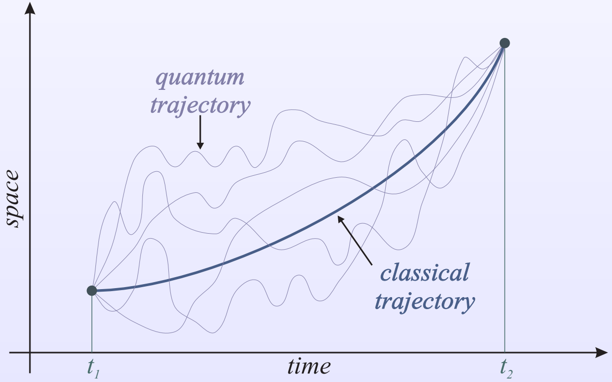

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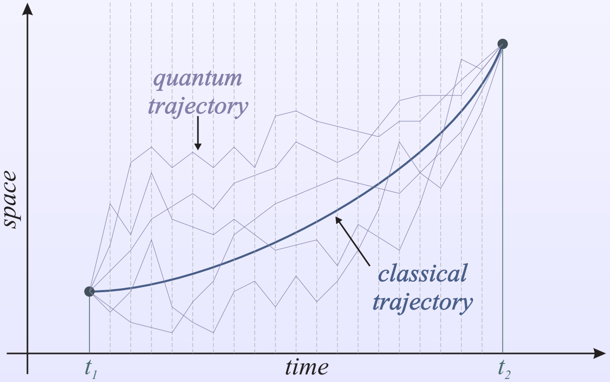
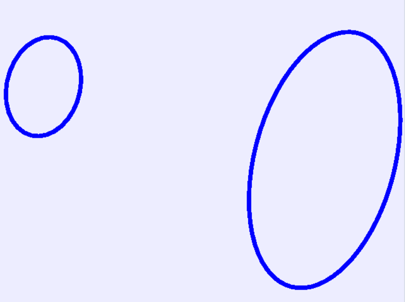


Path integral formulation of quantum gravity

- **General Relativity**: gravity is encoded in space-time geometry.
- The role of a trajectory is now played by the geometry of four-dimensional space-time.
- All space-time histories contribute to the transition amplitude.

1+1D Example: State of system: one-dimensional spatial geometry



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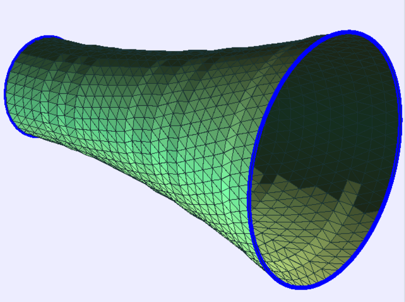
1+1D Example: Evolution of one-dimensional closed universe



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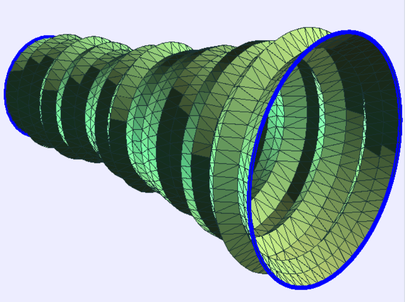
Sum over all two-dimensional surfaces joining the in- and out-state



Transition amplitude

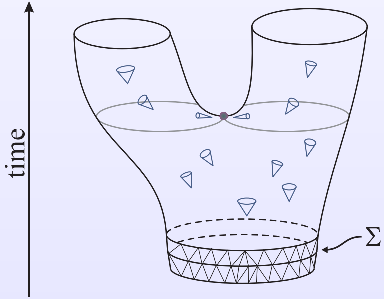
Our aim is to calculate the amplitude of a transition between two geometric states:

$$G(\mathbf{g}_i, \mathbf{g}_f, t) \equiv \int_{\mathbf{g}_i \to \mathbf{g}_f} \mathrm{D}[g] \mathrm{e}^{iS^{EH}[g]}$$

To define this path integral we have to specify the *measure* $\mathrm{D}[g]$ and the *domain of integration* - **a class of admissible space-time geometries** joining the in- and out- geometries.

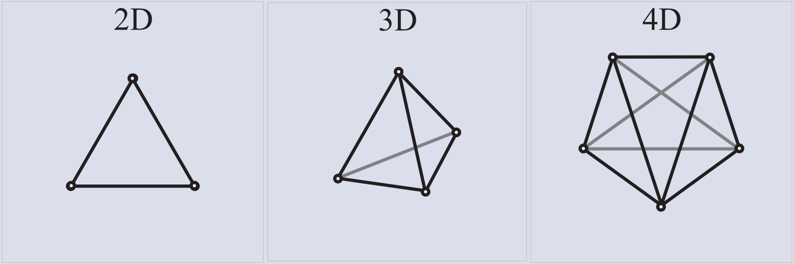
Causality - difference between DT and CDT

- **Causal D**ynamical **T**riangulations assume global proper-time foliation. Spatial slices (leaves) have fixed topology and are not allowed to split in time.
- Foliation distinguishes between time-like and spatial-like links.
- In **Euclidean DT** one cannot avoid introducing causal singularities, which lead to creation of baby universes.
- **EDT** and **CDT** differ in a class of admissible space-time geometries.



Regularization by triangulation

1. **Discretization** is a the standard regularization used in QFT.
2. Spatial states are 3D geometries with a topology S^3. Discretized states are made of equilateral **tetrahedra**.
3. 4D simplicial manifold is obtained by gluing pairs of 4-simplices along their 3-faces.
4. The metric is **flat** inside each 4-simplex.
5. Length of time links a_t and space links a_s is constant.
6. **Curvature** (angle deficit) is localized at triangles.



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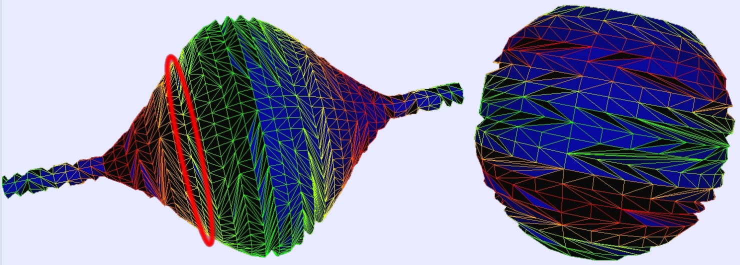

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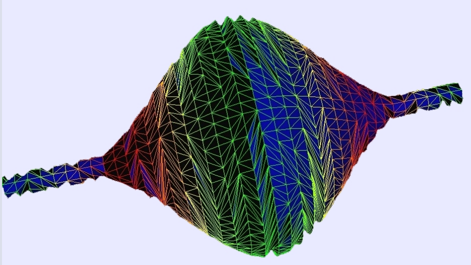

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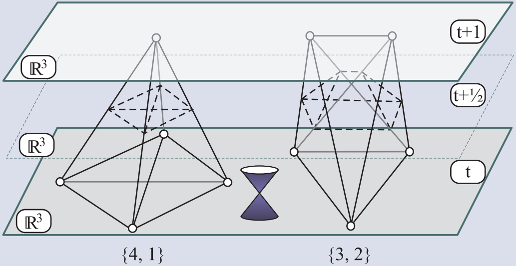

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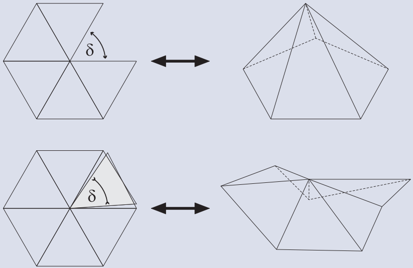


Regge action

The Einstein-Hilbert action has a natural realization on piecewise linear geometries called Regge action

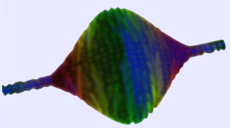
$$S^E[g] = -\frac{1}{G} \int \mathrm{d}t \int \mathrm{d}^D x \sqrt{g} (R - 2\Lambda)$$

N_0 number of vertices

N_4 number of simplices

N_{14} number of simplices of type $\{1,4\}$

K_0 K_4 Δ bare coupling constants $(G, \Lambda, a_t/a_s)$



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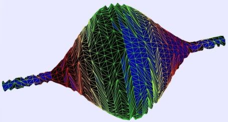

- The partition function of quantum gravity is defined as a formal integral over all geometries weighted by the Einstein-Hilbert action.

$$Z = \int \mathrm{D}[g] e^{i S^{EH}[g]}$$

- To make sense of the gravitational path integral one uses the standard method of regularization - discretization.
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Causal Dynamical Triangulations

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Numerical setup

- To calculate the **expectation value of an observable**, we approximate the path integral by a sum over a finite set of Monte Carlo configurations

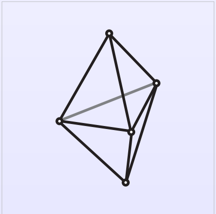
$$
\begin{aligned}
\langle \mathcal{O}[g] \rangle &= \frac{1}{Z} \int \mathcal{D}[g] \mathcal{O}[g] e^{-S[g]} \\
&\downarrow \\
\langle \mathcal{O}[\mathcal{T}] \rangle &= \frac{1}{Z} \sum_{\mathcal{T}} \mathcal{O}[\mathcal{T}] e^{-S[\mathcal{T}]} \\
&\downarrow \\
\langle \mathcal{O}[\mathcal{T}] \rangle &\approx \frac{1}{K} \sum_{i=1}^{K} \mathcal{O}[\mathcal{T}^{(i)}]
\end{aligned}
$$

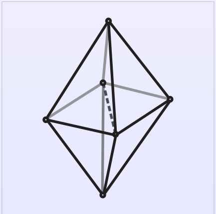
- **Monte Carlo** algorithm probes the space of configurations with the probability $P[\mathcal{T}] = \frac{1}{Z} e^{-S[\mathcal{T}]}$.

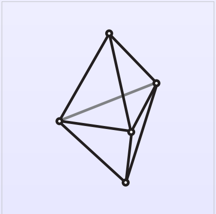
Monte Carlo simulations

- Random walk over configuration space, consisting of a series of Monte Carlo moves.
- Ergodicity - all configurations can be generated by the set of Pachner moves.
- Detailed balance condition - $P(\mathcal{A})W(\mathcal{A} \to \mathcal{B}) = P(\mathcal{B})W(\mathcal{B} \to \mathcal{A})$
- Fixed topology - moves don't change the topology.
- Causality - moves preserve the foliation.
- 4D CDT - set of 7 moves.

Moves in 3D



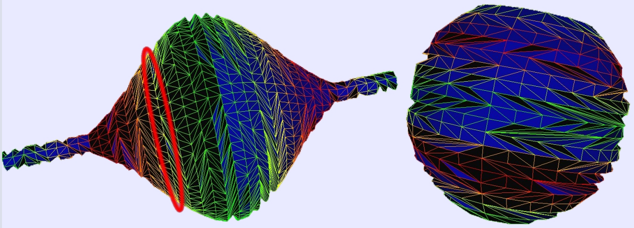




Spatial slices

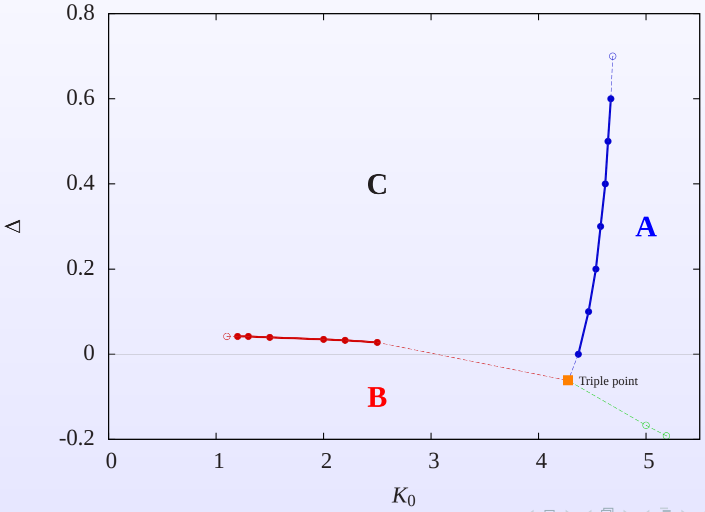
- The simplest observable giving information about the geometry, is the **spatial volume** n_i defined as a number of tetrahedra building a three-dimensional slice $i = 1 \ldots T$.
- Restricting our considerations to the spatial volume n_i we reduce the problem to one-dimensional quantum mechanics.

3D spatial slices with topology S^3



Phase diagram

$$S[\mathcal{T}] = -K_0 N_0 + K_4 N_4 + \Delta(N_{14} - 6N_0)$$



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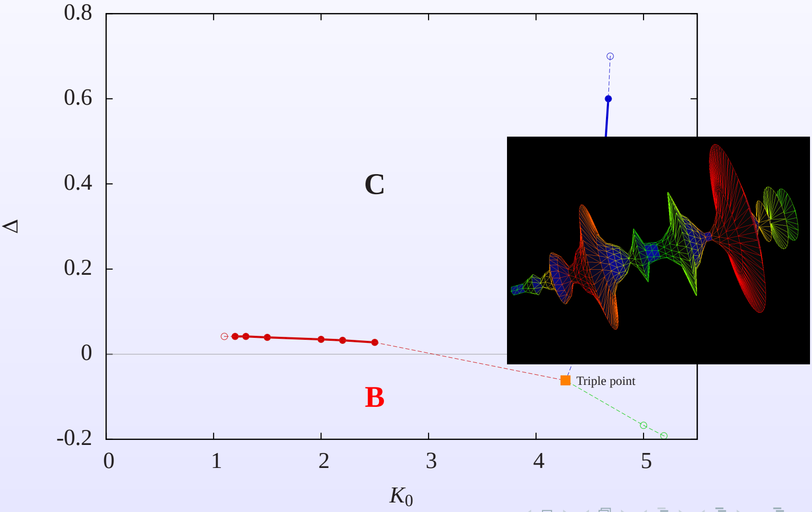

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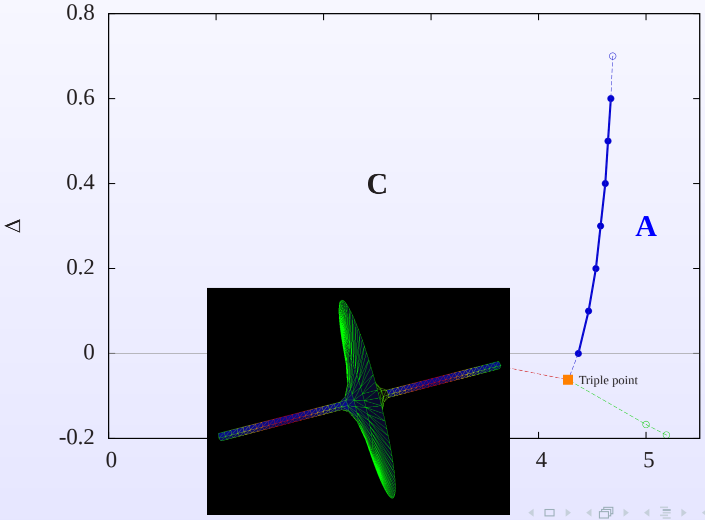

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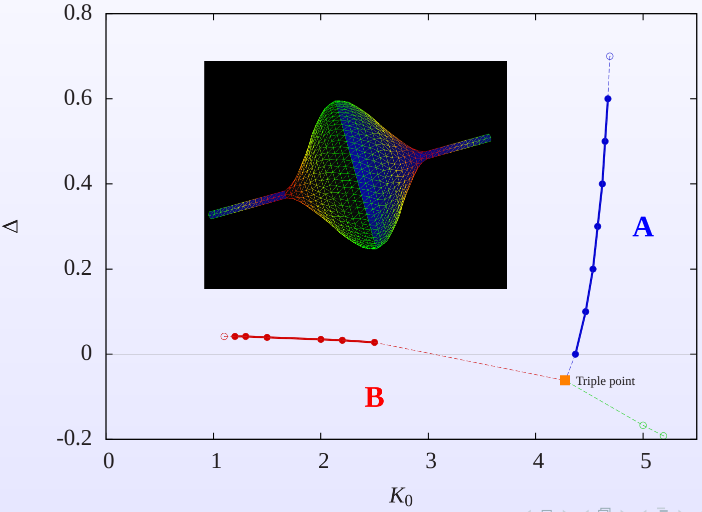
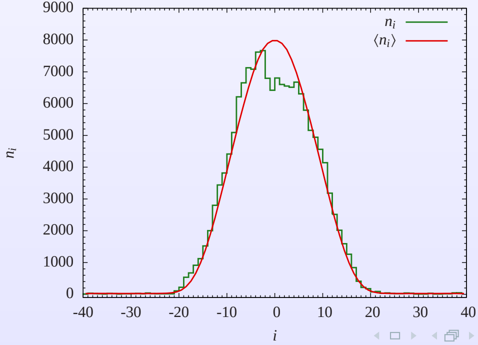


De Sitter space-time as background geometry

- In phase **C** the time translation symmetry is spontaneously broken and the distribution n_i is bell-shaped.
- The average volume $\langle n_i \rangle$ is with high accuracy given by formula

$$\langle n_i \rangle = H \cos^3 \left(\frac{i}{W} \right)$$

- It describes Euclidean **de Sitter** space (S^4), a classical **vacuum solution**.



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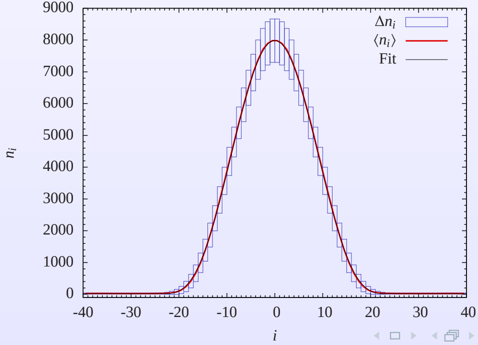
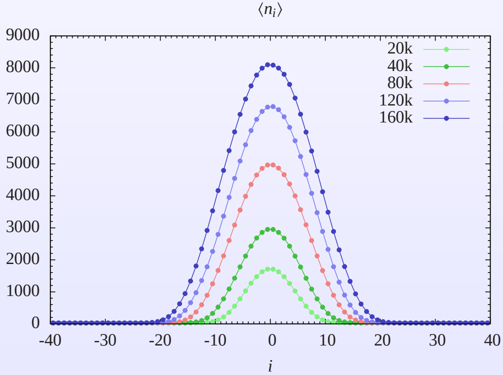


Hausdorff dimension

The spatial volume $\langle n_i \rangle$ scales with total volume N_4 as

$$t = N_4^{-1/4} i, \quad \bar{v}(t) = N_4^{-3/4} \langle n_i \rangle = \frac{3}{4\omega} \cos^3\left(\frac{t}{\omega}\right).$$

Such result is expected for a genuine four-dimensional *Universe*.



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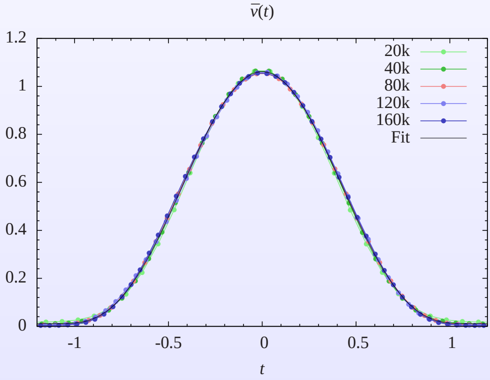
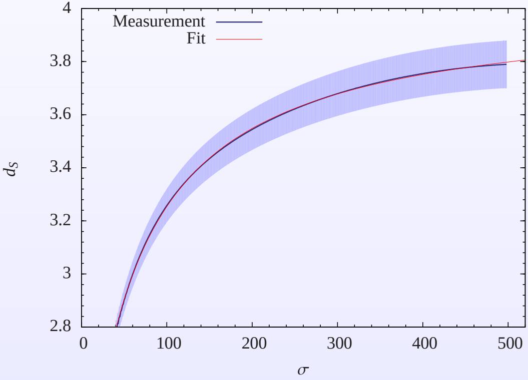


Spectral dimension

Simulations of the diffusion process allow to compute spectral dimension d_s.



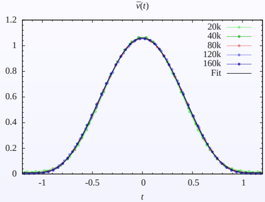
Extrapolation of the results gives short and long range behavior

$$d_s(\sigma \to 0) = 1.95 \pm 0.10, \quad d_s(\sigma \to \infty) = 4.02 \pm 0.10,$$

where σ is a fictitious diffusion time.

Minisuperspace model

$$\bar{v}(t) = \frac{3}{4\omega}\cos^3\left(\frac{t}{\omega}\right)$$



- Classical trajectory $\bar{v}(t)$ corresponds to Euclidean de Sitter space (S^4), a spatially homogeneous and isotropic vacuum solution.
- We ,,freeze'' all degrees of freedom except spatial volume and assume that metric on $S^3 \times S^1$ space-time has form

$$\mathrm{d}s^2 = \mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Omega_3^2, \quad v(t) = a^3(t)$$

- In this particular case, the Einstein-Hilbert action takes form

$$S = \frac{1}{G}\int \mathrm{d}t \int \mathrm{d}\Omega\sqrt{g}(R - 6\lambda)$$

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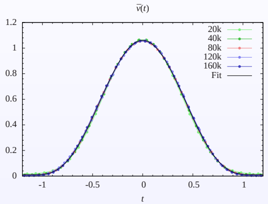

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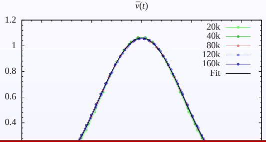

Question?

How well does the minisuperspace model describe quantum fluctuations of spatial volume in CDT?

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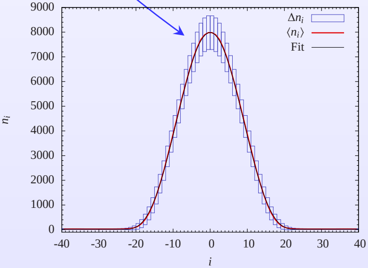
Quantum fluctuations

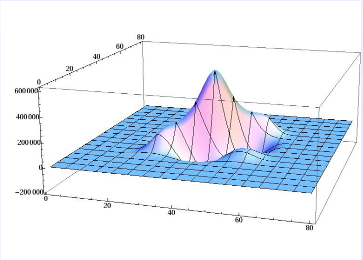
- We can measure the correlation matrix of spatial volume fluctuations around the classical solution $\bar{n} = \langle n \rangle$,

$$\mathbf{C}_{ij} \equiv \langle (n_i - \langle n_i \rangle)(n_j - \langle n_j \rangle) \rangle$$

- The propagator **C** appears in the **semiclassical** expansion of the **effective action** describing quantum fluctuations

$$S[n = \bar{n} + \eta] = S[\bar{n}] + \frac{1}{2} \sum_{i,j} \eta_i \, [\mathbf{C}^{-1}]_{ij} \, \eta_j + O(\eta^3), \quad [\mathbf{C}^{-1}]_{ij} = \left. \frac{\partial^2 S[n]}{\partial n_i \partial n_j} \right|_{n=\bar{n}}$$





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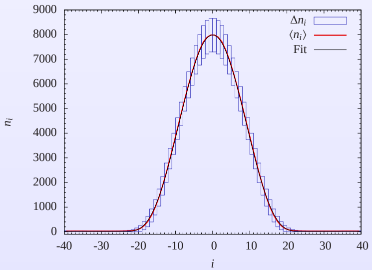

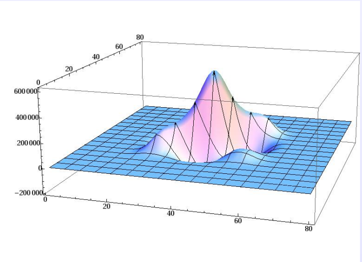

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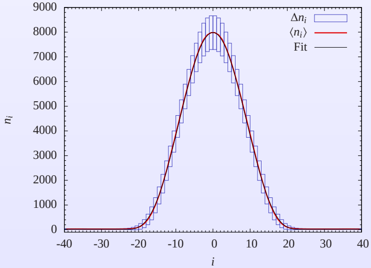
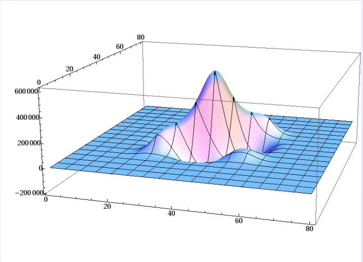
- The propagator **C** appears in the **semiclassical** expansion of the **effective action** describing quantum fluctuations

$$S[n = \bar{n} + \eta] = S[\bar{n}] + \frac{1}{2} \sum_{i,j} \eta_i \, [\mathbf{C}^{-1}]_{ij} \, \eta_j + O(\eta^3), \quad [\mathbf{C}^{-1}]_{ij} = \left. \frac{\partial^2 S[n]}{\partial n_i \partial n_j} \right|_{n = \bar{n}}$$





Effective action

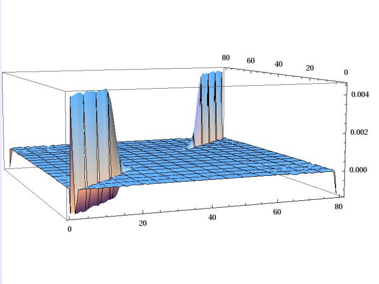
- Minisuperspace action $S[v] = \frac{1}{G}\int \frac{\dot{v}^2}{v} + v^{\frac{1}{3}} - \lambda v \mathrm{d}t$

$$\Downarrow$$

- Discretization $S[n] = \frac{1}{\Gamma}\sum_t \left(\frac{(n_{t+1}-n_t)^2}{n_{t+1}+n_t} + \mu n_t^{1/3} - \lambda n_t \right)$

- Inverse of propagator **C**

$$\eta_i \left[\mathbf{C}^{-1}\right]_{ij} \eta_j = \frac{(\eta_{i+1}-\eta_i)^2}{k_i} - u_i \eta_i^2, \quad k_i = g_1 \bar{n}_i, \quad u_i = g_2 \bar{n}_i^{-5/3}$$



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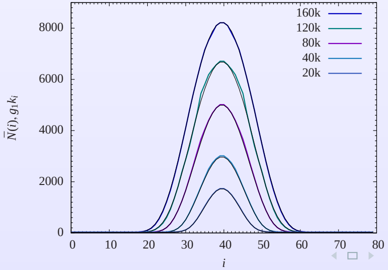

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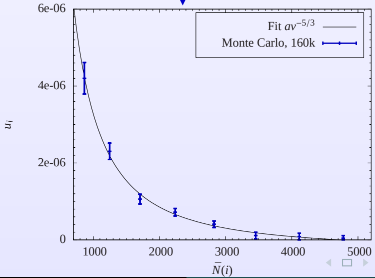


We can measure the discrete effective action

$$S[n] = \frac{1}{\Gamma} \sum_t \left(\frac{(n_{t+1} - n_t)^2}{n_{t+1} + n_t} + \mu n_t^{1/3} - \lambda n_t \right), \quad \Gamma = g_1, \quad \mu = \frac{9}{2} g_1 g_2$$

and relate the effective coupling constant Γ and the **cut-off** a with the Newton's gravitational constant G,

$$G = \mathrm{const} \cdot \Gamma \cdot a^2$$

We may express the lattice constant in term of Planck length and estimate that the Universe built of 362000 simplices has a radius of about 20 Planck lengths.

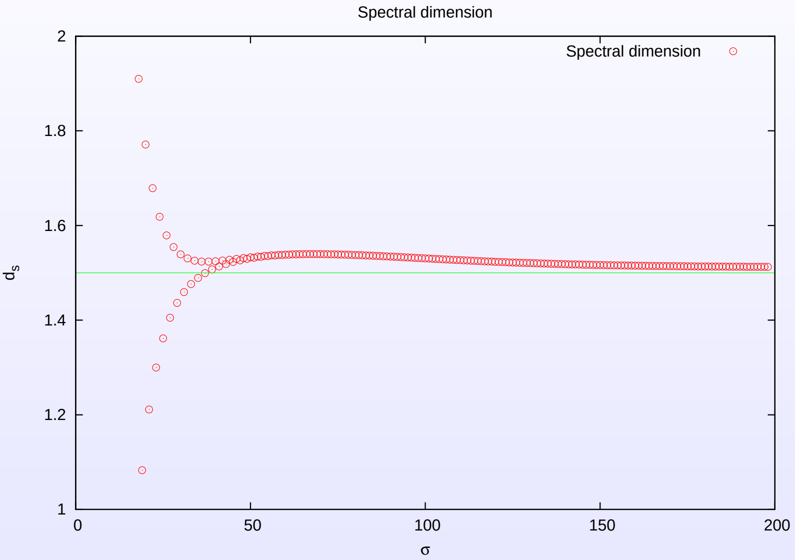
Summary

Causal **D**ynamical **T**riangulations is a **background independent** approach to quantum gravity.

1. Only geometric invariants like length and angles are involved. While no coordinates are introduced, the model is manifestly **diffeomorphism-invariant**.
2. Phase diagram consists of three phases. In phase **C** emerges a **four-dimensional** universe with well defined time and space extent.
3. The **background** geometry corresponds to the Euclidean **de Sitter** space, i.e. **classical solution** of the *minisuperspace* model. However, in CDT no degrees of freedom are frozen.
4. **Quantum fluctuations** of the spatial volume are also properly described by this simple model.

Thank you for your attention!

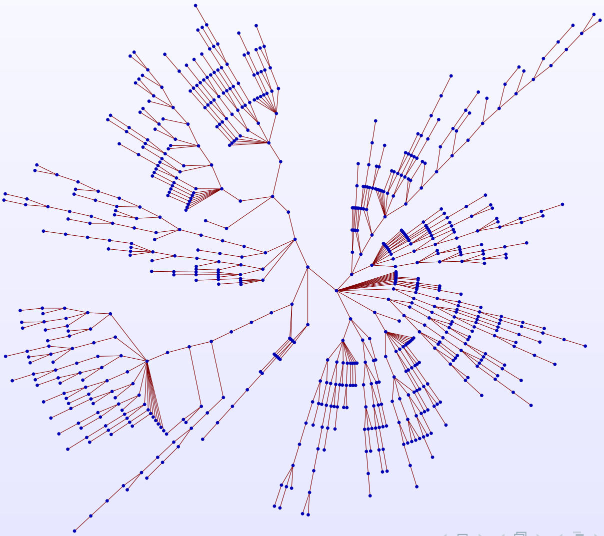
Spectral dimension of spatial slices



Spectral dimension of spatial slices

- The measured spectral dimension is almost a half smaller then the expected *classical* value.
  Such behaviour suggests a fractal nature of constant-time slices (determined by foliation)
- **Is the character of quantum geometry fractal?**
- **Can it be described by Gaussian fluctuations around an average geometry?**

Tree of minimal necks.



Structure of spatial slices

- The quantum geometry has a nontrivial microstructure
- Spatial slices reveal a fractal nature, completely different from smooth S^3
- Similarity to *branched polymers*
- Quantum fluctuations can not be described by Gaussian deviations from background geometry

Monte Carlo simulations - Alexander moves

- We construct a starting space-time manifold with given topology ($S^3 \times S^1$) and perform a random walk over configuration space.

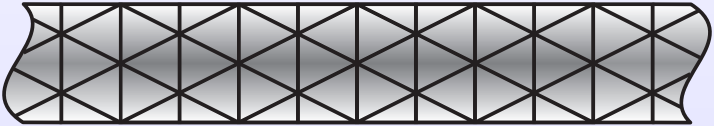
Ergodicity In the dynamical triangulation approach all possible configurations are generated by the set of Alexander moves.

Fixed topology The moves don't change the topology.

Causality Only moves that preserve the foliation are allowed.

4D CDT We have 4 types of moves.

Minimal configuration



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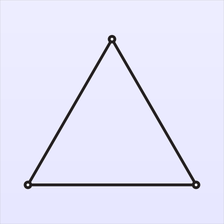
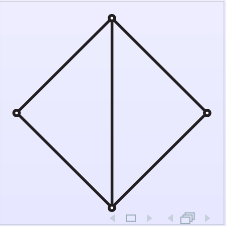
Moves in 2D





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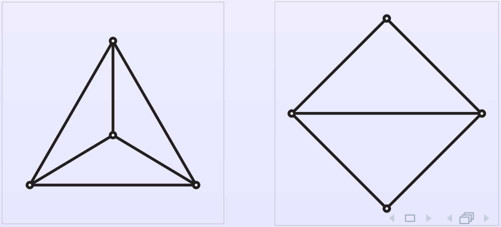

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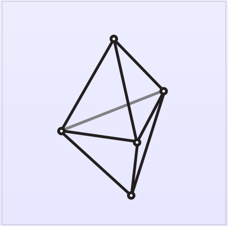

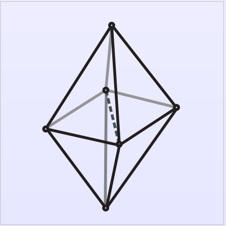

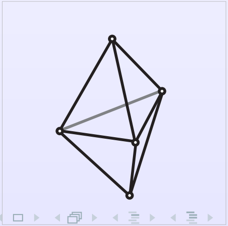

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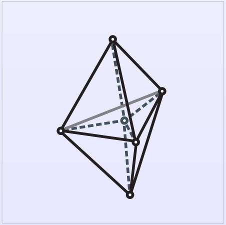

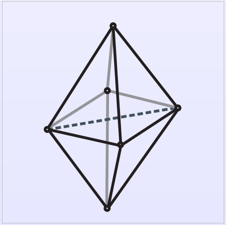

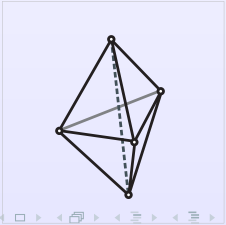

Monte Carlo Markov Chain

- We perform a random walk in the phase-space of configurations (space of piecewise linear geometries).
- Each step is one of the 4D CDT moves.
- The weight (acceptance probability) $W(\mathcal{A} \to \mathcal{B})$ of a move from configuration $\mathcal{A}$ to $\mathcal{B}$ is determined (not uniquely) by the detailed balance condition:

$$P(\mathcal{A})W(\mathcal{A} \to \mathcal{B}) = P(\mathcal{B})W(\mathcal{B} \to \mathcal{A})$$

- The Monte Carlo algorithm ensures that we probe the configurations with the probability $P(\mathcal{A})$.
- After sufficiently long time, the configurations are independent.
- All we need, is the probability functional for configurations $P(\mathcal{A})$.