Spinfoam gravity: progress and perspectives - Lecture 2

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Outline

Lecture 1:

Path integral and the Spinfoam amplitude

Lecture 2:

⇨ Quantum geometry in Spinfoams

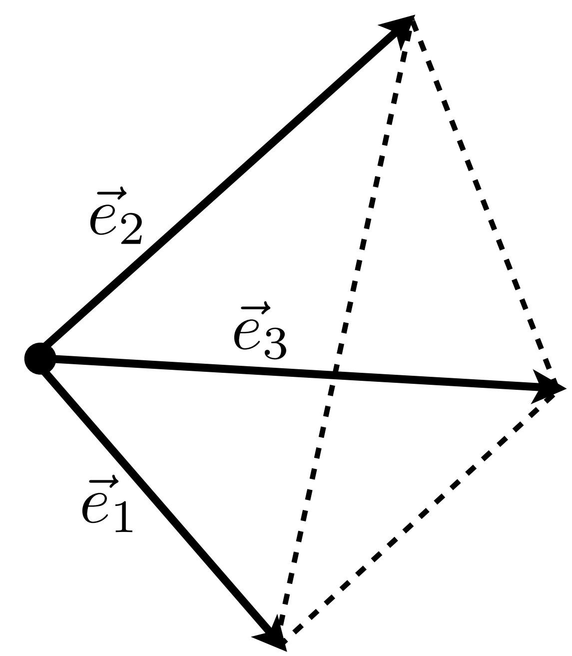
Lecture 3:

- gravitons
- quantum cosmology
- black hole entropy

Classical geometry of a tetrahedron in $\mathbb{R}^3$

Edge vectors (*triads*)

$$\vec{e}_a \qquad a = 1, 2, 3$$



Metric $\quad q_{ab} = \vec{e}_a \cdot \vec{e}_b$

Volume $\quad V = \frac{1}{3!}\sqrt{\det q} = \frac{1}{3!}|\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)|$

Area vectors (*Ashtekar electric field*)

$$\vec{E}^a = \frac{1}{2}\epsilon^{abc}\,\vec{e}_b \times \vec{e}_c$$

e.g.: $\vec{E}_3 = \frac{1}{2}\vec{e}_1 \times \vec{e}_2 \;=\; A_3\,\vec{n}_3$

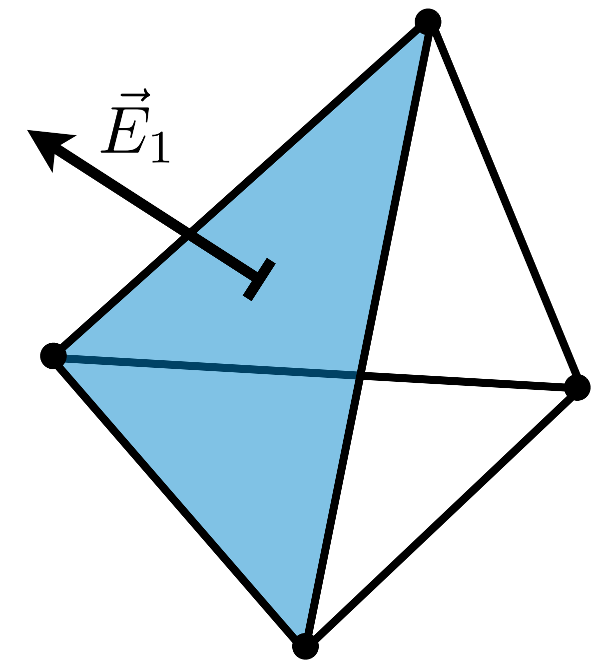
Metric (inverse, densitized)

$$\vec{E}^a \cdot \vec{E}^b = (\det q)\,q^{ab}$$

* Area vectors can be used as fundamental variables:
  tetrahedron specified by *four vectors* $\quad \vec{E}_a \qquad a = 1, 2, 3, 4$
  + *closure* $\quad \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 \;=\; 0$

Area vectors $\vec{E}_a$ $\quad a = 1, 2, 3, 4$

Closure $\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0$



- area of a face $A_a = |\vec{E}_a|$

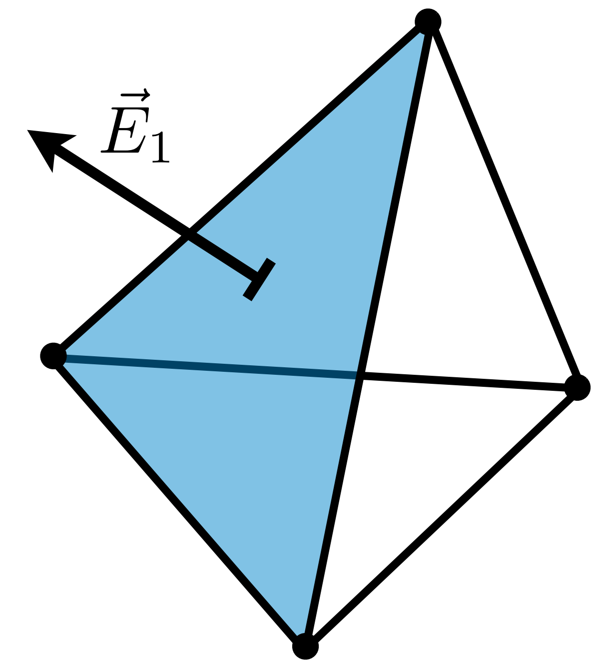
- angle between two faces $\vec{E}_a \cdot \vec{E}_b = A_a A_b \cos\theta_{ab}$

- volume of the tetrahedron

$$V = \frac{\sqrt{2}}{3}\sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|}$$

The phase space of a tetrahedron (face-areas A_a fixed)

$\vec{E}_a = A_a \, \vec{n}_a \qquad\qquad a = 1,2,3,4$



Fuction $\quad f: \; S^2 \times S^2 \times S^2 \times S^2 \; \to \mathbb{R}$

Poisson brackets

$$\{f(\vec{E}_a),\, g(\vec{E}_a)\} \; = \sum_{a=1}^{4} \vec{E}_a \cdot \left(\frac{\partial f}{\partial \vec{E}_a} \times \frac{\partial g}{\partial \vec{E}_a}\right)$$

Fuctions invariant under rotations

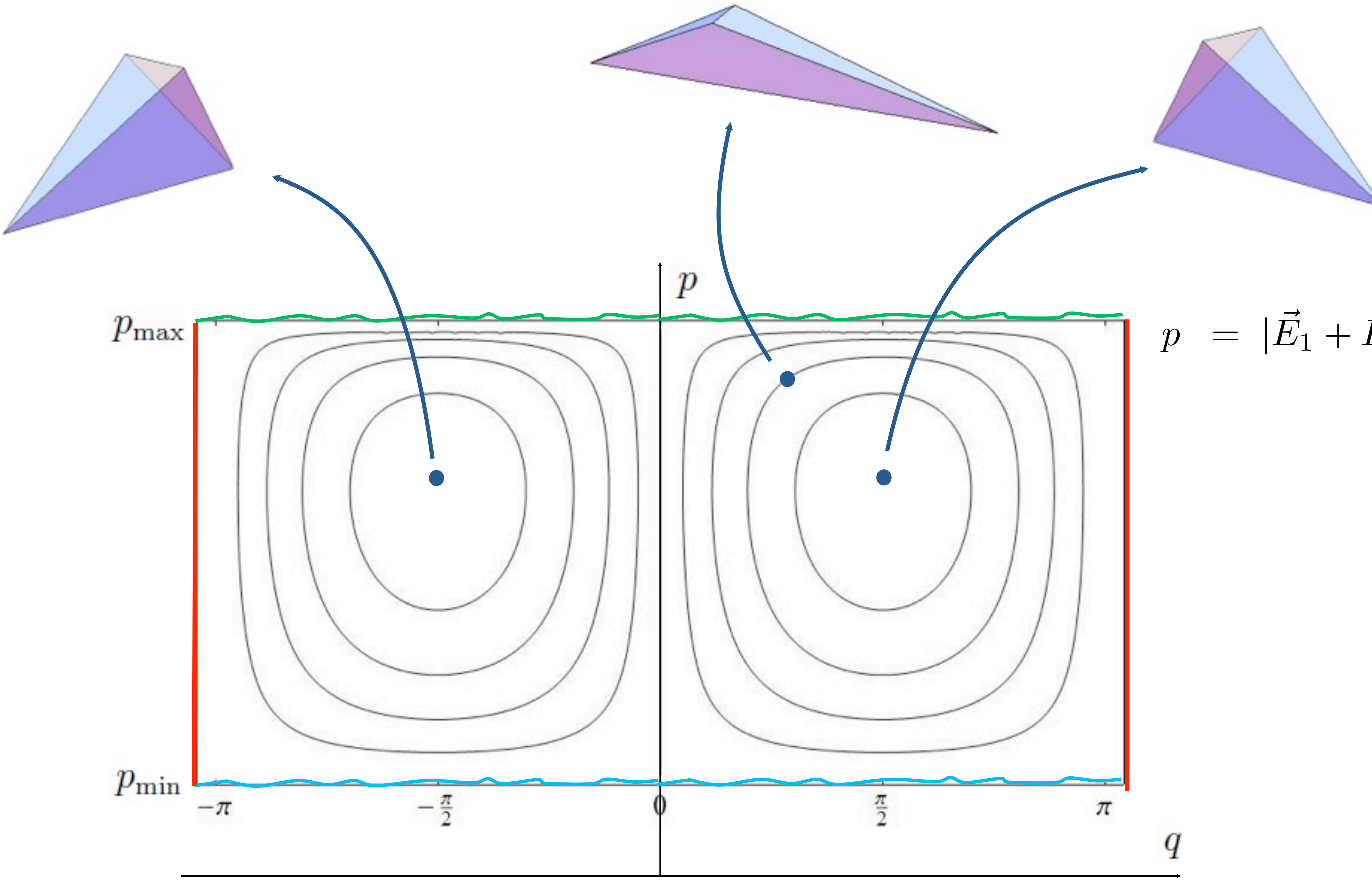
q = angle between $\vec{E}_1 \times \vec{E}_2$ and $\vec{E}_3 \times \vec{E}_4$

$p \; = \; |\vec{E}_1 + \vec{E}_2|$

Canonical variables $\quad \{q, p\} = 1$

Volume as a function of *q* and *p* (equal areas)

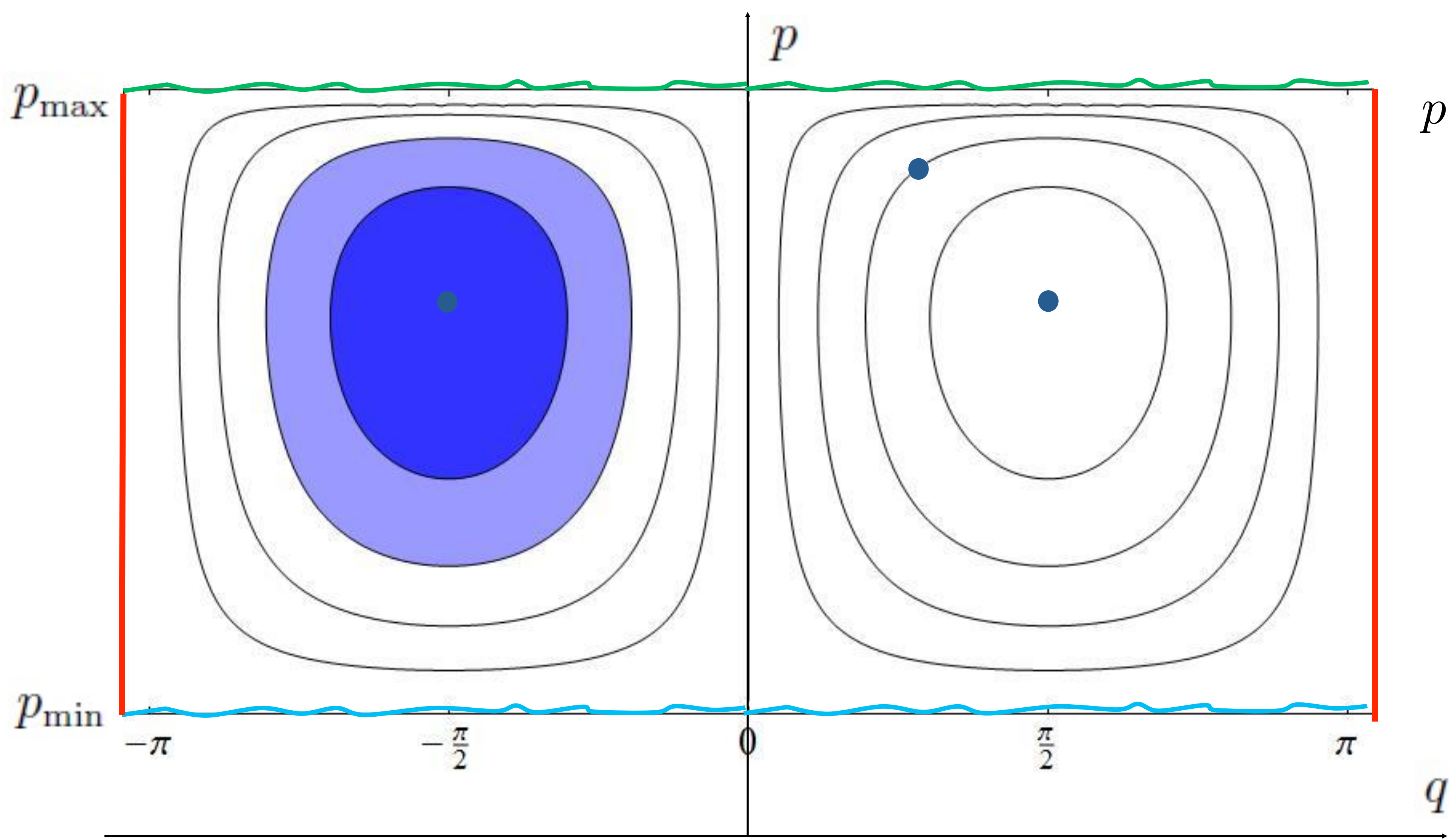
$$V = \frac{\sqrt{2}}{3}\sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|} \; = \; \frac{1}{3\sqrt{2}}\sqrt{p\,(p^2 - 4A^2)|\sin q|}$$



$p = |\vec{E}_1 + \vec{E}_2|$

Quantization condition:

orbits of constant volume enclose an integer number of phase-space cells of area $2\pi\hbar$



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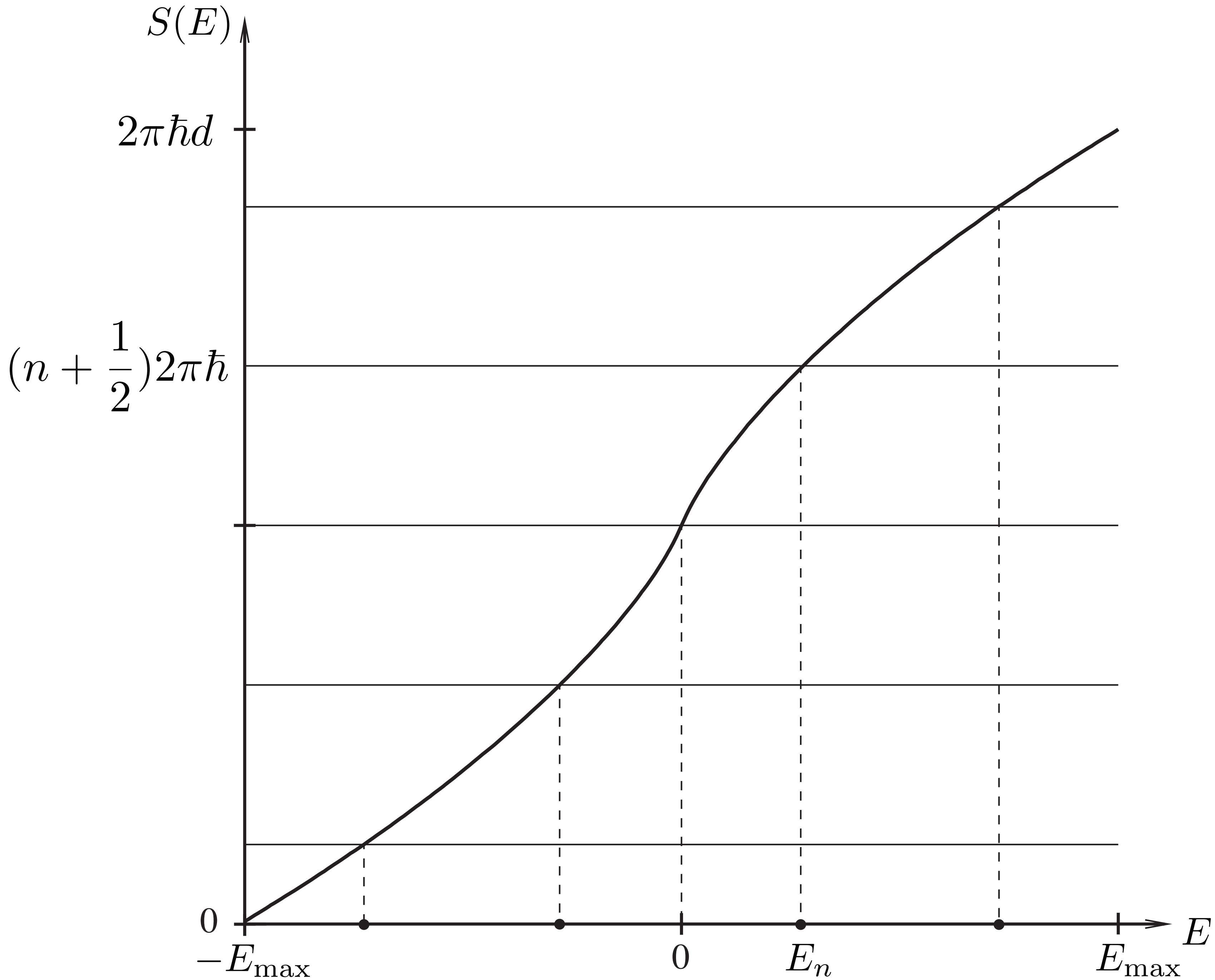

Bohr-Sommerfeld quantization of the Volume

[Bianchi-Haggard, *PRL* '11]

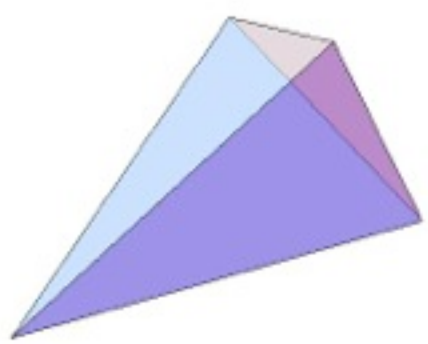

Table: Volume spectrum

j_1 j_2 j_3 j_4	Loop gravity	Bohr-Sommerfeld	Accuracy
$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	0.310	0.252	19%
$\frac{1}{2}$ $\frac{1}{2}$ 1 1	0.396	0.344	13%
$\frac{1}{2}$ $\frac{1}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	0.464	0.406	12%
$\frac{1}{2}$ 1 1 $\frac{3}{2}$	0.498	0.458	8%
1 1 1 1	0	0	exact
	0.620	0.566	9%
$\frac{1}{2}$ $\frac{1}{2}$ 2 2	0.522	0.458	12%
$\frac{1}{2}$ 1 $\frac{3}{2}$ 2	0.577	0.535	7%
1 1 1 2	0.620	0.598	4%
$\frac{1}{2}$ $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$	0.620	0.598	4%
1 1 $\frac{3}{2}$ $\frac{3}{2}$	0	0	exact
	0.753	0.707	6%
...			
6 6 6 7	1.828	1.795	1.8%
	3.204	3.162	1.3%
	4.225	4.190	0.8%
	5.133	5.105	0.5%
	5.989	5.967	0.4%
	6.817	6.799	0.3%

Bohr-Sommerfeld quantization of the Volume

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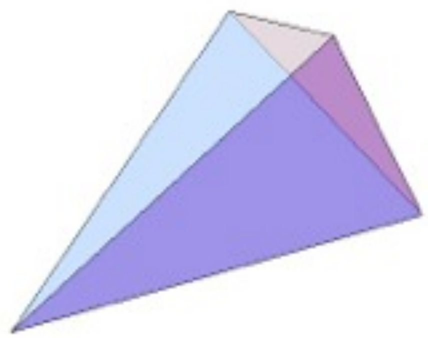
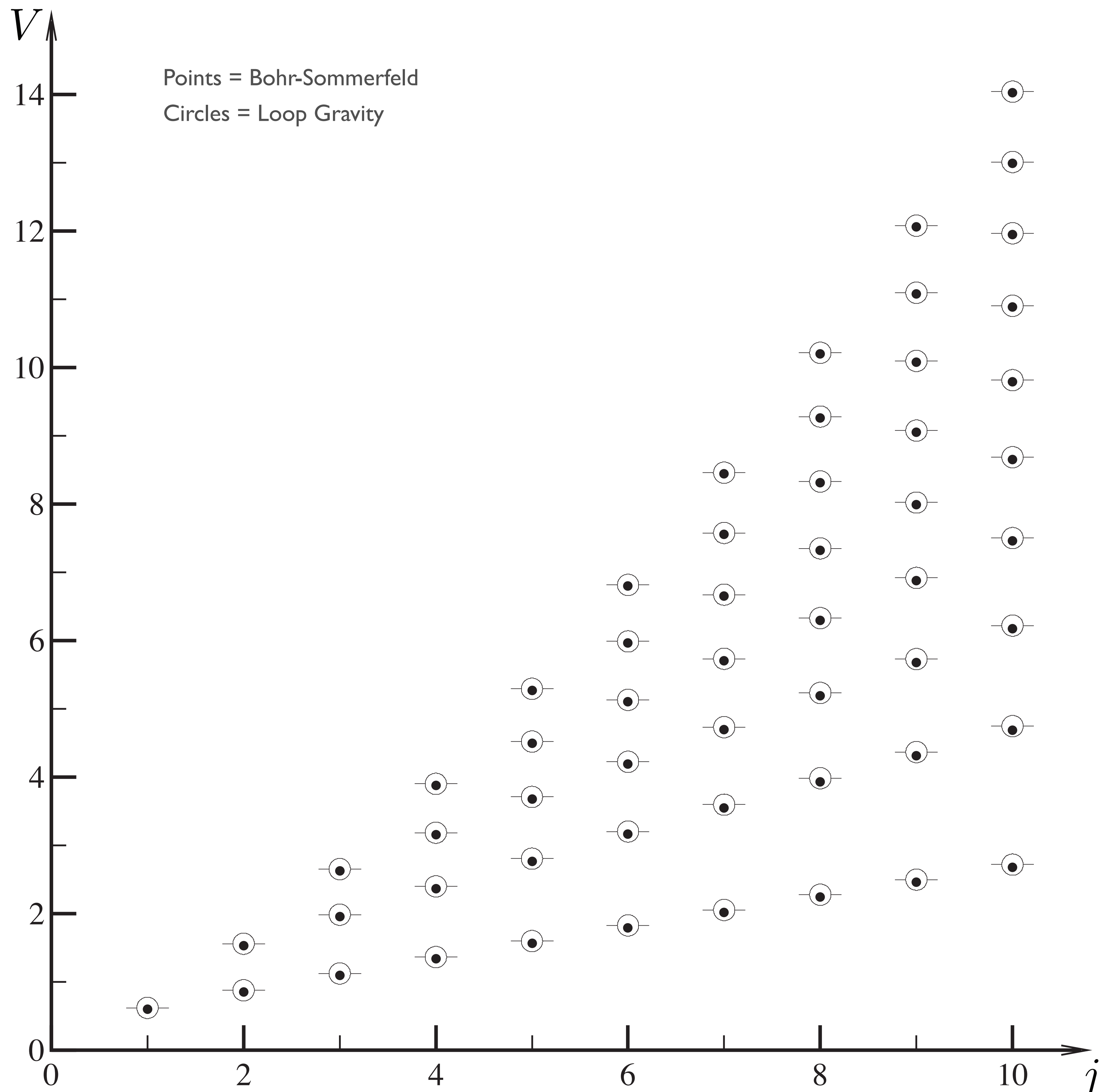


$$A_1 = j + 1/2$$
$$A_2 = j + 1/2$$
$$A_3 = j + 1/2$$
$$A_4 = j + 3/2$$



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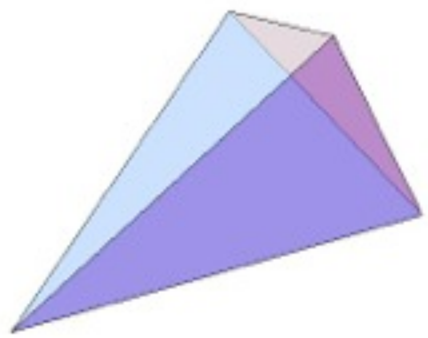
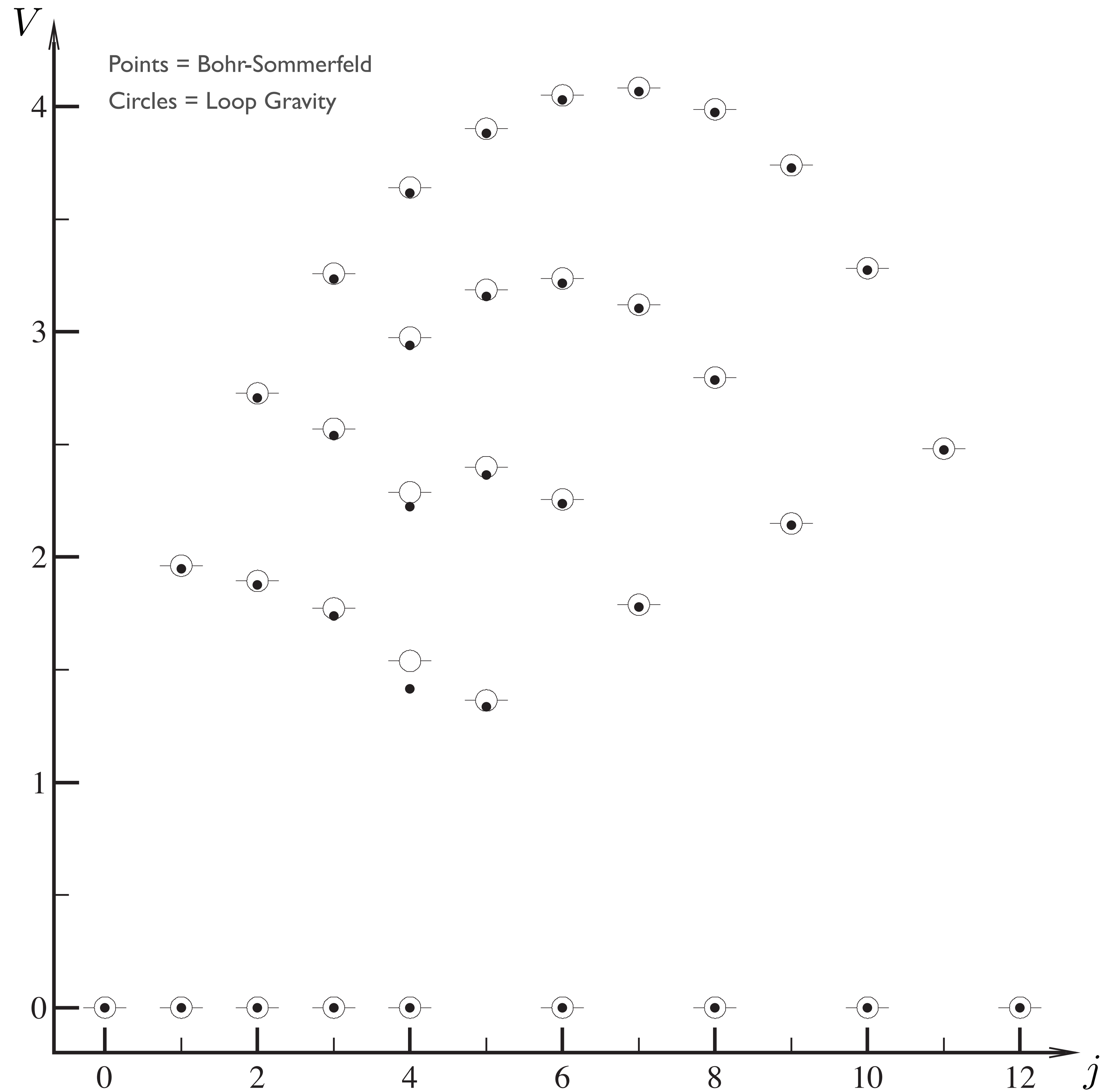


$A_1 = 9/2$

$A_2 = 9/2$

$A_3 = 9/2$

$A_4 = j + 1/2$



 [Barbieri, *NPB* '97]

Spin: irreps of SU(2) $|j,m\rangle \in \mathcal{H}_j$

Intertwiner: invariant tensor $|i\rangle \in \mathrm{Inv}_{SU(2)}\,(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4})$

$$|i\rangle = \sum_{m_1 m_2 m_3 m_4} i_{m_1 m_2 m_3 m_4}\, |j_1,m_1\rangle |j_2,m_2\rangle |j_3,m_3\rangle |j_4,m_4\rangle$$

Rovelli-Smolin '95

Ashtekar-Lewandowski '95

Quantum Geometry

- area normals $\vec{E}_a \;=\; 8\pi G\hbar\gamma\, \vec{L}_a$ $\quad a = 1,2,3,4$

- area operator $A_a \;=\; |\vec{E}_a|$

  spectrum $A_a\,|i\rangle \;=\; 8\pi G\hbar\gamma\,\sqrt{j_a(j_a+1)}\;|i\rangle$

- angle operator $\vec{E}_a \cdot \vec{E}_b$
  (*Penrose metric*)

- Volume operator $V = \dfrac{\sqrt{2}}{3}\sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|}$

Exercise: Volume spectrum in $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$

Basis of intertwiner space $\quad |0\rangle \; , \quad |1\rangle$

Matrix elements of $\quad Q = \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3)$

$$Q_i{}^j = \langle i | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | j \rangle = \begin{pmatrix} 0 & \mathrm{i}\frac{\sqrt{3}}{4} \\ -\mathrm{i}\frac{\sqrt{3}}{4} & 0 \end{pmatrix}$$

Eigenvectors and Eigenvalues

$$Q|q_\pm\rangle = q_\pm |q_\pm\rangle \qquad |q_\pm\rangle = \frac{|0\rangle \pm \mathrm{i}|0\rangle}{\sqrt{2}} \qquad q_\pm = \pm\frac{\sqrt{3}}{4}$$

Volume spectrum

$$V = (8\pi G\hbar\gamma)^{3/2}\, \frac{\sqrt{2}}{3} \sqrt{|Q|}$$

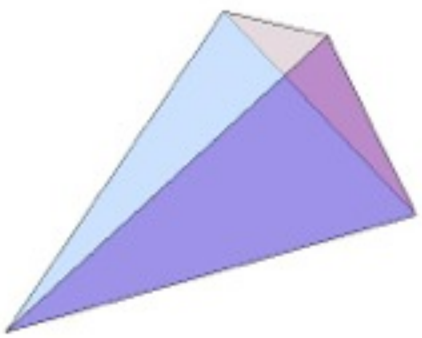
$$V|q_\pm\rangle = v_\pm |q_\pm\rangle$$

$$v_\pm = (8\pi G\hbar\gamma)^{3/2}\, \frac{\sqrt{2}}{3} \sqrt{\frac{\sqrt{3}}{4}}$$

$$\approx (8\pi G\hbar\gamma)^{3/2} \times 0.310$$

Bohr-Sommerfeld quantization of the Volume

[Bianchi-Haggard, *PRL* '11]



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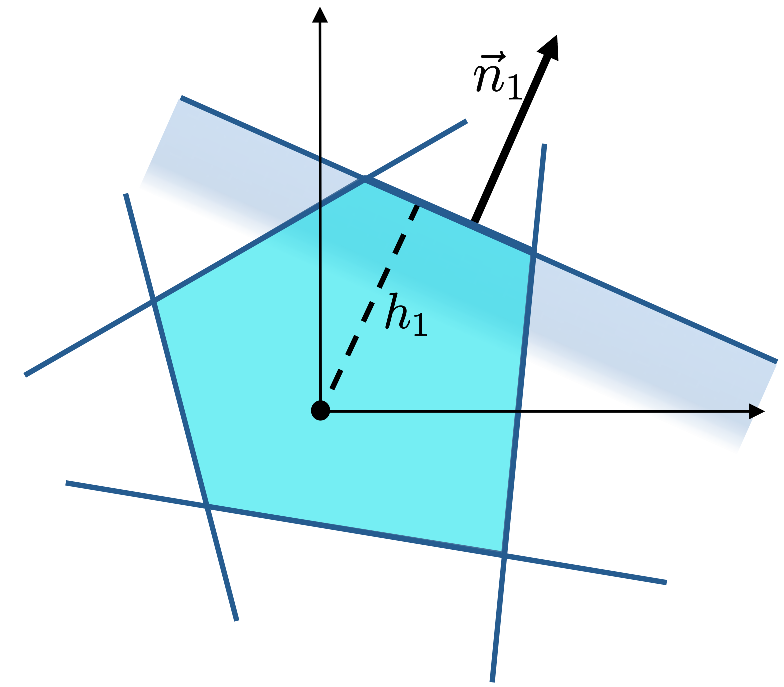
Beyond tetrahedra: number of faces N >4

- Minkowski theorem [1897]

up to rotations, there is a unique convex polyhedron in 3d Euclidean space having faces with normals $\vec{E}_a = A_a\,\vec{n}_a$



A_a = areas

$\vec{n}_a$ = unit vectors

$$\sum_a A_a \vec{n}_a = 0$$

$$\mathcal{P}_N = \{\vec{E}_a,\ a = 1\,..\,N \,|\, \sum_a \vec{E}_a = 0\,,\ \|\vec{E}_a\| = A_a\}/SO(3)$$

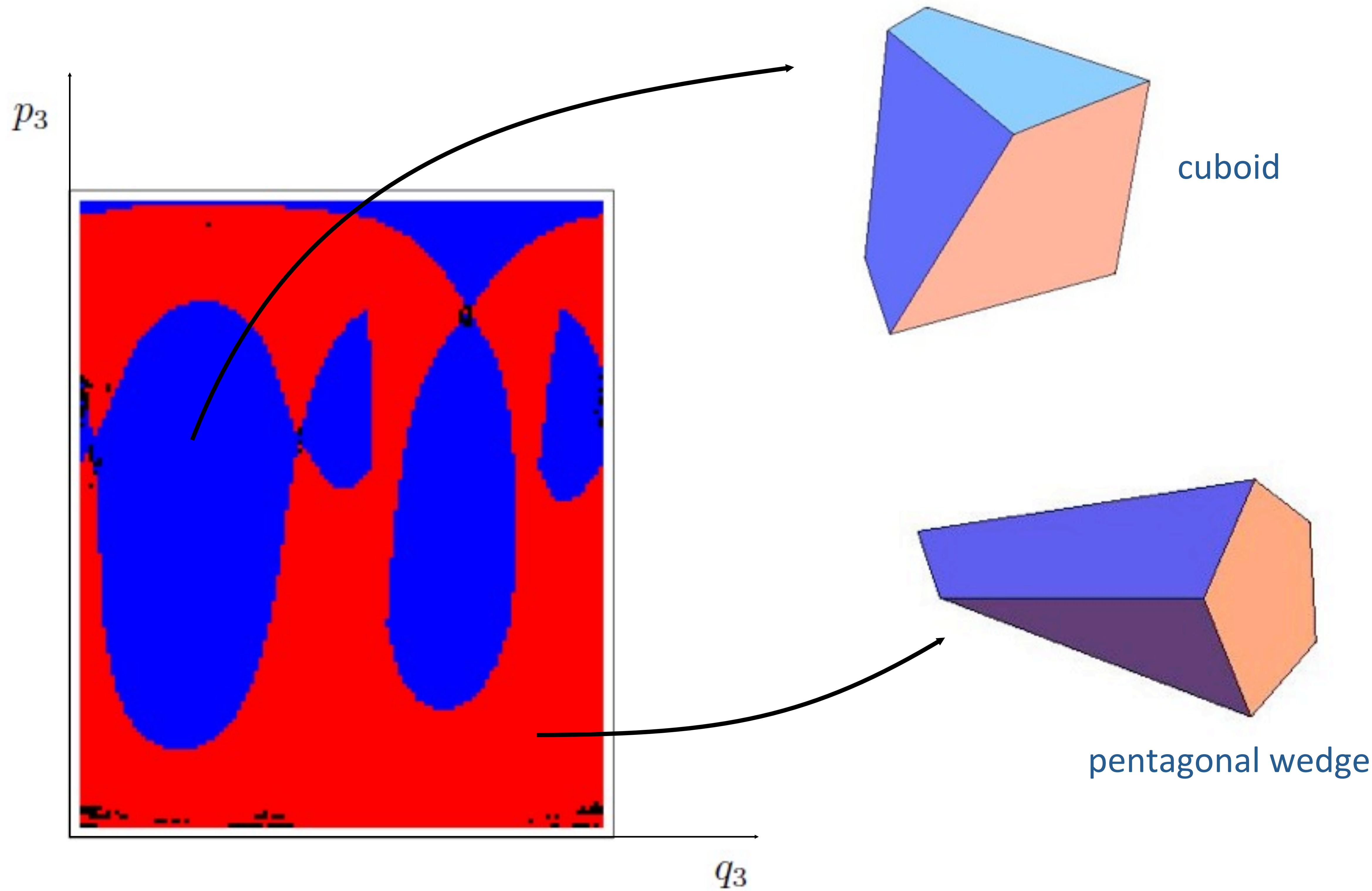
- Kapovich-Millson theorem [1996]

$\mathcal{P}_N$ has naturally the structure of a phase space

Poisson brackets $$\{f(\vec{E}_a),\, g(\vec{E}_a)\} = \sum_{a=1}^{N} \vec{E}_a \cdot \Big(\frac{\partial f}{\partial \vec{E}_a} \times \frac{\partial g}{\partial \vec{E}_a}\Big)$$

⇒ Convex Euclidean polyhedra form a phase space

Quantization ⇒ Hilbert space of intertwiners = nodes of a spin-network graph



Volume spectrum with Quantum Chaos behavior

Haggard *PRD*'13
ColemanSmith-Muller *PRD*'13

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Path integral and the Spinfoam amplitude

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Quantum geometry in Spinfoams

Lecture 3:

- gravitons
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